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A remarkable transformations group on the tangent bundle

by
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Abstract

In the present paper starting from the notion of metrical structure on the tangent bundle, we determine all metrical d-linear connections in the case when the nonlinear connection is arbitrary and we find important particular cases. We study the role of the torsion tensor fields: T and S in this theory. We find the group of transformations of semi-symmetric metrical d-linear connections, corresponding to the same nonlinear connection N, and its important invariants.

Key Words: Tangent bundle, d-linear connection, curvature, torsion, metrical structure, metrical d-linear connection, semi-symmetric metrical d-linear connection, invariants.
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1 Introduction

The geometry of the tangent bundle (TM, π, M) has been studied by M.Matsumoto in [4], by R.Miron and M.Anastasiei in [5], [6], by R.Miron and M.Hashiguchi in [7], by V.Oproiu in [8], by Gh.Atanasiu and I.Ghinea in [1], by R.Bowman in [2], by K.Yano and S.Ishihara in [10],etc.

In the present section we recall the basic notions which are needed. For more detalies see [5-6].

Let M be a real C$^\infty$-differentiable manifold with dimension n, and (TM, π, M) its tangent bundle. If (x^i) is a local coordinates system on a domain U of a chart on M, the induced system of coordonates on $\pi^{-1}(U)$ is $(x^i, y^i), (i = 1, ..., n)$.

Let $V(TM) = ker\ \pi_* \subset T(TM)$ be the vertical bundle, spanned locally by $\{\frac{\partial}{\partial y^i}\}$. A nonlinear connection N determines a supplementary subbundle to V(TM) in T(TM), i.e. $T(TM) = H(TM) \oplus V(TM)$. The adapted basis is $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_{\ i}\frac{\partial}{\partial y^j}$ where $N^i_{\ j}(x, y)$ are the coefficients of the nonlinear connection.

A metric structure on TM is a tensor field G which satisfies the conditions: it is nondegenerate, symmetric and with constant signature. In the adapted basis, the metric structure G is:

$$G(x,y)=\frac{1}{2}g_{ij}(x,y)dx^i\otimes dx^j+\frac{1}{2}\tilde{g}_{ij}(x,y)\delta y^i\otimes\delta y^j, \tag{1}$$

where $\{dx^i,\delta y^i\}$, is the dual basis of $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i}\}$, and $(g_{ij}(x,y),\ \tilde{g}_{ij}(x,y))$ is a pair of given d-tensor fields on TM, of the type (0,2), each of them symmetric and nondegenerate.

The Obata's operators associated to the metric structure G are Ω^{ir}_{sj}, $\tilde{\Omega}^{ir}_{sj}$, Ω^{*ir}_{sj}, $\tilde{\Omega}^{*ir}_{sj}$,(see [5] p.96), which have the same properties as the ones associated with a Finsler space [7].

Let D be a d-linear connection on TM with the local coefficients: $D\Gamma(N)=(L^i{}_{jk},\tilde{L}^i{}_{jk},\tilde{C}^i{}_{jk},C^i{}_{jk})$. It is called metrical d-linear connection with respect to G if D preserves by parallelism the vertical distribution V(TM) and DG=0. In locally coordinate, these mean:

$$g_{ij|k}=0,\ g_{ij}|_k=0,\ \tilde{g}_{ij|k}=0,\ \tilde{g}_{ij}|_k=0, \tag{2}$$

where |, | denote the h-and respective v-covariant derivatives with respect to D.

A d-linear connection, D, on TM, is called semi-symmetric d-linear connection if the torsion tensor fields $T_{(0)}{}^i{}_{jk}$ and $S^i{}_{jk}$ have the form:

$$\begin{aligned} T_{(0)}{}^i{}_{jk}&=\tfrac{1}{n-1}(T_j\delta^i_k-T_k\delta^i_j)=\sigma_j\delta^i_k-\sigma_k\delta^i_j,\\ S^i{}_{jk}&=\tfrac{1}{n-1}(S_j\delta^i_k-S_k\delta^i_j)=\tau_j\delta^i_k-\tau_k\delta^i_j,\end{aligned} \tag{3}$$

where $T_j=T_{(0)}{}^i{}_{ji}$, $S_j=S^i{}_{ji}$ and $\sigma_j=\frac{T_j}{n-1},\tau_j=\frac{S_j}{n-1}$.

2 Metrical d-linear connections on TM.

We shall determine the set of all metrical d-linear connections with respect to G.

Let $\overset{0}{N}$ be another nonlinear connection on TM, with the coefficients: $\overset{0}{N}{}^i{}_j\ (x,y),(i,j=1,...,n)$. Let $\overset{0}{D}\Gamma(\overset{0}{N})=(\overset{0}{L}{}^i{}_{jk},\overset{0}{\tilde{L}}{}^i{}_{jk},\overset{0}{\tilde{C}}{}^i{}_{jk},\overset{0}{C}{}^i{}_{jk})$ be the local coefficients of a fixed d-linear connection $\overset{0}{D}$ on TM. Then any d-linear connection, D, on TM, with local coefficients: $D\Gamma(N)=(L^i{}_{jk},\tilde{L}^i{}_{jk},\tilde{C}^i{}_{jk},C^i{}_{jk})$, can be expresed in the form (see[9]):

$$
\begin{aligned}
&N^i{}_j = \overset{0}{N}{}^i{}_j - A^i{}_j, \quad A^i{}_{j\overset{0}{|}k} = 0,\\
&L^i{}_{jk} = \overset{0}{L}{}^i{}_{jk} + A^l{}_k \, \overset{0}{\tilde{C}}{}^i{}_{jl} - B^i{}_{jk}, \quad C^i{}_{jk} = \overset{0}{C}{}^i{}_{jk} - D^i{}_{jk},\\
&\tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}}{}^i{}_{jk} + A^l{}_k \, \overset{0}{C}{}^i{}_{jl} - \tilde{B}^i{}_{jk}, \quad \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}}{}^i{}_{jk} - \tilde{D}^i{}_{jk},
\end{aligned}
\tag{4}
$$

where $(A^i{}_j, B^i{}_{jk}, \tilde{B}^i{}_{jk}, \tilde{D}^i{}_{jk}, D^i{}_{jk})$ are components of the difference tensor fields of $D\Gamma(N)$ from $\overset{0}{D}\,\Gamma(\overset{0}{N})$.

By extension of the R.Miron-M.Hashiguchi method given for the case of Finsler connections in [7], from (4) and (2) we have one of the most important results concerning to the metrical d-linear connections:

Theorem 2.1. *The set of all metrical d-linear connections on TM, with local coefficients $D\Gamma(N) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$ is given by:*

$$
\begin{aligned}
&N^i_j = \overset{0}{N}{}^i_j - X^i_j,\\
&L^i{}_{jk} = \overset{0}{L}{}^i{}_{jk} + \overset{0}{\tilde{C}}{}^i{}_{jm} X^m{}_k + \tfrac{1}{2} g^{is} (g_{sj\overset{0}{|}k} + g_{sj} \overset{0}{|}_m X^m{}_k) + \Omega^{ir}_{hj} X^h{}_{rk},\\
&\tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}}{}^i{}_{jk} + \overset{0}{C}{}^i{}_{jm} X^m{}_k + \tfrac{1}{2} \tilde{g}^{is} (\tilde{g}_{sj\overset{0}{|}k} + \tilde{g}_{sj} \overset{0}{|}_m X^m{}_k) + \tilde{\Omega}^{ir}_{hj} \tilde{X}^h{}_{rk},\\
&\tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}}{}^i{}_{jk} + \tfrac{1}{2} g^{is} g_{sj} \overset{0}{|}_k + \Omega^{ir}_{hj} \tilde{Y}^h{}_{rk},\\
&C^i{}_{jk} = \overset{0}{C}{}^i{}_{jk} + \tfrac{1}{2} \tilde{g}^{is} \tilde{g}_{sj} \overset{0}{|}_k + \tilde{\Omega}^{ir}_{hj} Y^h{}_{rk}, \quad X^i{}_{j\overset{0}{|}k} = 0,
\end{aligned}
\tag{5}
$$

where X^i_j, X^i_{jk}, $\tilde{X}^i_{jk}$, $\tilde{Y}^i_{jk}$, Y^i_{jk} are arbitrary tensor fields on TM.

Particular cases:

1. If we take $X^i_j = X^i_{jk} = \tilde{X}^i_{jk} = \tilde{Y}^i_{jk} = Y^i_{jk} = 0$ in Theorem 2.1., we obtain an example of metrical d-linear connection on TM, given in (1.12) p.96 from [5].

2. If we take a metrical d-linear connection on TM (e.g. canonical d-linear connection of G, with local coefficients: $C\Gamma(N) = (\overset{c}{L}{}^i{}_{jk}, \overset{c}{\tilde{L}}{}^i{}_{jk}, \overset{c}{\tilde{C}}{}^i{}_{jk}, \overset{c}{C}{}^i{}_{jk})$, (see (1.11) p.96 from [5]) as $\overset{0}{D}$, in Theorem 2.1., we have:

Proposition 2.1. *The set of all metrical d-linear connections on TM, with local coefficients: $D\Gamma(N) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$ is given by:*

$$\begin{aligned}
&N^i_{\ j} = \overset{c}{N}{}^i_{\ j} - X^i_{\ j},\\
&L^i_{\ jk} = \overset{c}{L}{}^i_{\ jk} + \overset{c}{\tilde{C}}{}^i_{\ jm}\, X^m_{\ k} + \Omega^{ir}_{hj} X^h_{\ rk}, \quad \tilde{L}^i_{\ jk} = \overset{c}{\tilde{L}}{}^i_{\ jk} + \overset{c}{C}{}^i_{\ jm}\, X^m_{\ k} + \tilde{\Omega}^{ir}_{hj} \tilde{X}^h_{\ rk},\\
&\tilde{C}^i_{\ jk} = \overset{c}{\tilde{C}}{}^i_{\ jk} + \Omega^{ir}_{hj} \tilde{Y}^h_{\ rk}, \quad C^i_{\ jk} = \overset{c}{C}{}^i_{\ jk} + \tilde{\Omega}^{ir}_{hj} Y^h_{\ rk}, \quad X^i_{\underset{j|k}{0}} = 0.
\end{aligned} \tag{6}$$

3. If we take $X^i_{\ j} = 0$ in Proposition 2.1 we obtain: Theorem 1.3 p.96 from [5].

4. If we shall tray replace the arbitrary tensor fields X^i_{jk}, Y^i_{jk} in Theorem 2.1 with $X^i_j = 0$, by the torsion fields $T_{(0)}{}^i_{\ jk}$, S^i_{jk} we find a result obtained by R.Miron and M.Anastasiei in (2.6) p.98 from [5].

Taking into account (3) and (2.6) p.98, [5] we obtain:

Proposition 2.2. *The set of all semi-symmetric metrical d-linear connections with local coefficients:* $D\Gamma(N) = (L^i_{\ jk}, \tilde{L}^i_{\ jk}, \tilde{C}^i_{\ jk}, C^i_{\ jk})$, *is given by:*

$$\begin{cases} L^i_{\ jk} = \overset{c}{L}{}^i_{\ jk} + \sigma_j \delta^i_k - g_{jk} g^{im} \sigma_m, & \tilde{L}^i_{\ jk} = \overset{c}{\tilde{L}}{}^i_{\ jk},\\ \tilde{C}^i_{\ jk} = \overset{c}{\tilde{C}}{}^i_{\ jk}, \quad C^i_{\ jk} = \overset{c}{C}{}^i_{\ jk} + \tau_j \delta^i_k - \tilde{g}_{jk} \tilde{g}^{im} \tau_m. \end{cases} \tag{7}$$

3 The group of transformations of semi-symmetric metrical d-linear connections.

We study the transformations $t(\sigma_j, \tau_j) : D\Gamma(N) \to D\bar{\Gamma}(N)$ of the semi-symmetric metrical d-linear connections, on TM, with respect to G.

Let N be a given nonlinear connection. Then any semi-symmetric metrical d-linear connection with local coefficients $\bar{D}\Gamma(N) = (\bar{L}^i_{\ jk}, \bar{\tilde{L}}^i_{\ jk}, \bar{\tilde{C}}^i_{\ jk}, \bar{C}^i_{\ jk})$ is given by (7). We have:

Proposition 3.1. *Two semi-symmetric metrical d-linear connections with local coefficients* $D\Gamma(N) = (L^i_{\ jk}, \tilde{L}^i_{\ jk}, \tilde{C}^i_{\ jk}, C^i_{\ jk})$ *and* $\bar{D}\Gamma(N) = (\bar{L}^i_{\ jk}, \bar{\tilde{L}}^i_{\ jk}, \bar{\tilde{C}}^i_{\ jk}, \bar{C}^i_{\ jk})$ *are related as follows:*

$$\begin{array}{lcl}
\bar{L}^i_{\ jk} = L^i_{\ jk} + \sigma_j \delta^i_k - g_{jk} g^{im} \sigma_m, & & \bar{L}^i_{\ jk} = L^i_{\ jk} + 2\Omega^{im}_{kj} \sigma_m,\\
\bar{\tilde{L}}^i_{\ jk} = \tilde{L}^i_{\ jk}, & & \bar{\tilde{L}}^i_{\ jk} = \tilde{L}^i_{\ jk},\\
\bar{\tilde{C}}^i_{\ jk} = \tilde{C}^i_{\ jk}, & or: & \bar{\tilde{C}}^i_{\ jk} = \tilde{C}^i_{\ jk},\\
\bar{C}^i_{\ jk} = C^i_{\ jk} + \tau_j \delta^i_k - \tilde{g}_{jk} \tilde{g}^{im} \tau_m, & & \bar{C}^i_{\ jk} = C^i_{\ jk} + 2\tilde{\Omega}^{im}_{kj} \tau_m.
\end{array} \tag{8}$$

Conversely, given $\sigma_j \in \mathcal{X}^*(M), \tau_j \in \mathcal{X}^*(M)$ the above (8) is thought to be a transformation of a semi-symmetric metrical d-linear connection D, with local coefficients $D\Gamma(N) = (L^{i}{}_{jk}, \tilde{L}^{i}{}_{jk}, \tilde{C}^{i}{}_{jk}, C^{i}{}_{jk})$ to a semi-symmetric metrical d-linear connection $\bar{D}$, with local coefficients $\bar{D}\Gamma(N) = (\bar{L}^{i}{}_{jk}, \bar{\tilde{L}}^{i}{}_{jk}, \bar{\tilde{C}}^{i}{}_{jk}, \bar{C}^{i}{}_{jk})$.

We shall denote this transformation by: $t(\sigma_j, \tau_j)$.

Thus we have:

Proposition 3.2. *The set* $\overset{ms}{\mathcal{T}}_N$ *of all transformations* $t(\sigma_j, \tau_j) : D\Gamma(N) \to \bar{D}\Gamma(N)$ *of semi-symmetric metrical d-linear connections given by (8) is an abelian group, together with the mapping product:*

$t(\bar{\sigma}_j, \bar{\tau}_j) \circ t(\sigma_j, \tau_j) = t(\sigma_j + \bar{\sigma}_j, \tau_j + \bar{\tau}_j)$.

This group acts on the set of all semi-symmetric metrical d-linear connections, corresponding to the same nonlinear connection N, transitively.

In order to find invariants of the group $\overset{ms}{\mathcal{T}}_N$, let us consider the transformation formulas of the torsion and the curvature tensor fields by a transformation of d-linear connections corresponding to the same nonlinear connection N ($A^{i}{}_{j} = 0$):

$t(0, B^{i}{}_{jk}, \tilde{B}^{i}{}_{jk}, \tilde{D}^{i}{}_{jk}, D^{i}{}_{jk}) : D\Gamma(N) \to \bar{D}\Gamma(N)$

$$\begin{aligned} &\bar{N}^{j}_{i} = N^{j}_{i},\ \bar{L}^{i}{}_{jk} = L^{i}{}_{jk} - B^{i}{}_{jk},\ \bar{\tilde{L}}^{i}{}_{jk} = \tilde{L}^{i}{}_{jk} - \tilde{B}^{i}{}_{jk}, \\ &\bar{\tilde{C}}^{i}{}_{jk} = \tilde{C}^{i}{}_{jk} - \tilde{D}^{i}{}_{jk},\ \bar{C}^{i}{}_{jk} = C^{i}{}_{jk} - D^{i}{}_{jk}. \end{aligned} \tag{9}$$

Proposition 3.3. *By a transformation (9) of d-linear connections, corresponding to the same nonlinear connection N,* $D\Gamma(N) \to \bar{D}\Gamma(N)$*, the torsion and curvature tensor fields,* ${T_{(0)}}^{i}{}_{jk}, {T_{(1)}}^{i}{}_{jk}, {P_{(1)}}^{i}{}_{jk}, {P_{(2)}}^{i}{}_{jk}, S^{i}{}_{jk}, {R_{(0)j}}^{i}{}_{kl}, {R_{(1)j}}^{i}{}_{kl}, {P_{(0)j}}^{i}{}_{kl}$, ${P_{(1)j}}^{i}{}_{kl}, {S_{(0)j}}^{i}{}_{kl}, {S_{(1)j}}^{i}{}_{kl}$ *are transformed as follows:*

$$\bar{T}_{(0)}{}^{i}{}_{jk} = {T_{(0)}}^{i}{}_{jk} + (B^{i}{}_{kj} - B^{i}{}_{jk}),\ \bar{T}_{(1)}{}^{i}{}_{jk} = {T_{(1)}}^{i}{}_{jk}, \tag{10}$$

$$\bar{P}_{(1)}{}^{i}{}_{jk} = {P_{(1)}}^{i}{}_{jk} - \tilde{D}^{i}{}_{jk},\ \bar{P}_{(2)}{}^{i}{}_{jk} = {P_{(2)}}^{i}{}_{jk} + \tilde{B}^{i}{}_{kj}, \tag{11}$$

$$\bar{S}^{i}{}_{jk} = S^{i}{}_{jk} + (D^{i}{}_{kj} - D^{i}{}_{jk}). \tag{12}$$

$$\bar{R}_{(0)j}{}^{i}{}_{kl} = {R_{(0)j}}^{i}{}_{kl} - \tilde{D}^{i}{}_{jh}R^{h}_{kl} + B^{i}{}_{jh}{T_{(0)}}^{h}{}_{kl} + \mathcal{A}_{kl}\{-B^{i}{}_{jk|l} + B^{h}_{jk}B^{i}{}_{hl}\}, \tag{13}$$

$$\bar{R}_{(1)j}{}^{i}{}_{kl} = {R_{(1)j}}^{i}{}_{kl} - D^{i}{}_{jh}R^{h}_{kl} - \tilde{B}^{i}{}_{jh}{T_{(0)}}^{h}{}_{kl} + \mathcal{A}_{kl}\{-\tilde{B}^{i}{}_{jk|l} + \tilde{B}^{h}_{jk}\tilde{B}^{i}{}_{hl}\}, \tag{14}$$

$$\begin{aligned} &\bar{P}_{(0)j}{}^{i}{}_{kl} = {P_{(0)j}}^{i}{}_{kl} - \tilde{D}^{i}{}_{jh}{P_{(2)}}^{h}{}_{kl} - B^{i}_{jh}\tilde{C}^{h}_{kl} - B^{i}_{jk}|_{l} + \tilde{D}^{i}{}_{jl|k} + \\ &+ B^{h}_{jk}\tilde{D}^{i}{}_{hl} - \tilde{D}^{h}_{jl}B^{i}{}_{hk}, \end{aligned} \tag{15}$$

$$\begin{aligned} &\bar{P}_{(1)j}{}^{i}{}_{kl} = {P_{(1)j}}^{i}{}_{kl} - D^{i}{}_{jh}{P_{(2)}}^{h}{}_{kl} - \tilde{B}^{i}_{jh}\tilde{C}^{h}_{kl} - \tilde{B}^{i}_{jk}|_{l} + D^{i}{}_{jl|k} + \\ &+ \tilde{B}^{h}_{jk}D^{i}{}_{hl} - D^{h}_{jl}\tilde{B}^{i}{}_{hk}, \end{aligned} \tag{16}$$

$$\bar{S}_{(0)j}{}^{i}{}_{kl} = {S_{(0)j}}^{i}{}_{kl} - \tilde{D}^{i}{}_{jh}S^{h}_{kl} + \mathcal{A}_{kl}\{-\tilde{D}^{i}{}_{jk}|_{l} + \tilde{D}^{h}_{jk}\tilde{D}^{i}{}_{hl}\}, \tag{17}$$

$$\bar{S}_{(1)j}{}^{i}{}_{kl} = {S_{(1)j}}^{i}{}_{kl} - D^{i}{}_{jh}S^{h}_{kl} + \mathcal{A}_{kl}\{D^{i}{}_{jk}|_{l} + D^{h}_{jk}D^{i}{}_{hl}\}. \tag{18}$$

We consider the tensor fields:

$$\mathcal{K}_{(0)j\ kl}^{\ \ \ \ i} = R_{(0)j\ kl}^{\ \ \ \ i} - \tilde{C}^i_{\ jh} R^h_{\ kl}, \tag{19}$$

$$\mathcal{K}_{(1)j\ kl}^{\ \ \ \ i} = R_{(1)j\ kl}^{\ \ \ \ i} - C^i_{\ jh} R^h_{\ kl}, \tag{20}$$

$$\mathcal{P}_{(0)j\ kl}^{\ \ \ \ i} = \mathcal{A}_{kl}\{P_{(0)j\ kl}^{\ \ \ \ i} - \tilde{C}^i_{\ jh} \frac{\partial N^h_{\ k}}{\partial y^l}\}, \tag{21}$$

$$\mathcal{P}_{(1)j\ kl}^{\ \ \ \ i} = \mathcal{A}_{kl}\{P_{(1)j\ kl}^{\ \ \ \ i} - C^i_{\ jh} \frac{\partial N^h_{\ k}}{\partial y^l}\}. \tag{22}$$

Proposition 3.4. *By a transformation (9) of d-linear connections corresponding to the same nonlinear connection N, the tensor fields* $\mathcal{K}_{(0)j\ kl}^{\ \ \ \ i}, \mathcal{K}_{(1)j\ kl}^{\ \ \ \ i}, \mathcal{P}_{(0)j\ kl}^{\ \ \ \ i}, \mathcal{P}_{(1)j\ kl}^{\ \ \ \ i}$ *are transformed as follows:*

$$\bar{\mathcal{K}}_{(0)j\ kl}^{\ \ \ \ i} = \mathcal{K}_{(0)j\ kl}^{\ \ \ \ i} + B^i_{\ jh} T_{(0)\ kl}^{\ \ \ h} + \mathcal{A}_{kl}\{-B^i_{\ jk|l} + B^h_{\ jk} B^i_{\ hl}\}, \tag{23}$$

$$\bar{\mathcal{K}}_{(1)j\ kl}^{\ \ \ \ i} = \mathcal{K}_{(1)j\ kl}^{\ \ \ \ i} - \tilde{B}^i_{\ jh} T_{(0)\ kl}^{\ \ \ h} + \mathcal{A}_{kl}\{-\tilde{B}^i_{\ jk|l} + \tilde{B}^h_{\ jk} \tilde{B}^i_{\ hl}\}, \tag{24}$$

$$\begin{aligned} \bar{\mathcal{P}}_{(0)j\ kl}^{\ \ \ \ i} &= \mathcal{P}_{(0)j\ kl}^{\ \ \ \ i} + \mathcal{A}_{kl}\{-B^i_{\ jk}|_l + \tilde{D}^i_{\ jl|k} + B^h_{\ jk}\tilde{D}^i_{hl} - \tilde{D}^h_{\ jl} B^i_{\ hk} + \\ &+ \tilde{D}^i_{\ jh} \tilde{L}^h_{\ lk} - B^i_{\ jh} \tilde{C}^h_{\ kl}\}, \end{aligned} \tag{25}$$

$$\begin{aligned} \bar{\mathcal{P}}_{(1)j\ kl}^{\ \ \ \ i} &= \mathcal{P}_{(1)j\ kl}^{\ \ \ \ i} + \mathcal{A}_{kl}\{-\tilde{B}^i_{\ jk}|_l + D^i_{\ jl|k} + \tilde{B}^h_{\ jk} D^i_{hl} - D^h_{\ jl} \tilde{B}^i_{\ hk} + \\ &+ D^i_{\ jh} \tilde{L}^h_{\ lk} - \tilde{B}^i_{\ jh} \tilde{C}^h_{\ kl}\}, \end{aligned} \tag{26}$$

Substituting in (9):

$$B^i_{\ jk} = -2\Omega^{im}_{kj}\sigma_m,\ \tilde{B}^i_{\ jk} = 0,\ \tilde{D}^i_{\ jk} = 0,\ D^i_{jk} = -2\tilde{\Omega}^{im}_{kj}\tau_m, \tag{27}$$

we have the transformation (8)

Proposition 3.5. *By a transformation (8) of semi-symmetric metrical d-linear connections corresponding to the same nonlinear connection N, the tensor fields* $\mathcal{K}_{(0)j\ kl}^{\ \ \ \ i}, \mathcal{K}_{(1)j\ kl}^{\ \ \ \ i}, S_{(0)j\ kl}^{\ \ \ \ i}, S_{(1)j\ kl}^{\ \ \ \ i}$ *are transformed as follows:*

$$\bar{\mathcal{K}}_{(0)j\ kl}^{\ \ \ \ i} = \mathcal{K}_{(0)j\ kl}^{\ \ \ \ i} + 2\mathcal{A}_{kl}\{\Omega^{im}_{kj}\sigma_{ml}\}, \tag{28}$$

$$\bar{\mathcal{K}}_{(1)j\ kl}^{\ \ \ \ i} = \mathcal{K}_{(1)j\ kl}^{\ \ \ \ i}, \tag{29}$$

$$\bar{S}_{(0)j\ kl}^{\ \ \ \ i} = S_{(0)j\ kl}^{\ \ \ \ i}, \tag{30}$$

$$\bar{S}_{(1)j\ kl}^{\ \ \ \ i} = S_{(1)j\ kl}^{\ \ \ \ i} + 2\mathcal{A}_{kl}\{\tilde{\Omega}^{im}_{kj}\tau_{ml}\}, \tag{31}$$

where:

$$\sigma_{ml} = \sigma_{m|l} - \sigma_m \sigma_l + \tfrac{1}{2} g_{ml} \sigma - \tfrac{\sigma_m T_{(0)l}}{n-1}, \ (\sigma = g^{rm} \sigma_r \sigma_m), \tag{32}$$

$$\tau_{ml} = \tau_m|_l - \tau_m \tau_l + \tfrac{1}{2} \tilde{g}_{ml} \tau - \tfrac{\tau_m S_l}{n-1}, \ (\tau = \tilde{g}^{rm} \tau_r \tau_m). \tag{33}$$

Using this results we can determine the invariants of the group $\overset{ms}{\mathcal{T}}_N$ using a well-known elimination method:

Theorem 3.1. *For $n > 2$ the following tensor fields:*

$$H_{(0)j}{}^{i}{}_{kl}, H_{(1)j}{}^{i}{}_{kl}, M_{(0)j}{}^{i}{}_{kl}, M_{(1)j}{}^{i}{}_{kl}$$

are invariants of the group $\overset{ms}{\mathcal{T}}_N$, of transformations, of semi-symmetric metrical d-linear connections on TM, corresponding to the same nonlinear connection N:

$$H_{(0)j}{}^{i}{}_{kl} = \mathcal{K}_{(0)j}{}^{i}{}_{kl} + \tfrac{2}{n-2} \mathcal{A}_{kl} \{ \Omega^{ir}_{kj} (\mathcal{K}_{(0)rl} - \tfrac{\mathcal{K}_{(0)} g_{rl}}{2(n-1)}) \}, \tag{34}$$

$$H_{(1)j}{}^{i}{}_{kl} = \mathcal{K}_{(1)j}{}^{i}{}_{kl}, \tag{35}$$

$$M_{(0)j}{}^{i}{}_{kl} = S_{(0)j}{}^{i}{}_{kl}, \tag{36}$$

$$M_{(1)j}{}^{i}{}_{kl} = S_{(1)j}{}^{i}{}_{kl} + \tfrac{2}{n-2} \mathcal{A}_{kl} \{ \Omega^{ir}_{kj} (S_{(1)rl} - \tfrac{S_{(1)} \tilde{g}_{rl}}{2(n-1)}) \}, \tag{37}$$

where:

$$\mathcal{K}_{(0)jk} = \mathcal{K}_{(0)j}{}^{i}{}_{ki}, \quad \mathcal{K}_{(0)} = g^{jk} \mathcal{K}_{(0)jk}, \quad S_{(1)jk} = S_{(1)j}{}^{i}{}_{ki}, \quad S_{(1)} = \tilde{g}^{jk} S_{(1)jk}.$$

We note that the results obtained from Theorem 3.1 in the particular case of the normal d-linear connections support the findings of R.Miron and M.Hashiguchi in their paper [7].

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