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On γ-normal spaces

by
Erdal Ekici

Abstract

The aim of this paper is to introduce and study a new class of spaces, called γ-normal spaces. The relationships among s-normal spaces, p-normal spaces and γ-normal spaces are investigated. Moreover, we introduce the forms of generalized γ-closed functions. We obtain characterizations of γ-normal spaces, properties of the forms of generalized γ-closed functions and preservation theorems.

Key Words: p-normal space, s-normal space, *γ-normal space, gγ-closed function, γ-gγ-closed function, separation axioms.*
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1 Introduction

First step in normality was taken by Viglino [30] who defined seminormal spaces. Then Singal and Arya [28] introduced the class of almost normal spaces and proved that a space is normal if and only if it is both a semi-normal space and an almost normal space. Normality is an important topological property and hence it is of significance both from intrinsic interest as well as from applications view point to obtain factorizations of normality in terms of weaker topological properties. In recent years, many authors have studied several forms of normality [10, 14, 16, 24]. On the other hand, the notions of p-normal spaces, s-normal spaces were introduced by Paul and Bhattacharyya [27]; Maheshwari and Prasad [19], respectively. Levine [18] initiated the investigation of g-closed sets in topological spaces, since then many modifications of g-closed sets were defined and investigated by a large number of topologists [6, 25]. In 1996, Maki and et al [20] introduced the concepts of gp-closed sets and Arya and Nour [3] introduced the concepts of gs-closed sets. The purpose of this paper is to introduce a new class of normal spaces, namely γ-normal spaces, which is a generalizing of the classes of p-normal spaces and s-normal spaces. The relations among γ-normal spaces, p-normal spaces and s-normal spaces and also properties of γ-normal spaces are investigated. Moreover, we introduce and study new forms of generalized γ-closed functions. We obtain properties of these new forms of generalized γ-closed functions and preservation theorems.

2 Preliminaries

In what follows, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f of a space (X, τ) into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

Definition 1. *A subset A of a space X is said to be:*
(1) regular open [29] if $A = int(cl(A))$,
(2) α-open [22] if $A \subset int(cl(int(A)))$,
(3) semiopen [17] if $A \subset cl(int(A))$,
(4) preopen [21] or nearly open [13] if $A \subset int(cl(A))$,
(5) b-open [1] or sp-open [9] or γ-open [12] if $A \subset cl(int(A)) \cup int(cl(A))$.

It is shown in [22] that the class of α-open sets is a topology and it is stronger than given topology on X.

The complement of an α-open (resp. semiopen, preopen, γ-open, regular open) set is called α-closed [22] (resp. semiclosed [8], preclosed [21], γ-closed [12], regular closed [29]).

The intersection of all α-closed (resp. semiclosed, preclosed, γ-closed) sets containing A is called the α-closure (resp. semiclosure, preclosure, γ-closure) of A and is denoted by $\alpha\text{-}cl(A)$ (resp. $s\text{-}cl(A)$, $p\text{-}cl(A)$, $\gamma\text{-}cl(A)$).

Dually, the α-interior (resp. semi-interior, preinterior, γ-interior) of A, denoted by $\alpha\text{-}int(A)$ (resp. $sint(A)$, $pint(A)$, $\gamma\text{-}int(A)$) is defined to be the union of all α-open (resp. semiopen, preopen, γ-open) sets contained in A.

The family of all γ-open (γ-closed, α-open, regular open, regular closed, semiopen, preopen) sets of a space X is denoted by $\gamma O(X)$ (resp. $\gamma C(X)$, $\alpha O(X)$, $RO(X)$, $RC(X)$, $SO(X)$, $PO(X)$). The family of all γ-open sets containing a point x is denoted by $\gamma O(X, x)$.

Definition 2. *A space X is said to be prenormal [26] or p-normal [27] (resp. s-normal [19]) if for any pair of disjoint closed sets A and B, there exist disjoint preopen (resp. semiopen) sets U and V such that $A \subset U$ and $B \subset V$.*

Definition 3. *A subset A of a space (X, τ) is said to be g-closed [18] (resp. gs-closed [3], gp-closed [20]) if $cl(A) \subset U$ (resp. $s\text{-}cl(A) \subset U$, $p\text{-}cl(A) \subset U$) whenever $A \subset U$ and $U \in \tau$. The complement of g-closed (resp. gs-closed, gp-closed) set is said to be g-open (resp. gs-open, gp-open).*

Definition 4. *A subset A of a space (X, τ) is said to be sg-closed [4] (resp. pg-closed [5]) if $s\text{-}cl(A) \subset U$ (resp. $p\text{-}cl(A) \subset U$) whenever $A \subset U$ and $U \in SO(X)$ (resp. $U \in PO(X)$). The complement of sg-closed (resp. pg-closed) set is said to be sg-open (resp. pg-open).*

3 γ-normal spaces

Definition 5. *A space X is said to be γ-normal if for any pair of disjoint closed sets A and B, there exist disjoint γ-open sets U and V such that $A \subset U$ and $B \subset V$.*

Remark 6. *The following diagram holds for a topological space (X, τ):*

$$\begin{array}{ccc} normal & \Rightarrow & s\text{-}normal \\ \Downarrow & & \Downarrow \\ p\text{-}normal & \Rightarrow & \gamma\text{-}normal \end{array}$$

None of these implications is reversible as shown by the following examples.

Example 7. *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$. Then the space (X, τ) is γ-normal but not s-normal.*

Example 8. *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then the space (X, τ) is γ-normal but not p-normal.*

For the other implications the examples can be seen in [19, 26, 27].

Theorem 9. *For a space X the following are equivalent:*

(1) X is γ-normal,

(2) For every pair of open sets U and V whose union is X, there exist γ-closed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$,

(3) For every closed set H and every open set K containing H, there exists a γ-open set U such that $H \subset U \subset \gamma\text{-}cl(U) \subset K$.

Proof: (1) $\Longrightarrow$ (2) : Let U and V be a pair of open sets in a γ-normal space X such that $X = U \cup V$. Then $X \setminus U$, $X \setminus V$ are disjoint closed sets. Since X is γ-normal there exist disjoint γ-open sets U_1 and V_1 such that $X \setminus U \subset U_1$ and $X \setminus V \subset V_1$. Let $A = X \setminus U_1$, $B = X \setminus V_1$. Then A and B are γ-closed sets such that $A \subset U$, $B \subset V$ and $A \cup B = X$.

(2) $\Longrightarrow$ (3) : Let H be a closed set and K be an open set containing H. Then $X \setminus H$ and K are open sets whose union is X. Then by (2), there exist γ-closed sets M_1 and M_2 such that $M_1 \subset X \setminus H$ and $M_2 \subset K$ and $M_1 \cup M_2 = X$. Then $H \subset X \setminus M_1$, $X \setminus K \subset X \setminus M_2$ and

$$(X \setminus M_1) \cap (X \setminus M_2) = \emptyset.$$

Let $U = X \setminus M_1$ and $V = X \setminus M_2$. Then U and V are disjoint γ-open sets such that

$$H \subset U \subset X \setminus V \subset K.$$

As $X \setminus V$ is γ-closed set, we have γ-$cl(U) \subset X \setminus V$ and $H \subset U \subset \gamma - cl(U) \subset K$
(3) $\implies$ (1) : Let H_1 and H_2 be any two disjoint closed sets of X. Put $K = X \setminus H_2$, then $H_2 \cap K = \emptyset$. $H_1 \subset K$ where K is an open set. Then by (3), there exists a γ-open set U of X such that

$$H_1 \subset U \subset \gamma - cl(U) \subset K.$$

It follows that $H_2 \subset X \setminus \gamma - cl(U) = V$, say, then V is γ-open and $U \cap V = \emptyset$. Hence H_1 and H_2 are separated by γ-open sets U and V. Therefore X is γ-normal. □

4 The related functions with γ-normal spaces

Definition 10. *A function $f : X \to Y$ is called*

(1) R-map [7] if $f^{-1}(V)$ is regular open in X for every regular open set V of Y,

(2) completely continuous [2] if $f^{-1}(V)$ is regular open in X for every open set V of Y,

(3) rc-continuous [15] if for each regular closed set F in Y, $f^{-1}(F)$ is regular closed in X.

Definition 11. *A function $f : X \to Y$ is called*

(1) strongly γ-open if $f(U) \in \gamma O(Y)$ for each $U \in \gamma O(X)$,

(2) strongly γ-closed if $f(U) \in \gamma C(Y)$ for each $U \in \gamma C(X)$,

(3) almost-γ-irresolute if for each x in X and each γ-neighbourhood V of $f(x)$, γ-$cl(f^{-1}(V))$ is a γ-neighbourhood of x.

Theorem 12. *A function $f : X \to Y$ is strongly γ-closed if and only if for each subset A in Y and for each γ-open set U in X containing $f^{-1}(A)$, there exists a γ-open set V containing A such that $f^{-1}(V) \subset U$.*

Proof: $(\Rightarrow)$: Suppose that f is strongly γ-closed. Let A be a subset of Y and $U \in \gamma O(X)$ containing $f^{-1}(A)$. Put $V = Y \setminus f(X \setminus U)$, then V is a γ-open set of Y such that $A \subset V$ and $f^{-1}(V) \subset U$.

$(\Leftarrow)$: Let K be any γ-closed set of X. Then $f^{-1}(Y \setminus f(K)) \subset X \setminus K$ and $X \setminus K \in \gamma O(X)$. There exists a γ-open set V of Y such that $Y \setminus f(K) \subset V$ and $f^{-1}(V) \subset X \setminus K$. Therefore, we have $f(K) \supset Y \setminus V$ and $K \subset f^{-1}(Y \setminus V)$. Hence, we obtain $f(K) = Y \setminus V$ and $f(K)$ is γ-closed in Y. This shows that f is strongly γ-closed. □

Lemma 13. *For a function $f : X \to Y$, the following are equivalent:*
(1) f is almost γ-irresolute,
(2) $f^{-1}(V) \subset \gamma - int(\gamma - cl(f^{-1}(V)))$ for every $V \in \gamma O(Y)$.

Theorem 14. *A function $f : X \to Y$ is almost γ-irresolute if and only if $f(\gamma - cl(U)) \subset \gamma - cl(f(U))$ for every $U \in \gamma O(X)$.*

Proof: $(\Rightarrow)$: Let $U \in \gamma O(X)$. Suppose $y \notin \gamma - cl(f(U))$. Then there exists $V \in \gamma O(Y, y)$ such that $V \cap f(U) = \emptyset$. Hence, $f^{-1}(V) \cap U = \emptyset$. Since $U \in \gamma O(X)$, we have

$$\gamma - int(\gamma - cl(f^{-1}(V))) \cap \gamma - cl(U) = \emptyset.$$

Then by Lemma 13, $f^{-1}(V) \cap \gamma - cl(U) = \emptyset$ and hence $V \cap f(\gamma - cl(U)) = \emptyset$. This implies that $y \notin f(\gamma - cl(U))$.

$(\Leftarrow)$: If $V \in \gamma O(Y)$, then $M = X \setminus \gamma - cl(f^{-1}(V)) \in \gamma O(X)$. By hypothesis, $f(\gamma - cl(M)) \subset \gamma - cl(f(M))$ and hence

$$\begin{aligned} X \setminus \gamma - int(\gamma - cl(f^{-1}(V))) &= \gamma - cl(M) \\ &\subset f^{-1}(\gamma - cl(f(M))) \\ &\subset f^{-1}(\gamma - cl(f(X \setminus f^{-1}(V)))) \\ &\subset f^{-1}(\gamma - cl(Y \setminus V)) \\ &= f^{-1}(Y \setminus V) \\ &= X \setminus f^{-1}(V). \end{aligned}$$

Therefore , $f^{-1}(V) \subset \gamma - int(\gamma - cl(f^{-1}(V)))$. By Lemma 13, f is almost γ-irresolute. □

Theorem 15. *If $f : X \to Y$ is a strongly γ-open continuous almost γ-irresolute function from a γ-normal space X onto a space Y, then Y is γ-normal.*

Proof: Let A be a closed subset of Y and B be an open set containing A. Then by continuity of f, $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of X such that $f^{-1}(A) \subset f^{-1}(B)$. As X is γ-normal, there exists a γ-open set U in X such that $f^{-1}(A) \subset U \subset \gamma - cl(U) \subset f^{-1}(B)$ by Theorem 9. Then, $f(f^{-1}(A)) \subset f(U) \subset f(\gamma - cl(U)) \subset f(f^{-1}(B))$. Since f is strongly γ-open almost γ-irresolute surjection, we obtain $A \subset f(U) \subset \gamma - cl(f(U)) \subset B$. Then again by Theorem 9 the space Y is γ-normal. □

Theorem 16. *If $f : X \to Y$ is an strongly γ-closed continuous function from a γ-normal space X onto a space Y, then Y is γ-normal.*

Proof: Let M_1 and M_2 be disjoint closed sets. Then $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are closed sets. Since X is γ-normal, then there exist disjoint γ-open sets U and V such that $f^{-1}(M_1) \subset U$ and $f^{-1}(M_2) \subset V$. By Theorem 12, there exist γ-open sets A and B such that $M_1 \subset A$, $M_2 \subset B$, $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Also, A and B are disjoint. Thus, Y is γ-normal. □

Definition 17. *A topological space (X, τ) is called an α-space [15] if $\tau = \alpha O(X)$.*

Definition 18. *A function $f : X \to Y$ is called α-closed [23] if for each closed set in X, $f(U)$ is α-closed set in Y.*

Theorem 19. *If $f : X \to Y$ is an α-closed continuous surjection and X is normal, then Y is γ-normal.*

Proof: Let A and B be disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X by continuity of f. As X is normal, there exist disjoint open sets U and V in X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. By Proposition 6 in [23], there are disjoint α-open sets G and H in Y such that $A \subset G$ and $B \subset H$. Since every α-open set is γ-open, G and H are disjoint γ-open sets containing A and B, respectively. Therefore, Y is γ-normal. □

5 Generalized γ-functions

Definition 20. *A subset A of a space (X, τ) is said to be $g\gamma$-closed if γ-cl$(A) \subset U$ whenever $A \subset U$ and $U \in \tau$. The complement of $g\gamma$-closed set is said to be $g\gamma$-open.*

Definition 21. *A subset A of a space (X, τ) is said to be γg-closed if γ-cl$(A) \subset U$ whenever $A \subset U$ and $U \in \gamma O(X)$. The complement of γg-closed set is said to be γg-open.*

Remark 22. *The following diagram holds for any subset of a topological space (X, τ):*

closed		$\Longrightarrow$		*g-closed*
$\Downarrow$				$\Downarrow$
semiclosed	$\Rightarrow$	*sg-closed*	$\Rightarrow$	*gs-closed*
$\Downarrow$				$\Downarrow$
γ-closed	$\Rightarrow$	*γg-closed*	$\Rightarrow$	*$g\gamma$-closed*
$\Uparrow$				$\Uparrow$
preclosed	$\Rightarrow$	*pg-closed*	$\Rightarrow$	*gp-closed*
$\Uparrow$				$\Uparrow$
closed		$\Longrightarrow$		*g-closed*

None of these implications is reversible as shown by the following examples.

Example 23. *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{b, c\}, \{a, b, c\}\}$. Then the set $\{a\}$ is $g\gamma$-closed but not gp-closed.*

The set $\{b, c\}$ is γg-closed but not pg-closed. This example shows that γg-closedness and pg-closedness are not equivalent.

Example 24. *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$. Then the set $\{a, b\}$ is γg-closed and so $g\gamma$-closed but it is neither gs-closed nor sg-closed.*

Also, this example shows that γg-closedness and sg-closedness are not equivalent.

Example 25. *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{b\}, \{d\}, \{b, d\}\}$. Then the set $\{a, b, d\}$ is $g\gamma$-closed but it is not γg-closed.*

Question: *Does there exist a subset of a space which is γg-closed (resp. sg-closed, pg-closed) and it is not γ-closed (resp. γg-closed)?*

For the other implications the examples can be seen in [3, 4, 5, 8, 12, 20, 21].

Definition 26. *A function $f : X \to Y$ is said to be*
(1) γ-closed if $f(A)$ is γ-closed in Y for each closed set A of X,
(2) γg-closed if $f(A)$ is γg-closed in Y for each closed set A of X,
(3) $g\gamma$-closed if $f(A)$ is $g\gamma$-closed in Y for each closed set A of X.

Definition 27. *A function $f : X \to Y$ is said to be*
(1) quasi γ-closed if $f(A)$ is closed in Y for each $A \in \gamma C(X)$,
(2) γ-γg-closed if $f(A)$ is γg-closed in Y for each $A \in \gamma C(X)$,
(3) γ-$g\gamma$-closed if $f(A)$ is $g\gamma$-closed in Y for each $A \in \gamma C(X)$,
(4) almost $g\gamma$-closed if $f(A)$ is $g\gamma$-closed in Y for each $A \in RC(X)$.

Remark 28. *The following diagram holds for a function $f : (X, \tau) \to (Y, \sigma)$:*

γ-$g\gamma$-closed	$\Rightarrow$	*$g\gamma$-closed*	$\Rightarrow$	*almost $g\gamma$-closed*
$\Uparrow$		$\Uparrow$		
γ-γg-closed	$\Rightarrow$	*γg-closed*		
$\Uparrow$		$\Uparrow$		
strongly γ-closed	$\Rightarrow$	*γ-closed*		
$\Uparrow$		$\Uparrow$		
quasi γ-closed	$\Rightarrow$	*closed*		

None of these implications is reversible as shown by the following examples.

Example 29. *Let $X = Y = \{a, b, c, d\}$ and $\tau = \sigma = \{X, \varnothing, \{a\}, \{b, c\}, \{a, b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as follows: $f(a) = b$, $f(b) = b$, $f(c) = a$ and $f(d) = d$. Then f is γ-closed and so γg-closed, $g\gamma$-closed but it is not γ-$g\gamma$-closed, γ-γg-closed, strongly γ-closed, closed.*

If we define the function $f : (X, \tau) \to (Y, \sigma)$ as a identity function, then f is strongly γ-closed and closed but it is not quasi γ-closed.

Example 30. *Let $X = Y = \{a, b, c, d\}$ and $\tau = \sigma = \{X, \varnothing, \{b\}, \{d\}, \{b, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as follows: $f(a) = b$, $f(b) = a$, $f(c) = d$ and $f(d) = a$. Then f is almost gγ-closed but it is not gγ-closed.*

If we define the function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = b$, $f(b) = d$, $f(c) = c$ and $f(d) = a$. Then f is gγ-closed but it is not γg-closed.

Question: *Does there exist a function between topological spaces which is γ-gγ-closed (resp. γ-γg-closed, γg-closed) and it is not γ-γg-closed (resp. strongly γ-closed, γ-closed)?*

Definition 31. *A function $f : X \to Y$ is said to be γ-gγ-continuous if $f^{-1}(K)$ is gγ-closed in X for every $K \in \gamma C(Y)$.*

Definition 32. *A function $f : X \to Y$ is said to be γ-irresolute [11] if $f^{-1}(V) \in \gamma O(X)$ for every $V \in \gamma O(Y)$.*

Theorem 33. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then*

(1) the composition $gof : X \to Z$ is γ-gγ-closed if f is γ-gγ-closed and g is continuous γ-gγ-closed.

(2) the composition $gof : X \to Z$ is γ-gγ-closed if f is strongly γ-closed and g is γ-gγ-closed.

(3) the composition $gof : X \to Z$ is γ-gγ-closed if f is quasi γ-closed and g is gγ-closed.

Theorem 34. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition $gof : X \to Z$ be γ-gγ-closed. If f is a γ-irresolute surjection, then g is γ-gγ-closed.*

Proof: Let $K \in \gamma C(Y)$. Since f is γ-irresolute and surjective, $f^{-1}(K) \in \gamma C(X)$ and $(gof)(f^{-1}(K)) = g(K)$. Hence, $g(K)$ is gγ-closed in Z and hence g is γ-gγ-closed. □

Lemma 35. *A function $f : X \to Y$ is γ-gγ-closed if and only if for each subset B of Y and each $U \in \gamma O(X)$ containing $f^{-1}(B)$, there exists a gγ-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.*

Proof: ($\Rightarrow$) : Suppose that f is γ-gγ-closed. Let B be a subset of Y and $U \in \gamma O(X)$ containing $f^{-1}(B)$. Put $V = Y \backslash f(X \backslash U)$, then V is a gγ-open set of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

($\Leftarrow$) : Let K be any γ-closed set of X. Then $f^{-1}(Y \backslash f(K)) \subset X \backslash K$ and $X \backslash K \in \gamma O(X)$. There exists a g$\gamma$-open set V of Y such that $Y \backslash f(K) \subset V$ and $f^{-1}(V) \subset X \backslash K$. Therefore, we have $f(K) \supset Y \backslash V$ and $K \subset f^{-1}(Y \backslash V)$. Hence, we obtain $f(K) = Y \backslash V$ and $f(K)$ is gγ-closed in Y. This shows that f is γ-gγ-closed. □

Theorem 36. *If $f : X \to Y$ is continuous γ-gγ-closed, then $f(H)$ is gγ-closed in Y for each gγ-closed set H of X.*

Proof: Let H be any gγ-closed set of X and V an open set of Y containing $f(H)$. Since $f^{-1}(V)$ is an open set of X containing H, $\gamma\text{-}cl(H) \subset f^{-1}(V)$ and hence $f(\gamma\text{-}cl(H)) \subset V$. Since f is γ-gγ-closed and $\gamma\text{-}cl(H) \in \gamma C(X)$, we have $\gamma\text{-}cl(f(H)) \subset \gamma\text{-}cl(f(\gamma\text{-}cl(H))) \subset V$. Therefore, $f(H)$ is gγ-closed in Y. □

Remark 37. *The following example shows that a γ-irresolute function is γ-gγ-continuous but not conversely.*

Example 38. *Let $X = Y = \{a, b, c, d\}$ and $\tau = \sigma = \{X, \varnothing, \{a, b\}, \{a, b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as follows: $f(a) = a$, $f(b) = a$, $f(c) = c$ and $f(d) = a$. Then f is γ-gγ-continuous but it is not γ-irresolute.*

Theorem 39. *A function $f : X \to Y$ is γ-gγ-continuous if and only if $f^{-1}(V)$ is gγ-open in X for every $V \in \gamma O(Y)$.*

Theorem 40. *If $f : X \to Y$ is closed γ-gγ-continuous, then $f^{-1}(K)$ is gγ-closed in X for each gγ-closed set K of Y.*

Proof: Let K be a gγ-closed set of Y and U an open set of X containing $f^{-1}(K)$. Put $V = Y - f(X - U)$, then V is open in Y, $K \subset V$, and $f^{-1}(V) \subset U$. Therefore, we have $\gamma\text{-}cl(K) \subset V$ and hence $f^{-1}(K) \subset f^{-1}(\gamma\text{-}cl(K)) \subset f^{-1}(V) \subset U$. Since f is γ-gγ-continuous, $f^{-1}(\gamma\text{-}cl(K))$ is gγ-closed in X and hence $\gamma\text{-}cl(f^{-1}(K)) \subset \gamma\text{-}cl(f^{-1}(\gamma\text{-}cl(K))) \subset U$. This shows that $f^{-1}(K)$ is gγ-closed in X. □

Corollary 41. *If $f : X \to Y$ is closed γ-irresolute, then $f^{-1}(K)$ is gγ-closed in X for each gγ-closed set K of Y.*

Theorem 42. *If $f : X \to Y$ is an open γ-gγ-continuous bijection, then $f^{-1}(K)$ is gγ-closed in X for every gγ-closed set K of Y.*

Proof: Let K be a gγ-closed set of Y and U an open set of X containing $f^{-1}(K)$. Since f is an open surjective, $K = f(f^{-1}(K)) \subset f(U)$ and $f(U)$ is open. Therefore, $\gamma\text{-}cl(K) \subset f(U)$. Since f is injective,

$$f^{-1}(K) \subset f^{-1}(\gamma - cl(K)) \subset f^{-1}(f(U)) = U.$$

Since f is γ-gγ-continuous, $f^{-1}(\gamma\text{-}cl(K))$ is gγ-closed in X and hence $\gamma\text{-}cl(f^{-1}(K)) \subset \gamma\text{-}cl(f^{-1}(\gamma\text{-}cl(K))) \subset U$. This shows that $f^{-1}(K)$ is gγ-closed in X. □

Theorem 43. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition $gof : X \to Z$ be γ-gγ-closed. If g is an open γ-gγ-continuous bijection, then f is γ-gγ-closed.*

Proof: Let $H \in \gamma C(X)$. Then $(gof)(H)$ is gγ-closed in Z and $g^{-1}((gof)(H)) = f(H)$. By Theorem 42, $f(H)$ is gγ-closed in Y and hence f is γ-gγ-closed. □

Theorem 44. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition $gof : X \to Z$ be γ-gγ-closed. If g is a closed γ-gγ-continuous injection, then f is γ-gγ-closed.*

Proof: Let $H \in \gamma C(X)$. Then $(gof)(H)$ is gγ-closed in Z and $g^{-1}((gof)(H)) = f(H)$. By Theorem 40, $f(H)$ is gγ-closed in Y and hence f is γ-gγ-closed. □

6 Preservation theorems and other characterizations of γ-normal spaces

Theorem 45. *For a topological space X, the following are equivalent :*

(a) X is γ-normal,

(b) for any pair of disjoint closed sets A and B of X, there exist disjoint gγ-open sets U and V of X such that $A \subset U$ and $B \subset V$,

(c) for each closed set A and each open set B containing A, there exists a gγ-open set U such that $cl(A) \subset U \subset \gamma\text{-}cl(U) \subset B$,

(d) for each closed A and each g-open set B containing A, there exists a γ-open set U such that $A \subset U \subset \gamma\text{-}cl(U) \subset int(B)$,

(e) for each closed A and each g-open set B containing A, there exists a gγ-open set G such that $A \subset G \subset \gamma\text{-}cl(G) \subset int(B)$,

(f) for each g-closed set A and each open set B containing A, there exists a γ-open set U such that $cl(A) \subset U \subset \gamma\text{-}cl(U) \subset B$,

(g) for each g-closed set A and each open set B containing A, there exists a gγ-open set G such that $cl(A) \subset G \subset \gamma\text{-}cl(G) \subset B$.

Proof: $(a) \Leftrightarrow (b) \Leftrightarrow (c)$: Since every γ-open set is gγ-open, it is obvious.

$(d) \Rightarrow (e) \Rightarrow (c)$ and $(f) \Rightarrow (g) \Rightarrow (c)$: Since every closed (resp. open) set is g-closed (resp. g-open), it is obvious.

$(c) \Rightarrow (e)$: Let A be a closed subset of X and B be an g-open set such that $A \subset B$. Since B is g-open and A is closed, $A \subset int(A)$. Then, there exists a gγ-open set U such that $A \subset U \subset \gamma\text{-}cl(U) \subset int(B)$.

$(e) \Rightarrow (d)$: Let A be any closed subset of X and B be a g-open set containing A. Then there exists a gγ-open set G such that $A \subset G \subset \gamma\text{-}cl(G) \subset int(B)$. Since G is gγ-open, $A \subset \gamma\text{-}int(G)$. Put $U = \gamma\text{-}int(G)$, then U is γ-open and $A \subset U \subset \gamma\text{-}cl(U) \subset int(B)$.

$(c) \Rightarrow (g)$: Let A be any g-closed subset of X and B be an open set such that $A \subset B$. Then $cl(A) \subset B$. Therefore, there exists a gγ-open set U such that $cl(A) \subset U \subset \gamma\text{-}cl(U) \subset B$.

$(g) \Rightarrow (f)$: Let A be any g-closed subset of X and B be an open set containing A. Then there exists a gγ-open set G such that $cl(A) \subset G \subset \gamma\text{-}cl(G) \subset B$. Since G is gγ-open and $cl(A) \subset G$, we have $cl(A) \subset \gamma\text{-}int(G)$, put $U = \gamma\text{-}int(G)$, then U is γ-open and $cl(A) \subset U \subset \gamma\text{-}cl(U) \subset B$. □

Theorem 46. *If $f : X \to Y$ is a continuous quasi γ-closed surjection and X is γ-normal, then Y is normal.*

Proof: Let M_1 and M_2 be any disjoint closed sets of Y. Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X. Since X is γ-normal, there exist disjoint U_1, $U_2 \in \gamma O(X)$ such that $f^{-1}(M_i) \subset U_i$ for $i = 1, 2$. Put $V_i = Y - f(X - U_i)$, then V_i is open in Y, $M_i \subset V_i$ and $f^{-1}(V_i) \subset U_i$ for $i = 1$, 2. Since $U_1 \cap U_2 = \varnothing$ and f is surjective; we have $V_1 \cap V_2 = \varnothing$. This shows that Y is normal. □

Lemma 47. *A subset A of a space X is gγ-open if and only if $F \subset \gamma\text{-}int(A)$ whenever F is closed and $F \subset A$.*

Theorem 48. *Let $f : X \to Y$ be a closed γ-gγ-continuous injection. If Y is γ-normal, then X is γ-normal.*

Proof: Let N_1 and N_2 be disjoint closed sets of X, Since f is a closed injection, $f(N_1)$ and $f(N_2)$ are disjoint closed sets of Y. By the γ-normality of Y, there exist disjoint V_1, $V_2 \in \gamma O(Y)$ such that $f(N_i) \subset V_i$ for $i = 1, 2$. Since f is γ-gγ-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint gγ-open sets of X and $N_i \subset f^{-1}(V_i)$ for $i = 1, 2$. Now, put $U_i = \gamma\text{-}int(f^{-1}(V_i))$ for $i = 1, 2$. Then, $U_i \in \gamma O(X)$, $N_i \subset U_i$ and $U_1 \cap U_2 = \varnothing$. This shows that X is γ-normal. □

Corollary 49. *If $f : X \to Y$ is a closed γ-irresolute injection and Y is γ-normal, then X is γ-normal.*

Proof: This is an immediate consequence since every γ-irresolute function is γ-gγ-continuous. □

Lemma 50. *A function $f : X \to Y$ is almost gγ-closed if and only if for each subset B of Y and each $U \in RO(X)$ containing $f^{-1}(B)$, there exists a gγ-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.*

Lemma 51. *If $f : X \to Y$ is almost $g\gamma$-closed, then for each closed set M of Y and each $U \in RO(X)$ containing $f^{-1}(M)$, there exists $V \in \gamma O(Y)$ such that $M \subset V$ and $f^{-1}(V) \subset U$.*

Theorem 52. *Let $f : X \to Y$ be a continuous almost $g\gamma$-closed surjection. If X is normal, then Y is γ-normal.*

Proof: Let M_1 and M_2 be any disjoint, closed sets of Y. Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X. By the normality of X, there exist disjoint open sets U_1 and U_2 such that $f^{-1}(M_i) \subset U_i$, where $i = 1, 2$. Now, put $G_i = int(cl(U_i))$ for $i = 1, 2$, then $G_i \in RO(X)$, $f^{-1}(M_i) \subset U_i \subset G_i$ and $G_1 \cap G_2 = \varnothing$. By Lemma 51, there exists $V_i \in \gamma O(Y)$ such that $M_i \subset V_i$ and $f^{-1}(V_i) \subset G_i$, where $i = 1, 2$. Since $G_1 \cap G_2 = \varnothing$ and f is surjective, we have $V_1 \cap V_2 = \varnothing$. This shows that Y is γ-normal. ∎

Corollary 53. *If $f : X \to Y$ is a continuous γ-closed surjection and X is normal, then Y is γ-normal.*

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Department of Mathematics,
Canakkale Onsekiz Mart University,
Terzioglu Campus,
17020 Canakkale/TURKEY
E-mail: `eekici @comu.edu.tr`