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Regular refraction property for a conic lens

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Abstract

We obtain the regular refraction interval for a conic Γ with respect to its focus. In all cases this interval is $[0, \sqrt{1+e^2}/e]$, where e is the eccentricity of the conic. Also we obtain the regular refraction interval for an arc of sinusoid $\overset{\frown}{AB}$ with respect to the midpoint of the line segment $[AB]$. Finally we determine the starlike arcs Γ for which the regular refraction interval with respect to the origin is the same for every arc $\Gamma' \subseteq \Gamma$. Such curves are arcs of the ellipses with the origin of axis in a focus and a class of starlike arcs of logarithmic spirals.

Key Words: Starlike curves, regular refraction property.
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1 Introduction and preliminaries

Let Γ be a smooth curve in the complex plane with a parametrization

$$z = z(t) = x(t) + iy(t), \ \ t \in [\alpha, \beta].$$

We suppose that Γ is a directed arc, the direction being that determined as t increases. The arc Γ is said to be starlike, with respect to a given point $p \notin \Gamma$, if $\arg[z(t) - p]$ is an increasing function of t. This means that

$$\frac{d}{dt} \arg[z(t) - p] > 0, \ \ t \in [\alpha, \beta].$$

In the sequel, we will suppose that $p = 0$ and that Γ is a starlike curve. Let R be the radius vector from the origin to the point $z(t) \in \Gamma$ and let N be the outer normal to Γ at the point $z(t)$. We denote by ω the angle between R and N. For a fixed nonnegative number γ we consider the vector V starting from $z(t)$ such that

$$\sin \psi = \gamma \sin \omega. \tag{1}$$

where ψ is the angle between V and N.

From the optical point of view we remark that if Γ separates two media with different indices of refraction and R and V are trajectories of the light in these two media, then the equation (1) is the well-known refraction law.

Definition 1. [1], [2]. We say that the curve Γ has the regular refraction property of index γ, with respect to the point $p = 0$, written $\Gamma \in RP(\gamma, 0)$, if the argument of the vector V is an increasing function of t on $[\alpha, \beta]$, i.e.

$$\frac{d}{dt} \arg V(t) \geq 0,\ t \in [\alpha, \beta]. \tag{2}$$

For a given starlike curve with respect to the origin, the problem is to obtain the largest interval $[\gamma_0, \gamma_1] \subseteq \mathbb{R}$ such that $\Gamma \in RP(\gamma, 0)$, for all $\gamma \in [\gamma_0, \gamma_1]$. It is clear that $0 \leq \gamma_0 < 1 < \gamma_1 \leq \infty$ for all starlike curves.

If we denote by $\chi = \arg V(t)$ and by $\varphi = \arg z(t)$, then it is obvious that

$$\chi = \varphi + \omega - \psi. \tag{3}$$

As in [5] we have

$$\varphi' = \mathrm{Im}\frac{z'}{z},\ \omega' = \mathrm{Im}\left[\frac{z''}{z'} - \frac{z'}{z}\right], \psi' = \frac{\gamma \cos\omega}{(1-\gamma^2+\gamma^2\cos^2\omega)^{1/2}}\omega',$$

where $\cos\omega = \frac{\mathrm{Im}(z'/z)}{|z'/z|}$ and using (3), the inequality (2) becomes

$$\gamma\left(\mathrm{Im}\frac{z'}{z}\right)^2 + \left[\sqrt{\Delta} - \gamma\mathrm{Im}\frac{z'}{z}\right]\mathrm{Im}\frac{z''}{z'} \geq 0,\ t \in [\alpha, \beta], \tag{4}$$

where

$$\Delta = \left(1-\gamma^2\right)\left|\frac{z'}{z}\right|^2 + \gamma^2\left(\mathrm{Im}\frac{z'}{z}\right)^2 \geq 0. \tag{5}$$

The condition (5) is equivalent to $\gamma|\sin\omega| = |\sin\psi| \leq 1$ which, in its turn, is equivalent to the existence of the refraction. Otherwise at the points $z(t)$, where $\Delta < 0$ we have total reflection. We also remark that since Γ is starlike we necessarily have $\varphi' = \mathrm{Im}\frac{z'}{z} > 0$. If we denote

$$F = F(\gamma, t) = 1 - \gamma\frac{\mathrm{Im}\frac{z'}{z}}{\sqrt{\Delta}}, \tag{6}$$

then we have $F(1, t) = 0$ and the function F defined by (6) is decreasing with respect to γ. The condition (4) is now equivalent to

$$(1-F)\ \mathrm{Im}\frac{z'}{z} + F\ \mathrm{Im}\frac{z''}{z'} \geq 0, \tag{7}$$

and from (7) it follows that for a given starlike Γ, there exists the largest interval $I = [\gamma_0, \gamma_1]$ with $1 \in I$, such that Γ has the regular refraction property for all $\gamma \in I$.

Definition 2. The regular refraction interval of the starlike curve Γ, with respect to the point $p = 0$, written $RRI[\Gamma, 0]$, is the largest interval $[\gamma_0, \gamma_1],\ 0 \leq \gamma_0 < 1 < \gamma_1 \leq \infty$, such that $\Gamma \in RP(\gamma, 0)$, for all $\gamma \in [\gamma_0, \gamma_1]$.

In [1] and [2] it was proved that any convex curve has the regular refraction interval with respect to the origin of the form $[0, \gamma_1]$ with $\gamma_1 > 1$.

In [3,4,5] it was find the regular refraction interval for an ellipse with respect to its center, for the parabolic lens with respect to its focus and for a circle, with respect to an arbitrary interior point. The following theorem is a complete variant of Theorem 2 in [5]:

Theorem 3. *Let Γ be a starlike curve, with respect to the point $p = 0 \notin \Gamma$, defined by the equation $z = z(t), t \in [\alpha, \beta]$. Let $RRI[\Gamma, 0] = [\gamma_0, \gamma_1]$ and let denote*

$$A = A(t) = \mathrm{Re}\frac{z'(t)}{z(t)},\ B = B(t) = \mathrm{Im}\frac{z'(t)}{z(t)},\ C = C(t) = \mathrm{Im}\frac{z''(t)}{z'(t)}.$$

Let

$$\delta = \delta(t) = \sqrt{1 + \frac{B^2}{A^2}}, \tag{8}$$

and

$$\sigma = \sigma(t) = \sqrt{\frac{(A^2 + B^2)C^2}{B^2(B - C)^2 + A^2C^2}}. \tag{9}$$

a) If $B(t) > 0, C(t) \geq 0, B(t) - C(t) \geq 0$, for all $t \in [\alpha, \beta]$, then $\gamma_0 = 0$ and

$$\gamma_1 = \min\left\{\sqrt{1 + \frac{B^2}{A^2}} : t \in [\alpha, \beta]\right\} = \min\left\{\delta(t) : t \in [\alpha, \beta]\right\}.$$

b) If $B(t) > 0,\ C(t) \geq 0$, for all $t \in [\alpha, \beta]$ and the set

$$T_1 = \{t \in [\alpha, \beta] : B(t) - C(t) < 0\} \neq \emptyset,$$

then $\gamma_0 = 0$ and

$$\gamma_1 = \min\left\{\inf\left\{\sigma(t) : t \in T_1\right\}, \min\left\{\delta(t) : t \in [\alpha, \beta]\right\}\right\}.$$

c) If $B(t) > 0$, for all $t \in [\alpha, \beta]$ and the set

$$T_2 = \{t \in [\alpha, \beta] : C(t) < 0\} \neq \emptyset,$$

and $B(t) - C(t) \geq 0,\ t \in [\alpha, \beta]$, then

$$\gamma_0 = \sup\left\{\sigma(t) : t \in T_2\right\},\quad \gamma_1 = \min\left\{\delta(t) : t \in [\alpha, \beta]\right\}.$$

d) If $B(t) > 0$, for all $t \in [\alpha, \beta]$ and

$$T_1 = \{t \in [\alpha, \beta] : B(t) - C(t) < 0\} \neq \emptyset,$$

$$T_2 = \{t \in [\alpha, \beta] : C(t) < 0\} \neq \emptyset,$$

then $T_1 \cap T_2 = \emptyset,\ \gamma_0 = \sup\{\sigma(t) : t \in T_2\}$, *and:*

$$\gamma_1 = \min\{\inf\{\sigma(t) : t \in T_2\}, \min\{\delta(t) : t \in [\alpha, \beta]\}\}.$$

The behavior of the test functions (8) and (9) essentially determine the form and length of the regular refraction interval.

2 Main results

Let now Γ be one of the following conics with center: the ellipse, the hyperbola and the parabola written in the canonical form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1; \quad y^2 = 2p\,x, \ \ a, b > 0,$$

respectively. Let $(c, 0), c > 0$ be a focus of the conic Γ. Let d be the vertical line through $(c, 0)$ and let A, B be the points of the intersection of d with the conic. Then the little arc $AB \subseteq \Gamma$ will be named *a conic lens with the focus* $(c, 0)$.

1. *The case of the elliptic lens.*

A convenient parametrization of the elliptic lens such that the origin of axis lies at one of the foci and such that the curve is covered in the positive sense, will be given by

$$z = a\,\cos t - c + i\,b\,\sin t, \ \ t \in \left[-\arcsin\frac{b}{a}, \arcsin\frac{b}{a}\right], \ \ c = \sqrt{a^2 - b^2}.$$

Then

$$z' = -a\,\sin t + i\,b\,\cos t; \ \ z'' = -a\,\cos t - i\,b\,\sin t;$$

and we have

$$A(t) = \frac{c\,\sin t}{a - c\,\cos t}; \ \ B(t) = \frac{b}{a - c\,\cos t}; \ \ C(t) = \frac{ab}{a^2 - c^2\,\cos^2 t};$$

$$B(t) - C(t) = \frac{bc\,\cos t}{a^2 - c^2\,\cos^2 t}.$$

It is clear that $B(t) > 0;\ C(t) > 0$ and $B(t) - C(t) > 0$, for $t \in [-\arcsin\frac{b}{a}, \arcsin\frac{b}{a}]$. By using Theorem 3 we have: $\gamma_0 = 0$ and

$$\gamma_1 = \min\left\{\sqrt{1 + \frac{B^2}{A^2}} \ : \ \sin t \in \left[-\frac{b}{a}, \frac{b}{a}\right]\right\}$$

$$= \min\left\{\sqrt{1 + \frac{b^2}{c^2\sin^2 t}} \ : \ \sin t \in \left[-\frac{b}{a}, \frac{b}{a}\right]\right\} = \sqrt{1 + \frac{a^2}{c^2}} = \frac{\sqrt{1 + e^2}}{e},$$

Figure 1: a=5, b=4,$\gamma=\gamma_1=\sqrt{34}/3$

where $e=c/a$ is the eccentricity of the ellipse. Then the regular refraction interval for an elliptic lens with respect to its focus is given by $RRI[\Gamma,0]=[0,\sqrt{1+e^2}/e]$. (see Fig. 1.)

2. *The case of the parabolic lens.* This case was studied in [4], where it was shown that the regular refraction interval with respect to its focus is $RRI[\Gamma,0]=[0,\sqrt{2}]$. Since in this case the eccentricity is equal to 1, we have

$$RRI[\gamma,0]=\left[0,\frac{\sqrt{1+e^2}}{e}\right]=[0,\sqrt{2}].$$

3. *The case of the hyperbolic lens.*

Here a convenient parametrization will be given by

$$z=a\,\cosh t-c-i\,b\,\sinh t;\ \ t\in\left[-\ln\frac{b+c}{a},\ln\frac{b+c}{a}\right],c=\sqrt{a^2+b^2}.$$

Then

$$z'=a\,\sinh t-i\,b\,\cosh t;\ \ z''=a\,\cosh t-i\,b\,\sinh t;$$

and we have

$$A(t)=\frac{c\,\sinh t}{c\,\cosh t-a};\ \ B(t)=\frac{b}{c\,\cosh t-a};\ \ C(t)=\frac{ab}{c^2\cosh^2 t-a^2};$$

$$B(t)-C(t)=\frac{bc\,\cosh t}{c^2\cosh^2 t-a^2},\ \ t\in\left[-\ln\frac{b+c}{a},\ln\frac{b+c}{a}\right].$$

It is clear that $B(t)>0$, $C(t)>0$, $B(t)-C(t)>0$, for $t\in\left[-\ln\frac{b+c}{a},\ln\frac{b+c}{a}\right]$. Hence $\gamma_0=0$ and

$$\gamma_1=\min\left\{\sqrt{1+\frac{B^2(t)}{A^2(t)}}\ :\ t\in\left[-\ln\frac{b+c}{a},\ln\frac{b+c}{a}\right]\right\}$$

$$= \min\left\{\sqrt{1+\frac{b^2}{c^2\sinh^2 t}}\ :\ \sinh t \in \left[-\frac{b}{a},\frac{b}{a}\right]\right\}$$

$$= \sqrt{1+\frac{a^2}{c^2}} = \frac{\sqrt{1+e^2}}{e},$$

where e is the eccentricity of the hyperbola. The regular refraction interval for the hyperbolic lens with respect to its focus is

$$RRI[\Gamma,0] = \left[0,\frac{\sqrt{1+e^2}}{e}\right].$$

(see Fig. 2.)

Figure 2: a=4, b=3,$\gamma = \gamma_1 = \sqrt{41}/5$

Remark 4. In the case of elliptic lens we have $\gamma_1 \in (\sqrt{2},\infty)$; in the case of the parabolic lens we have $\gamma_1 = \sqrt{2}$; and in the case of hyperbolic lens we have $\gamma_1 \in (1,\sqrt{2})$. So we have obtained:

Theorem 5. *The regular refraction interval for a conic lens, with respect to its focus is* $[0,\sqrt{1+e^2}/e]$*, where e is the eccentricity of the conic.*

4. *The case of a sinusoidal lens.*

In this case the curve $\Gamma = \Gamma_a$ is $y = a\,\sin x$, $a \in (0,1]$, $x \in [0,\pi]$. When the origin of axis is translated in $(\pi/2,0)$, a convenient parametrization is given by:

$$z = z(t) = -t + ia\,\sin(t+\pi/2) = -t + ia\,\cos t;\ t \in [-\pi/2,\pi/2].$$

Then

$$z'(t) = -1 - ia\,\sin t;\ z''(t) = -ia\,\cos t;$$

$$A(t) = \frac{t - a^2 \sin t \cos t}{t^2 + a^2 \cos^2 t}; \ B(t) = \frac{a(t \sin t + \cos t)}{t^2 + a^2 \cos^2 t};$$

$$C(t) = \frac{a \cos t}{1 + a^2 \sin t}; \ B(t) - C(t) = \frac{a(t \sin t + \cos t)}{t^2 + a^2 \cos^2 t} - \frac{a \cos t}{1 + a^2 \sin t}.$$

We have $B(t) > 0, \ C(t) \geq 0$, for $t \in [-\pi/2, \pi/2]$ and $a \in (0, 1]$. We shall prove that $B(t) - C(t) \geq 0$, for $t \in [-\pi/2, \pi/2]$. This is equivalent to

$$f(t) := (t \sin t + \cos t)(1 + a^2 \sin^2 t) - \cos t(t^2 + a^2 \cos^2 t) \geq 0, \tag{10}$$

for $t \in [-\pi/2, \pi/2]$. Since f is an even function it is sufficient to prove the inequality (10) only for $t \in [0, \pi/2]$. But

$$-f(t) = t^2 \cos t - t \sin t(1 + a^2 \sin^2 t) - \cos t(1 - a^2 \cos(2t)),$$

and the inequality (10) becomes

$$t \leq \frac{\sin t(1 + a^2 \sin^2 t) + \sqrt{\Delta_1}}{2 \cos t}, \ t \in [0, \pi/2),$$

where

$$\Delta_1 = \sin^2 t(1 + a^2 \sin^2 t)^2 + \cos^2 t(1 - a^2 \cos^2 t) \geq \sin^2 t(1 + a^2 \sin^2 t)^2,$$

for $t \in [0, \pi/2], a \in (0, 1]$. It is sufficient to prove that:

$$t \leq \frac{\sin t(1 + a^2 \sin^2 t)}{\cos t}, \ t \in [0, \pi/2), \ a \in (0, 1],$$

which is true since $t \leq \tan t, \ t \in [0, \pi/2)$. So, by $B(t) > 0, \ C(t) > 0, \ B(t) - C(t) \geq 0$ we have $RRI[\Gamma, 0] = [0, \gamma_1]$, where

$$\gamma_1 = \min \left\{ \sqrt{1 + \frac{B^2}{A^2}} : \ t \in [-\pi/2, \pi/2] \right\}$$

$$= \min \left\{ \sqrt{1 + \frac{a^2(t \sin t + \cos t)^2}{(t - a^2 \sin t \cos t)^2}}; \ t \in [-\pi/2, \pi/2] \setminus \{0\} \right\}.$$

Denoting by

$$h(t) = \frac{t \sin t + \cos t}{t - a^2 \sin t \cos t}, \ t \in [-\pi/2, \pi/2] \setminus \{0\},$$

by a straightforward computation we obtain

$$h'(t) = \frac{-f(t)}{(t - a^2 \sin t \cos t)^2} \leq 0, \ t \in [-\pi/2, \pi/2] \setminus \{0\}, \ a \in (0, 1].$$

In this case $\gamma_1 = \sqrt{1 + a^2 h^2(\pi/2)} = \sqrt{1 + a^2}$. For $a = 1$ we have the sinusoidal lens and its regular refraction interval will be $[0, \sqrt{2}]$ as in the case of parabolic lens. (see Fig. 3.)

So, we have proved

Figure 3: a=1,$\gamma = \gamma_1 = \sqrt{2}$

Theorem 6. *For arbitrary* $a \in (0,1]$, *the regular refraction interval of the sinusoidal lens* Γ_a *with respect to the point* $(\pi/2, 0)$ *is* $[0, \sqrt{1+a^2}]$.

In the sequel we are interested to find the smooth starlike curves $\Gamma = \Gamma_{\alpha,\beta}$ having the parametrization

$$z(t) = x(t) + i\,y(t),\ t \in [\alpha, \beta],$$

such that

$$RRI[\Gamma_{\alpha,\beta}, 0] = RRI[\Gamma_{\alpha',\beta'}, 0] = [0, \gamma_1].$$

for any interval $[\alpha', \beta'] \subseteq [\alpha, \beta]$. By using Theorem 3 we have to study the following cases

a) $B(t) > 0,\ C(t) > 0,\ B(t) - C(t) \geq 0$ and $\delta(t) = const.$ for $t \in [\alpha, \beta]$;

b) $B(t) > 0,\ C(t) > 0,\ B(t) - C(t) < 0$ and $\sigma(t) = const.$ for $t \in [\alpha, \beta]$.

At the beginning we obtain some general relations concerning the functions $A, B, C : [\alpha, \beta] \to \mathbb{R}$. From the equality

$$\frac{z'(t)}{z(t)} = A(t) + iB(t),\ t \in [\alpha, \beta], \tag{11}$$

we deduce that

$$A(t) = \frac{x(t)x'(t) + y(t)y'(t)}{x^2(t) + y^2(t)} = \frac{1}{2}\frac{(x^2(t) + y^2(t))'}{x^2(t) + y^2(t)};$$

$$B(t) = \frac{x(t)y'(t) - x'(t)y(t)}{x^2(t) + y^2(t)} = \frac{(y(t)/x(t))'}{1 + (y(t)/x(t))^2}.$$

Denoting by $A_0(t)$, $B_0(t)$ two primitives for $A(t)$, $B(t)$ respectively we have

$$\begin{cases} B_0(t) + c_1 = \arctan(y(t)/x(t)), \\ A_0(t) = \ln((x^2(t) + y^2(t))^{1/2}/c_2), \end{cases} \tag{12}$$

where c_1, c_2 are constants and $c_2 > 0$, which implies

$$\begin{cases} x(t) = c_2 \cos(B_0(t) + c_1) e^{A_0(t)}, \\ y(t) = c_2 \sin(B_0(t) + c_1) e^{A_0(t)}, \end{cases} \tag{13}$$

or equivalently

$$\begin{cases} x(t) = (c_3 \cos B_0(t) - c_4 \sin B_0(t)) e^{A_0(t)}, \\ y(t) = (c_4 \cos B_0(t) + c_3 \sin B_0(t)) e^{A_0(t)}, \quad t \in [\alpha, \beta], \end{cases} \tag{14}$$

where c_3, c_4 are arbitrary constants. By (11) it follows that

$$z'(t) = (A(t) + iB(t)) z(t),$$
$$z''(t) = (A'(t) + iB'(t)) z(t) + (A(t) + iB(t)) z'(t).$$

Then

$$\begin{aligned} C(t) = \operatorname{Im} \frac{z''(t)}{z'(t)} &= B(t) + \operatorname{Im} \frac{A'(t) + iB'(t)}{z'(t)/z(t)} \\ &= B(t) + \frac{(B(t)/A(t))'}{1 + B^2(t)/A^2(t)}. \end{aligned} \tag{15}$$

In the case a) the condition $\delta(t) = const.$ is equivalent to $B(t)/A(t) = const. \neq 0$, on $[\alpha, \beta]$. Then there exist the constants $a, b \in \mathbb{R}$, $b > 0$, $a \neq 0$ and the function $f : [\alpha, \beta] \to (0, \infty)$ such that

$$B(t) = bf(t), \;\; A(t) = af(t), \;\; t \in [\alpha, \beta].$$

Since $(B(t)/A(t))' \equiv 0$, by (15) we deduce $B(t) \equiv C(t)$, $t \in [\alpha, \beta]$. We remark that the case $B > 0, C > 0, B/A = const.$ $B - C < 0$ is impossible. By (12), for $F(t)$ a primitive of f on $[\alpha, \beta]$ we obtain

$$\begin{cases} bF(t) = \arctan(y(t)/x(t)) - c_1, \\ aF(t) = \ln[(x^2(t) + y^2(t))^{1/2}/c_2]. \end{cases} \tag{16}$$

The equations (16) implies that the family of curves verifying the conditions a) are

$$\frac{b}{a} = \frac{\arctan(y/x) - c_1}{\ln[(x^2 + y^2)^{1/2}/c_2]}.$$

In polar coordinates we obtain the family of logarithmic spirals

$$\rho(\theta) = c_2 e^{(\theta - c_1)a/b} = c_2 e^{(\theta - c_1)c}.$$

By (13) and (14) a parametric representation of the family of logarithmic spirals is given by

$$\begin{cases} x(t) = (c_3 \cos F_1(t) - c_4 \sin F_1(t)) e^{c_5 F_1(t)}, \\ y(t) = (c_4 \cos F_1(t) + c_3 \sin F_1(t)) e^{c_5 F_1(t)}, \quad t \in [\alpha, \beta], \end{cases}$$

where c_3, c_4, c_5 are constants and $F_1(t)$ is a strictly increasing function on $[\alpha, \beta]$. So we have deduced:

Theorem 7. *The family of smooth starlike curves, verifying the conditions in the case a) is a family of starlike arcs of logarithmic spirals depending on 2 parameters*

$$\rho(\theta) = c_2 e^{c\cdot\theta},\ \ \theta \in [\theta_1, \theta_2], \tag{17}$$

where c_2, c *are constants,* $c_2 > 0$*.*

Remark 8. If the arc of logarithmic spiral is given by (17) then its regular refraction interval with respect to origin is given by

$$RRI[\Gamma, 0] = \left[0, \sqrt{1 + \frac{1}{c^2}}\right] = \left[0, \frac{\sqrt{c^2+1}}{|c|}\right].$$

In the case when $\gamma = \sqrt{c^2+1}/|c|$ all the refracted rays are tangent to the logarithmic spiral. (see Fig. 4)

Figure 4: c=c_2=1,$\gamma = \gamma_1 = \sqrt{2}$

In case b) the regular refraction interval is of the form $[0, \gamma]$ and γ is independent of the interval $[\alpha', \beta'] \subseteq [\alpha, \beta]$ if and only if $\sigma(t)$ is constant on $[\alpha, \beta]$. This interval is now given by $[0, \gamma_1]$ with $\gamma_1 = \sigma(\alpha) = \sigma(\beta)$.

Let $B(t) = f(t) > 0$, for $t \in [\alpha, \beta]$, and let $C(t) - B(t) = g(t) > 0$, for $t \in [\alpha, \beta]$. With this notation the relation (15) becomes

$$g(t) = \frac{(B(t)/A(t))'}{1 + (B(t)/A(t))^2},\ \ t \in [\alpha, \beta].$$

Denoting by G a primitive of g we have

$$G(t) + c_1 = \arctan\frac{f(t)}{A(t)},\ \ c_1 \in \mathbb{R},$$

and

$$A(t)=f(t)\cot(G(t)+c_1),\ t\in[\alpha,\beta]. \tag{18}$$

It yields

$$\begin{aligned}\sigma^2(t)=k^2&=\frac{(A^2(t)+B^2(t))C^2(t)}{B^2(t)(B(t)-C(t))^2+A^2(t)C^2(t)}\\&=\frac{(1+\cot^2(G(t)+c_1))(f(t)+g(t))^2}{g^2(t)+\cot^2(G(t)+c_1)(f(t)+g(t))^2}.\end{aligned}$$

We remark that by $B(t)-C(t)<0$ it follows that $k>1$. One obtains

$$f(t)+g(t)=\frac{kg(t)|\sin(G(t)+c_1)|}{\sqrt{1-k^2\cos^2(G(t)+c_1)}}.$$

At the moment we suppose that $G(t)+c_1=\arctan(f(t)/A(t))\in[0,\pi/2)$. We have

$$\begin{aligned}B(t)=f(t)&=-g(t)+(f(t)+g(t))\\&=-g(t)+\frac{kg(t)\sin(G(t)+c_1)}{\sqrt{1-k^2\cos^2(G(t)+c_1)}},\end{aligned}$$

and consequently

$$c_2+B_0(t)=-G(t)-c_1-\arcsin(k\cos(G(t)+c_1)).$$

On the other hand by (18) it follows

$$A(t)=-\frac{g(t)\cos(G(t)+c_1)}{\sin(G(t)+c_1)}+\frac{kg(t)\cos(G(t)+c_1)}{\sqrt{k^2\sin^2(G(t)+c_1)-(k^2-1)}};$$

and

$$A_0(t)=\ln(k+\sqrt{k^2-(k^2-1)/\sin^2(G(t)+c_1)})-\ln d,\ d>0.$$

For $G(t)+c_1\in(-\pi/2,0]$ the primitives of $A(t)$ are the same. By (13), after a straightforward computation we deduce

$$\begin{cases}x(t)=c(1-k^2)\cot(G(t)+c_1),\\ y(t)=-c(1+k\sqrt{1-(k^2-1)\cot^2(G(t)+c_1)}), \quad t\in[\alpha,\beta].\end{cases}$$

where c, c_1 are constants $c>0$, and G is an increasing function on $[\alpha,\beta]$. These are the parametric representations of the family of starlike curves verifying the conditions in b). The implicit equations for this family are

$$\frac{x^2}{(k^2-1)c^2}+\frac{(y+c)^2}{k^2c^2}=1,\ k>1,\ c>0. \tag{19}$$

which is a family of ellipses of semiminor axis $\sqrt{k^2-1}c$ and semimajor axis kc and having the foci $(0,0)$ and $(0,2c)$. On the other hand when $\sin(G(t)+c_1) \in (-\pi/2, 0]$ then we obtain the ellipse

$$\frac{x^2}{(k^2-1)c^2} + \frac{(y-c)^2}{k^2c^2} = 1, \;\; k > 1, \;\; c > 0.$$

Theorem 9. *The family of smooth starlike curves verifying the conditions in the case b) is a family of elliptic arcs depending on 2 parameters, given by equations* `(19)`.

It is of interest to determine all arcs $\Gamma_{\alpha',\beta'}$ on the previous ellipses such that $\sigma(t) = const.$ for $t \in [\alpha', \beta']$ and such that $B(t) - C(t) < 0$ for $t \in [\alpha', \beta']$. For this purpose it is sufficient to obtain the regular refraction interval for the entire ellipse with respect to one of its foci. As in the case of elliptic lens we have

$$z = a\cos t - c + ib\sin t; \;\; t \in [0, 2\pi], \;\; c = \sqrt{a^2-b^2};$$

$$A(t) = \frac{c\sin t}{a - c\cos t}; \;\; B(t) = \frac{b}{a - b\cos t}; \;\; C(t) = \frac{ab}{a^2 - c^2\cos^2 t};$$

$$B(t) - C(t) = \frac{bc\cos t}{a^2 - c^2\cos^2 t}, \;\; t \in [0, 2\pi].$$

We have $B > 0, C > 0, \; t \in [0, 2\pi]$; $B - C \geq 0$ for $t \in [0, \pi/2] \cup [3\pi/2, 2\pi]$ and $B - C < 0$ for $t \in (\pi/2, 3\pi/2)$;

$$\min\{\delta(t) : t \in [0, 2\pi]\} = \min\left\{\sqrt{1 + \frac{b^2}{c^2\sin^2 t}} \; : \; t \in [0, 2\pi]\right\} = \frac{\sqrt{1+e^2}}{e};$$

$$\inf\{\sigma(t) : \; t \in (\pi/2, 3\pi/2)\} = \inf\left\{\frac{a}{c} \; : \; t \in (\pi/2, 3\pi/2)\right\} = \frac{a}{c} = \frac{1}{e}.$$

By Theorem 3 $\gamma_0 = 0$ and

$$\gamma_1 = \min\left\{\frac{\sqrt{1+e^2}}{e}, \frac{1}{e}\right\} = \frac{1}{e} = \frac{a}{c}.$$

Theorem 10. *The regular refraction interval for entire ellipse* $x^2/a^2 + y^2/b^2 = 1, \; a > b > 0$ *with respect to the focus* $(c, 0)$ *is* $[0, a/c]$.

Compare this result with the regular refraction interval for entire ellipse with respect to its center which is $[0, a^2/c^2]$, see [3]. From the previous analysis we conclude that the arcs of the ellipse (19) having the properties in b) are only the arcs $\Gamma_{\alpha',\beta'}$ with $[\alpha', \beta'] \subseteq [\pi, 2\pi]$. In this case the regular refraction interval of $\Gamma_{\alpha',\beta'}$ is $[0, k]$. In the case when $\gamma = k$, for arcs contained in $\Gamma_{\pi,2\pi}$ all the refracted rays are parallel.

(see Fig. 5.)

Figure 5: c=1, k=$\sqrt{2}$,$\gamma = \gamma_1 = \sqrt{2}$

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