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Some classes of pseudo-MTL algebras

by
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Abstract

Pseudo-MTL algebras or weak pseudo-BL algebras are non-commutative fuzzy structures which arise from pseudo-t-norms, namely, pseudo-BL algebras without the pseudo-divisibility condition. The aim of this paper is to investigate the properties of pseudo-BL algebras that also hold for pseudo-MTL algebras. We will also study some classes of pseudo-MTL algebras such as good, local and Archimedean pseudo-MTL algebras and we show that, generally, an Archimedean pseudo-MTL algebra is not commutative. We prove that any locally finite pseudo-MTL algebra is Archimedean.

Key Words: Pseudo-MTL algebra, Local pseudo-MTL algebra, Good pseudo-MTL algebra, Perfect pseudo-MTL algebra, Archimedean pseudo-MTL algebra, Hyperarchimedean element.
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1 Introduction

In order to formalize the many-valued logics induced by continuous t-norms on the real unit interval $[0, 1]$, in 1998 P. Hájek introduced a very general many-valued logic, called *Basic Logic* ([14]). It is well known the result that a t-norm has residuum if and only if the t-norm is left-continuous, so this shows that the Basic Logic is not the most general t-norm based logic. In fact, a logic weaker than the Basic Logic, called *Monoidal t-norm based logic* (MTL for short) was defined by Esteva and Godo in [8] and proved in [17] to be the logic of left-continuous t-norms and their residua. Thus, the MTL is indeed the most general t-norm based logic and MTL algebra is an algebraic counterpart of this logic. Pseudo-BL algebras were introduced by G. Georgescu and A. Iorgulescu in [11] as a non-commutative extension of Hájek′s BL-algebras. Pseudo-BL algebras are bounded non-commutative residuated lattices $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ which satisfy the conditions :

$$x \wedge y = (x \to y) \odot x = x \odot (x \rightsquigarrow y) \qquad \text{(pseudo-divisibility)}$$
$$(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1 \quad \text{(pseudo-prelinearity).}$$

Depending on the above conditions, there are two directions to extend pseudo-BL algebras. One direction investigates the (bounded) non-commutative residuated lattices satisfying the pseudo-divisibility condition which were studied under the name *(bounded) divisible pseudo-residuated lattices* in [15] or *(bounded) $R\ell$-monoids* ([7], [19], [21]) and examples in the bounded case are given in [16]. The second direction deals with (bounded) non-commutative residuated lattices with the pseudo-prelinearity condition, that is *pseudo-MTL algebras*. Pseudo-MTL algebras were introduced in [9], under the name *weak pseudo-BL algebras* in order to obtain a structure on $[0, 1]$, since there are not pseudo-BL algebras on $[0, 1]$. They were studied in [15] (including the not-bounded case and the *good* pseudo-MTL algebras) and examples in the bounded case of finite good and not good pseudo-MTL algebras are given in [16].

In this paper we will study some properties for pseudo-BL algebras proved in [5] and [6] which are valid in the case of pseudo-MTL algebras, in other words, some properties of pseudo-BL algebras whose proofs don't need the pseudo-divisibility condition. We will also investigate some special classes of pseudo-MTL algebras, such as good, local and Archimedean pseudo-MTL algebras. It was proved that every locally finite pseudo-MV algebra is commutative ([20]) and that every locally finite pseudo-BL algebra is an MV algebra, so it is commutative ([12]). We show that in the case of pseudo-MTL algebras this fact is not true, namely we will give an example of locally finite pseudo-MTL algebra which is not commutative. Finally, we prove that any locally finite pseudo-MTL algebra is Archimedean.

2 Pseudo-MTL algebras and their basic properties

Definition 2.1. *A pseudo-MTL algebra is an algebra* $\mathcal{A} = (A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ *of the type* $(2, 2, 2, 2, 2, 0, 0)$ *satisfying the following conditions:*
(M_1) $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice;*
(M_2) $(A, \odot, 1)$ *is a monoid;*
(M_3) $x \odot y \leq z$ *iff* $x \leq y \to z$ *iff* $y \leq x \rightsquigarrow z$ *for any* $x, y, z \in A$*;*
(M_4) $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$ *(pseudo-prelinearity).*

Remark 2.2. (1) *If additionally for any* $x, y \in A$ *the structure* $\mathcal{A}$ *satisfies the axiom:*
(M_5) $(x \to y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$ *(pseudo-divisibility)*
then A *is a pseudo-BL algebra.*
(2) *If* A *satisfies the conditions* $(M_1), (M_2), (M_3)$ *and* (M_5)*, then it is a* bounded divisible residuated lattice *([15], [16]). These structures were also studied under the name bounded* $R\ell$*-monoids ([7], [19], [21]).*

A is called *commutative* if the operation $\odot$ is commutative. In this case $\to = \rightsquigarrow$ and thus, a commutative pseudo-MTL algebra is a MTL algebra.

A totally ordered (linear ordered) pseudo-MTL algebra is called *chain*.
In the sequel we will agree that the operations $\wedge, \vee, \odot$ have higher priority than the operations $\rightarrow, \rightsquigarrow$.
A pseudo-MTL algebra $\mathcal{A}$ will be also referred by its universe A.

Example 2.3. *Let's consider* $A = \{0, a, b, c, 1\}$ *with* $0 < a < b < c < 1$ *and the operations* $\odot, \rightarrow, \rightsquigarrow$ *given by the following tables:*

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	0	b	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

.

Then $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ *is a pseudo-MTL chain. One can easily prove that* $\mathcal{A}$ *is not a pseudo-BL algebra, because* $(b \rightarrow a) \odot b \neq b \odot (b \rightsquigarrow a)$*, so* (M_5) *doesn't hold.*

In a pseudo-MTL algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ we define for all $x \in A$: $x^- = x \rightarrow 0$ and $x^\sim = x \rightsquigarrow 0$.

The following proposition provides some rules of calculus in a pseudo-MTL algebra (see [5], [6], [9], [15]).

Proposition 2.4. *In any pseudo-MTL algebra* A *the following rules of calculus hold:*
(c_1) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$ *and* $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$;
(c_2) $x \leq y$ *iff* $x \rightarrow y = 1$ *iff* $x \rightsquigarrow y = 1$;
(c_3) $x \rightarrow x = x \rightsquigarrow x = 1$ *and* $x \rightarrow 1 = x \rightsquigarrow 1 = 1$;
(c_4) $0 \rightarrow x = 0 \rightsquigarrow x = 1$;
(c_5) $x \odot 0 = 0 \odot x = 0$;
(c_6) $x \odot y \leq x \wedge y$;
(c_7) $(x \rightarrow y) \odot x \leq y$ *and* $x \odot (x \rightsquigarrow y) \leq y$;
(c_8) $x \leq y \rightarrow (x \odot y)$ *and* $x \leq y \rightsquigarrow (y \odot x)$;
(c_9) $x \leq y$ *implies* $x \odot z \leq y \odot z$ *and* $z \odot x \leq z \odot y$ *for any* $z \in A$;
(c_{10}) $(x \rightarrow y) \odot x \leq x \wedge y$ *and* $x \odot (x \rightsquigarrow y) \leq x \wedge y$;
(c_{11}) $(x \rightarrow y) \odot x \leq x \leq y \rightarrow (x \odot y)$ *and* $(x \rightarrow y) \odot x \leq y \leq x \rightarrow (y \odot x)$;
(c_{12}) $x \odot (x \rightsquigarrow y) \leq y \leq x \rightsquigarrow (x \odot y)$ *and* $x \odot (x \rightsquigarrow y) \leq x \leq y \rightsquigarrow (y \odot x)$;
(c_{13}) *if* $x \leq y$ *then* $z \rightarrow x \leq z \rightarrow y$ *and* $z \rightsquigarrow x \leq z \rightsquigarrow y$;
(c_{14}) *if* $x \leq y$ *then* $y \rightarrow z \leq x \rightarrow z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$;
(c_{15}) $1 \rightarrow x = x$ *and* $1 \rightsquigarrow x = x$;
(c_{16}) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$ *and* $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$;
(c_{17}) $x \rightarrow y \leq (z \rightarrow x) \rightsquigarrow (z \rightarrow y)$ *and* $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightarrow (z \rightsquigarrow y)$;
(c_{18}) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$ *and* $x \rightsquigarrow (y \rightarrow z) = y \rightarrow (x \rightsquigarrow z)$;
(c_{19}) $x \rightarrow (x \rightsquigarrow y) = x \rightsquigarrow (x \rightarrow y)$ *and* $x \rightsquigarrow (x \rightarrow y) = x \rightarrow (x \rightsquigarrow y)$;

(c_{20}) $x \to y = x \to (x \wedge y)$ *and* $x \rightsquigarrow y = x \rightsquigarrow (x \wedge y)$;
(c_{21}) $y \leq x \to y$ *and* $y \leq x \rightsquigarrow y$;
(c_{22}) *if* $x \leq y$, *then* $x \leq z \to y$ *and* $x \leq z \rightsquigarrow y$;
(c_{23}) $z \odot (x \wedge y) \leq (z \odot x) \wedge (z \odot y)$ *and* $(x \wedge y) \odot z \leq (x \odot z) \wedge (y \odot z)$;
(c_{24}) $x \to y \leq (x \odot z) \to (y \odot z)$ *and* $x \rightsquigarrow y \leq (z \odot x) \rightsquigarrow (z \odot y)$;
(c_{25}) $(y \to z) \odot (x \to y) \leq x \to z$ *and* $(x \rightsquigarrow y) \odot (y \rightsquigarrow z) \leq x \rightsquigarrow z$;
(c_{26}) $x \odot (y \to z) \leq y \to (x \odot z)$ *and* $(y \rightsquigarrow z) \odot x \leq y \rightsquigarrow (z \odot x)$;
(c_{27}) $(x_{n-1} \to x_n) \odot (x_{n-2} \to x_{n-1}) \odot ... \odot (x_2 \to x_3) \odot (x_1 \to x_2) \leq x_1 \to x_n$ *and*
$(x_1 \rightsquigarrow x_2) \odot (x_2 \rightsquigarrow x_3) \odot ... \odot (x_{n-1} \rightsquigarrow x_n) \leq x_1 \rightsquigarrow x_n$;
(c_{28}) $(x \to y) \odot (x' \to y') \leq (x \vee x') \to (y \vee y')$ *and*
$(x \rightsquigarrow y) \odot (x' \rightsquigarrow y') \leq (x \vee x') \rightsquigarrow (y \vee y')$;
(c_{29}) $(x \to y) \odot (x' \to y') \leq (x \wedge x') \to (y \wedge y')$ *and*
$(x \rightsquigarrow y) \odot (x' \rightsquigarrow y') \leq (x \wedge x') \rightsquigarrow (y \wedge y')$;
(c_{30}) $1^- = 1^\sim = 0$ *and* $0^- = 0^\sim = 1$;
(c_{31}) $x^- \odot x = 0$ *and* $x \odot x^\sim = 0$;
(c_{32}) $x \leq y^-$ *iff* $x \odot y = 0$ *and* $x \leq y^\sim$ *iff* $y \odot x = 0$;
(c_{33}) $x \leq x^{-\sim}$ *and* $x \leq x^{\sim-}$;
(c_{34}) $x \to y^- = (x \odot y)^-$ *and* $x \rightsquigarrow y^\sim = (y \odot x)^\sim$;
(c_{35}) $x \leq y^-$ *iff* $y \leq x^\sim$;
(c_{36}) *if* $x \leq y$, *then* $y^- \leq x^-$ *and* $y^\sim \leq x^\sim$;
(c_{37}) $x \leq x^\sim \to y$ *and* $x \leq x^- \rightsquigarrow y$;
(c_{38}) $x \to y \leq y^- \rightsquigarrow x^-$ *and* $x \rightsquigarrow y \leq y^\sim \to x^\sim$;
(c_{39}) $x \to y^\sim = y \rightsquigarrow x^-$ *and* $x \rightsquigarrow y^- = y \to x^\sim$;
(c_{40}) $x^{\sim-\sim} = x^\sim$ *and* $x^{-\sim-} = x^-$;
(c_{41}) $x \to x^\sim = x \rightsquigarrow x^-$;
(c_{42}) $x \odot (\vee_{i\in I} y_i) = \vee_{i\in I}(x \odot y_i)$ *and* $(\vee_{i\in I} y_i) \odot x = \vee_{i\in I}(y_i \odot x)$;
(c_{43}) $(\vee_{i\in I} x_i) \to y = \wedge_{i\in I}(x_i \to y)$ *and* $(\vee_{i\in I} x_i) \rightsquigarrow y = \wedge_{i\in I}(x_i \rightsquigarrow y)$;
(c_{44}) $y \to (\wedge_{i\in I} x_i) = \wedge_{i\in I}(y \to x_i)$ *and* $y \rightsquigarrow (\wedge_{i\in I} x_i) = \wedge_{i\in I}(y \rightsquigarrow x_i)$;
(c_{45}) $(x \vee y) \to (x \wedge y) = (x \to y) \wedge (y \to x)$ *and*
$(x \vee y) \rightsquigarrow (x \wedge y) = (x \rightsquigarrow y) \wedge (y \rightsquigarrow x)$;
(c_{46}) $(x \vee y)^- = x^- \wedge y^-$ *and* $(x \vee y)^\sim = x^\sim \wedge y^\sim$;
(c_{47}) $(x \wedge y)^- = x^- \vee y^-$ *and* $(x \wedge y)^\sim = x^\sim \vee y^\sim$;
(c_{48}) $(x \vee y)^{-\sim} = x^{-\sim} \vee y^{-\sim}$ *and* $(x \vee y)^{\sim-} = x^{\sim-} \vee y^{\sim-}$;
(c_{49}) $y^- \rightsquigarrow x^- = x^{-\sim} \to y^{-\sim} = x \to y^{-\sim}$;
(c_{50}) $y^\sim \to x^\sim = x^{\sim-} \rightsquigarrow y^{\sim-} = x \rightsquigarrow y^{\sim-}$.

Corollary 2.5. *In any pseudo-MTL algebra we have:*
(c_{51}) $z \odot (x_1 \wedge x_2 \wedge \cdots \wedge x_n) \leq (z \odot x_1) \wedge (z \odot x_2) \wedge \cdots \wedge (z \odot x_n)$ *and*
$(x_1 \wedge x_2 \wedge \cdots \wedge x_n) \odot z \leq (x_1 \odot z) \wedge (x_2 \odot z) \wedge \cdots \wedge (x_n \odot z)$.

Proposition 2.6. *([15]) For any* $g, h, k \in A$ *we have*
(c_{52}) $g \vee (h \odot k) \geq (g \vee h) \odot (g \vee k)$.

Corollary 2.7. (c_{53}) $g \vee (h_1 \odot h_2 \odot ...h_n) \geq (g \vee h_1) \odot (g \vee h_2) \odot ... \odot (g \vee h_n)$.

Proposition 2.8. *([9]) In any pseudo-MTL algebra the following properties hold:*
(c_{54}) $x \vee y = [(x \rightsquigarrow y) \to y] \wedge [(y \to x) \rightsquigarrow x]$;
(c_{55}) $x \vee y = [(x \to y) \rightsquigarrow y] \wedge [(y \to x) \rightsquigarrow x]$.

Corollary 2.9. *If $x \vee y = 1$, then $x \to y = x \rightsquigarrow y = y$.*

For any $n \in \mathbb{N}$, $x \in A$ we put $x^0 = 1$ and $x^{n+1} = x^n \odot x = x \odot x^n$. The order of $x \in A$, denoted $ord(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If there is no such n, then $ord(x) = \infty$.

Proposition 2.10. *In any pseudo-MTL algebra the following hold:*
(1) *if $x \vee y = 1$, then for each $n \in N$, $n \geq 1$, $x^n \vee y^n = 1$;*
(2) $(x \to y)^n \vee (y \to x)^n = 1$ *and* $(x \rightsquigarrow y)^n \vee (y \rightsquigarrow x)^n = 1$*;*
(3) $x \vee y^n \geq (x \vee y)^n$.

Proof: (1) Similarly as in [5], Lemma 2.16;
(2) It follows from (1) and the pseudo-prelinearity condition;
(3) It follows from Corollary 2.7. □

Definition 2.11. *A pseudo-MTL algebra A is locally finite if for any $x \in A$, $x \neq 1$ implies $ord(x) < \infty$.*

Example 2.12. (1) *Consider the pseudo-MTL chain A from Example 2.3. Since $ord(c) = \infty$, it follows that A is not locally finite.*
(2) *Let's consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\odot, \to, \rightsquigarrow$ given by the following tables:*

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	a	a	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	b	c	c	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	1	1
c	a	c	c	1	1
1	0	a	b	c	1

Then $\mathcal{A} = (A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ is a pseudo-MTL chain. Because $(b \to a) \odot b = c \odot b = a$ and $b \odot (b \rightsquigarrow a) = b \odot c = 0$, it follows that $(b \to a) \odot b \neq b \odot (b \rightsquigarrow a)$, so $\mathcal{A}$ is not a pseudo-BL algebra. We have:

$$ord(0) = 1, \; ord(a) = 2, \; ord(b) = 2, \; ord(c) = 3,$$

so A is a locally finite pseudo-MTL algebra.

Remark 2.13. (1) *In [20] is proved that every locally finite pseudo-MV algebra is commutative.*
(2) *In [12] is proved that every locally finite pseudo-BL algebra is an MV algebra, so it is commutative.*
(3) *By the above example we proved that there exist locally finite pseudo-MTL algebras which are not MTL algebras.*

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. Recall (see [13]) that an element $a \in L$ is called *complemented* if there is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$; if such element b exists it is called a *complement* of a. We will denote $b = a'$ and the set of all complemented elements in L by $B(L)$. Complements are generally not unique, unless the lattice is distributive. In pseudo-MTL algebras however, although the underlying lattices need not be distributive, the complements are unique.

The next result is proved in [18] for the case of commutative residuated lattices, but the proof is also valid for the non-commutative case.

Lemma 2.14. *([18]) Let A be a pseudo-MTL algebra. Suppose that $a \in A$ has a complement $b \in A$. Then the following hold:*
(1) *If c is another complement of a in A, then $c = b$;*
(2) *$a' = b$ and $b' = a$;*
(3) *$a^2 = a$.*

Let $B(A)$ the set of all complemented elements of the lattice

$$L(A) = (A, \wedge, \vee, 0, 1).$$

Lemma 2.15. *Let A be a pseudo-MTL algebra. Then the following are equivalent:*
(a) *$x \in B(A)$;*
(b) *$x \vee x^{-} = 1$ and $x \wedge x^{-} = 0$;*
(c) *$x \vee x^{\sim} = 1$ and $x \wedge x^{\sim} = 0$.*

Proof: $(a) \Rightarrow (b)$. Since $x \in B(A)$, there exists $y \in A$ such that $x \vee y = 1$ and $x \wedge y = 0$. Hence, $x^{-} = x^{-} \odot 1 = x^{-} \odot (x \vee y) = (x^{-} \odot x) \vee (x^{-} \odot y) = x^{-} \odot y$, so $y \geq x^{-} \odot y = x^{-}$.
On the other hand, because $y \odot x \leq x \wedge y = 0$ it follows that $y \odot x = 0$, so $y \leq x^{-}$. Thus, $x^{-} = y$, that is $x \vee x^{-} = 1$ and $x \wedge x^{-} = 0$.
$(b) \Rightarrow (a)$. Obviously.
$(a) \Leftrightarrow (c)$. Similarly as $(a) \Leftrightarrow (b)$. □

Proposition 2.16. *Let A be a pseudo-MTL algebra, $x \in B(A)$ and $n \in \mathbb{N}$, $n \geq 1$. Then the following are equivalent:*
(a) *$x^n \in B(A)$;*
(b) *$x \vee (x^n)^{-} = 1$ and $x \vee (x^n)^{\sim} = 1$.*

Proof: $(a) \Rightarrow (b)$. Let $x^n \in B(A)$. By Lemma 2.15 we have $x^n \vee (x^n)^{-} = 1$. Since $x^n \leq x$, we get $1 = x^n \vee (x^n)^{-} \leq x \vee (x^n)^{-}$, so $x \vee (x^n)^{-} = 1$. Similarly, $x \vee (x^n)^{\sim} = 1$.
$(b) \Rightarrow (a)$. Since $x \vee (x^n)^{-} = 1$, by Proposition 2.10(1) we have $x^n \vee ((x^n)^{-})^n = 1$. Because $((x^n)^{-})^n \leq (x^n)^{-}$, we get $1 = x^n \vee ((x^n)^{-})^n \leq x^n \vee (x^n)^{-}$, so $x^n \vee$

$(x^n)^- = 1$.
Similarly, $x^n \vee (x^n)^\sim = 1$, so $(x^n)^- \wedge (x^n)^{\sim -} = 0$.
Because $x^n \leq (x^n)^{\sim -}$ we get $(x^n)^- \wedge x^n \leq (x^n)^- \wedge (x^n)^{\sim -} = 0$, so $(x^n)^- \wedge x^n = 0$.
From $x^n \vee (x^n)^- = 1$ and $x^n \wedge (x^n)^- = 0$ it follows that $x^n \in B(A)$. $\square$

Proposition 2.17. *([4]) If $x \in A$, $n \in \mathbb{N}$, $n \geq 1$ such that $x^n \in B(A)$ and $x^n \geq x^- \vee x^\sim$, then $x = 1$.*

3 Lattice of filters of a pseudo-MTL algebra

Recall that a nonempty subset F of a lattice L is a filter of L if it satisfies the conditions: (i) $x, y \in F$ implies $x \wedge y \in F$ and (ii) $x \in F$, $y \in L$, $x \leq y$ implies $y \in F$.

Definition 3.1. *Let A be a pseudo-MTL algebra. A nonempty set F of A is called filter of A if the following conditions hold:*
(F_1) *If $x, y \in F$, then $x \odot y \in F$;*
(F_2) *If $x \in F$, $y \in A$, $x \leq y$, then $y \in F$.*

We will denote by $\mathcal{F}(A)$ the set of all filters of A.

Remark 3.2. *If F is a filter of A, then:*
(F_3) $1 \in F$;
(F_4) *If $x \in F$, $y \in A$, then $y \to x \in F$, $y \rightsquigarrow x \in F$;*
(F_5) *If $x, y \in F$, then $x \wedge y \in F$.*

Example 3.3. *The subset $F = \{c, 1\}$ of the pseudo-MTL chain A from Example 2.3 is a filter of A.*

Remark 3.4. *Any filter of A is a filter for the lattice $(A, \vee, \wedge)$, but the converse is not true. Indeed, let F be a filter of a pseudo-MTL algebra A and $x, y \in A$. Since $x \odot y \in F$ and $x \odot y \leq x \wedge y$, we get $x \wedge y \in F$, so F is a filter of the lattice $(A, \vee, \wedge)$.*
Let's consider the Example 2.12 and $F = \{c, 1\}$. One can easily prove that F is a filter of the lattice $(A, \vee, \wedge)$, but F is not a filter of the pseudo-MTL algebra A.

Proposition 3.5. *([5]) For a subset F of A the following are equivalent:*
(a) *F is a filter of A;*
(b) *$1 \in A$ and if $x, x \to y \in A$, then $y \in A$;*
(c) *$1 \in F$ and if $x, x \rightsquigarrow y \in A$, then $y \in A$.*

Definition 3.6. *A filter F of A is proper if $F \neq A$.*

Remark 3.7. *If F is a proper filter, then $0 \notin F$.*

Proposition 3.8. *([12] If A is a pseudo-MTL algebra, then the sets*

$$A_0^- = \{x \in A \mid x^- = 0\} \text{ and } A_0^\sim = \{x \in A \mid x^\sim = 0\}$$

are proper filters of A.

Definition 3.9. *For every subset $X \subseteq A$, the smallest filter of A containing X (i.e. the intersection of all filters $F \in \mathcal{F}(A)$ such that $X \subseteq F$) is called the filter generated by X and will be denoted by $[X)$.*

Lemma 3.10. *([12]) Let A be a pseudo-MTL algebra and $x, y \in A$. Then:*
(1) *$[x)$ is proper iff $ord(x) = \infty$;*
(2) *if $x \leq y$ and $ord(y) < \infty$, then $ord(x) < \infty$;*
(3) *if $x \leq y$ and $ord(x) = \infty$, then $ord(y) = \infty$.*

Proposition 3.11. *([5]) If $X \subseteq A$, then*

$$[X) = \{y \in A \mid y \geq x_1 \odot x_2 \odot \cdots \odot x_n \textit{ for some } n \geq 1 \textit{ and } x_1, x_2, \ldots, x_n \in X\}.$$

Proposition 3.12. *([5]) If $X \subseteq A$, then*

$$\begin{aligned}[X) &= \{y \in A \mid x_1 \to (x_2 \to (\ldots(x_n \to y))\ldots) = 1 \\ &\quad \text{for some } n \geq 1 \text{ and } x_1, x_2, \ldots, x_n \in X\} = \\ &= \{y \in A \mid x_1 \rightsquigarrow (x_2 \rightsquigarrow (\ldots(x_n \rightsquigarrow y))\ldots) = 1 \\ &\quad \text{for some } n \geq 1 \text{ and } x_1, x_2, \ldots, x_n \in X\}.\end{aligned}$$

Remark 3.13. *([5])* (1) *If X is a filter of A, then $[X) = X$;*
(2) *If $X = \{x\}$ we write $[x)$ instead of $[\{x\})$ and $[x) = \{y \in X \mid y \geq x^n$ for some $n \geq 1\}$. $[x)$ is called principal filter.*
(3) *If F is a filter of A and $x \in A$, then*
$F(x) = [F \cup \{x\}) = \{y \in A \mid y \geq (f_1 \odot x^{n_1}) \odot (f_2 \odot x^{n_2}) \odot \cdots \odot (f_m \odot x^{n_m})$ for some $m \geq 1,\ n_1, n_2, ..., n_m \geq 0,\ f_1, f_2, ..., f_m \in F\}$.

If F_1 and F_2 are filters of A, we define $F_1 \wedge F_2 = F_1 \cap F_2$ and $F_1 \vee F_2 = [F_1 \cup F_2)$.

Proposition 3.14. *([5],[4]) In any pseudo-MTL algebra A the following hold:*
(1) *If F is a filter of A and $x \in A \setminus F$, then $F(x) = F \vee [x)$;*
(2) *$[x)$ is a proper filter iff $ord(x) = \infty$;*
(3) *$[x \vee y) = [x) \cap [y)$;*
(4) *If $x \leq y$, then $[y) \subseteq [x)$;*
(5) *$[x) \vee [y) = [x \vee y) = [x \odot y)$;*
(6) *$[x \odot y) = [y \odot x)$;*
(7) *$[x \to y) \vee [x) = [x \rightsquigarrow y) \vee [x)$.*

Proposition 3.15. *([4]) If F_1, F_2 are nonempty subsets of A such that $1 \in F_1 \cap F_2$, then $F_1 \vee F_2 = [F_1 \cup F_2) = \{x \in A \mid x \geq (f_1 \odot f_1') \odot (f_2 \odot f_2') \odot \cdots \odot (f_n \odot f_n')$ for some $n \geq 1, f_1, f_2, \ldots, f_n \in F_1,\ f_1', f_2', \ldots, f_n' \in F_2\}$.*

Definition 3.16. *([3]) Let $\mathcal{L} = (L, \wedge, \vee)$ be a lattice.*
(i) *For every $y, z \in L$, the* relative pseudocomplement of y with respect to z*, provided it exists, is the greatest element x such that $x \wedge y \leq z$. It is denoted by $y \Rightarrow z$ (i.e. $y \Rightarrow z = max\{x \mid x \wedge y \leq z\}$).*
(ii) *$\mathcal{L}$ is said to be* relatively pseudocomplemented *provided the relative pseudocomplement $y \Rightarrow z$ exists for every $y, z \in L$.*
(iii) *A* Heyting algebra *is a relatively pseudocomplemented lattice with* 0*, i.e a bounded one.*

If $\mathcal{L}$ is a relatively pseudocomplemented lattice, then $\Rightarrow$ can be viewed as a binary operation on L and there exists the greatest element, 1, of the lattice : $1 = x \Rightarrow x$ for all $x \in L$. Consequently, we have the following equivalent definition, with $\odot = \wedge$:

Definition 3.17. (1) *A* relatively pseudocomplemented lattice *is an algebra $\mathcal{L} = (L, \wedge, \vee, \Rightarrow, 1)$, where $(L, \wedge, \vee, 1)$ is a lattice with greatest element and the binary operation $\Rightarrow$ on L verifies : for all $x, y, z \in L, x \leq y \Rightarrow z$ if and only if $x \wedge y \leq z$.*
($1^{'}$) *A* Heyting algebra *is a duplicate name for bounded relatively pseudocomplemented lattice. For any $x \in L$, the element $x^* = x \Rightarrow 0$ is called the pseudocomplement of x.*

Remark 3.18. (1) *([1], [3]) A* Brouwer algebra *is the dual of a Heyting algebra ($\vee$ instead of $\wedge$, $\geq$ instead of $\leq$, $y \rightarrow z = min\{x \mid z \leq x \vee y\}$ instead of $y \Rightarrow z$).*
(2) *Recall that Gödel algebras are Heyting algebras verifying the condition $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ and that the Gödel t-norm and its associated residum (implication) on $[0, 1]$ are :*

$$x \odot_G y = min(x, y) = x \wedge y, \quad x \rightarrow_G y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y, \end{cases} \quad \text{(Gödel implication).}$$

Note also that a proper Heyting algebra (i.e. which is not a Gödel algebra) is not linearly ordered.

Theorem 3.19. *([2]) A complete lattice is a Heyting algebra if and only if it satisfies the identity*

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i).$$

Theorem 3.20. *([4]) $(\mathcal{F}(A), \wedge, \vee, \Rightarrow, \{1\}, A)$ is a complete Heyting algebra, that is*

$$F \wedge (\bigvee_{i \in I} G_i) = \bigvee_{i \in I} (F \wedge G_i)$$

for any filter F and for any family of filters $\{G_i\}_{i \in I}$ of A.

Definition 3.21. *A filter H of A is called normal if for any $x, y \in A$*

$$(N) \quad x \rightarrow y \in H \ \text{ iff } \ x \rightsquigarrow y \in H.$$

We will denote by $\mathcal{F}_n(A)$ the set of all normal filters of A.

Remark 3.22. *It is obvious that for any pseudo-MTL algebra A:*
(1) $\{1\}$ *and A are normal filters of A;*
(2) $\mathcal{F}_n(A) \subseteq \mathcal{F}(A)$.

Example 3.23. *Let us consider the filter $F = \{c, 1\}$ of the pseudo-MTL chain A from Example 2.3. Since $b \to a = c \in F$ and $b \rightsquigarrow a = b \notin F$, it follows that F is not a normal filter of A.*

Example 3.24. *([16]) Let's consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\odot, \to, \rightsquigarrow$ given by the following tables:*

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	b	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

.

Then $\mathcal{A} = (A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ is a pseudo-MTL chain. Since $(c \to b) \odot c = b \odot c = b \neq a = c \odot b = c \odot (c \rightsquigarrow b)$, it follows that $\mathcal{A}$ is not a pseudo-BL algebra. It is easy to check that $H = \{a, b, c, 1\}$ is a normal filter of A.

Remark 3.25. *([6]) Let H be a normal filter of A. Then:*
(1) $x^- \in H$ *iff* $x^\sim \in H$*;*
(2) $x \in H$ *implies* $(x^-)^- \in H$ *and* $(x^\sim)^\sim \in H$.

Remark 3.26. *In the case of a pseudo-BL algebra A, it is proved that a filter H is normal if and only if $x \odot H = H \odot x$ for any $x \in A$ ([6]). This equality doesn't hold in the case of pseudo-MTL algebras as we can see from Example 3.24. Indeed, in this case we have $c \odot H = \{a, c\}$ and $H \odot c = \{a, b, c\}$, so $c \odot H \neq H \odot c$.*

Lemma 3.27. *Let H be a normal filter of A. Then:*
(1) *For any $x \in A$ and $h \in H$ there is $h' \in H$ such that $x \odot h \geq h' \odot x$;*
(2) *For any $x \in A$ and $h \in H$ there is $h' \in H$ such that $h \odot x \geq x \odot h'$.*

Proof: (1) Let $y = x \odot h$. Then $x \odot h = y = x \wedge y \geq (x \to y) \odot x$.
But $h \leq x \rightsquigarrow x \odot h = x \rightsquigarrow y$. Since $h \in H$, it follows that $x \rightsquigarrow y \in H$.
Because H is a normal filter we have $h' = x \to y \in H$. Thus $x \odot h \geq h' \odot x$.
(2) Let $y = h \odot x$. Then $h \odot x = y = x \wedge y \geq x \odot (x \rightsquigarrow y)$. But $h \leq x \to h \odot x = x \to y$. Since $h \in H$, it follows that $x \to y \in H$. Because H is a normal filter we have $h' = x \rightsquigarrow y \in H$. Thus $h \odot x \geq x \odot h'$. $\square$

Proposition 3.28. *Let H be a normal filter of A and $x \in A$. Then*

$$\begin{aligned} H(x) = [H \cup \{x\}) &= \{y \in A \mid y \geq h \odot x^n \text{ for some } n \in \mathbb{N},\, h \in H\} \\ &= \{y \in A \mid y \geq x^n \odot h \text{ for some } n \in \mathbb{N},\, h \in H\} \\ &= \{y \in A \mid x^n \to y \in H \text{ for some } n \geq 1\} \\ &= \{y \in A \mid x^n \rightsquigarrow y \in H \text{ for some } n \geq 1\}. \end{aligned}$$

Proof: Let $y \in H(x)$. Then $y \geq (h_1 \odot x^{n_1}) \odot (h_2 \odot x^{n_2}) \odot \cdots \odot (h_m \odot x^{n_m})$ for some $m \geq 1$, $n_1, n_2, \ldots, n_m \geq 0$, $h_1, h_2, \ldots, h_m \in H$, by Remark 3.13(3).
If $m = 1$, then $y \geq h_1 \odot x^{n_1}$ and we take $h = h_1$ and $n = n_1$.
If $m = 2$, then $y \geq (h_1 \odot x^{n_1}) \odot (h_2 \odot x^{n_2}) = h_1 \odot (x^{n_1} \odot h_2) \odot x^{n_2}$.
According to Lemma 3.27, there is $h_2' \in H$ such that $x^{n_1} \odot h_2 \geq h_2' \odot x^{n_1}$.
Hence, $y \geq h_1 \odot (h_2' \odot x^{n_1}) \odot x^{n_2} = (h_1 \odot h_2') \odot x^{n_1+n_2}$ and we take $h = h_1 \odot h_2'$ and $n = n_1 + n_2$. By induction we get $y \geq h \odot x^n$ for some $n \in \mathbb{N}$, $h \in H$.
Similarly, $y \geq x^n \odot h$ for some $n \in \mathbb{N}$, $h \in H$. Thus,

$$H(x) = \{y \in A \mid y \geq h \odot x^n \text{ for some } n \in \mathbb{N},\, h \in H\} = \{y \in A \mid y \geq x^n \odot h \text{ for some } n \in \mathbb{N},\, h \in H\}.$$

If $y \in H(x)$, then $h \odot x^n \leq y$ for some $n \geq 1$, $h \in H$. Thus, $h \leq x^n \to y$, hence $x^n \to y \in H$.
Conversely, assume that $h = x^n \to y \in H$ for some $n \geq 1$.
We also have $(h \odot x^n) \to y = h \to (x^n \to y) = h \to h = 1$, hence $h \odot x^n \leq y$.
Therefore, $y \in H(x)$ and we conclude that

$$H(x) = \{y \in L \mid x^n \to y \in H \text{ for some } n \geq 1\}.$$

Similarly, $H(x) = \{y \in A \mid x^n \rightsquigarrow y \in H \text{ for some } n \geq 1\}$.
Note that the last two equalities are also proved in [6], Lemma 1.12 for the case of pseudo-BL algebras. □

Proposition 3.29. *If $F_1, F_2 \in \mathcal{F}_n(A)$ then,*

$$F_1 \vee F_2 = [F_1 \cup F_2) = \{x \in A \mid x \geq u \odot v \text{ for some } u \in F_1, v \in F_2\}.$$

Proof: By Proposition 3.15 we have:
$F_1 \vee F_2 = [F_1 \cup F_2) = \{x \in A \mid x \geq (f_1 \odot f_1') \odot (f_2 \odot f_2') \odot \cdots \odot (f_n \odot f_n') \text{ for some } n \geq 1, f_1, f_2, \ldots, f_n \in F_1, f_1', f_2', \ldots, f_n' \in F_2\}$.
Put $f = (f_1 \odot f_1') \odot (f_2 \odot f_2') \odot \cdots \odot (f_n \odot f_n') = f_1 \odot (f_1' \odot f_2) \odot \cdots \odot (f_{n-1}' \odot f_n) \odot f_n'$.
By Lemma 3.27, there is $f_2'' \in F_2$ such that $f_1' \odot f_2 \geq f_2 \odot f_2''$. Hence,

$$f \geq f_1 \odot f_2 \odot (f_2'' \odot f_3) \odot \cdots \odot (f_n \odot f_n').$$

Similarly, there is $f_3'' \in F_2$ such that $f_2'' \odot f_3 \geq f_3 \odot f_3''$, so

$$f \geq f_1 \odot f_2 \odot f_3 \odot (f_3'' \odot f_4) \odot \cdots \odot (f_n \odot f_n').$$

Finally, $f \geq f_1 \odot f_2 \odot f_3 \odot \cdots \odot f_n \odot f''_n$ with $f_1, f_2, \ldots, f_n \in F_1$, $f''_n \in F_2$. Taking $u = f_1 \odot f_2 \odot f_3 \odot \cdots \odot f_n$, $v = f''_n$, we get $x \geq f \geq u \odot v$ with $u \in F_1$, $v \in F_2$. $\square$

Proposition 3.30. *If $F_1, F_2 \in \mathcal{F}_n(A)$ then:*
(1) $F_1 \wedge F_2 \in \mathcal{F}_n(A)$*;*
(2) $F_1 \vee F_2 \in \mathcal{F}_n(A)$.

Proof: (1) We have $F_1 \wedge F_2 = F_1 \cap F_2$. Consider $x, y \in A$ such that $x \to y \in F_1 \cap F_2$, that is $x \to y \in F_1$ and $x \to y \in F_2$. It follows that $x \rightsquigarrow y \in F_1$ and $x \rightsquigarrow y \in F_2$, hence $x \rightsquigarrow y \in F_1 \cap F_2$. Similarly, $x \rightsquigarrow y \in F_1 \cap F_2$ implies $x \to y \in F_1 \cap F_2 = F_1 \wedge F_2$.
(2) Let $x, y \in A$ such that $x \to y \in F_1 \vee F_2$. By Proposition 3.29, there are $u \in F_1, v \in F_2$ such that $u \odot v \leq x \to y$. Hence, $(u \odot v) \odot x \leq y$ so $u \odot (v \odot x) \leq y$. Since there is $v' \in F_2$ such that $v \odot x \geq x \odot v'$, we get $y \geq (u \odot x) \odot v'$.
Similarly, there is $u' \in F_1$ such that $u \odot x \geq x \odot u'$, so $y \geq x \odot (u' \odot v')$. We get $u' \odot v' \leq x \rightsquigarrow y$, hence $x \rightsquigarrow y \in F_1 \vee F_2$. Similarly, $x \rightsquigarrow y \in F_1 \vee F_2$ implies $x \to y \in F_1 \vee F_2$. $\square$

Proposition 3.31. *If $(F_i)_{i \in I}$ is a family of normal filters of A, then:*
(1) $\bigwedge\limits_{i \in I} F_i \in \mathcal{F}_n(A)$*;*
(2) $\bigvee\limits_{i \in I} F_i \in \mathcal{F}_n(A)$*;*

Proof: Similarly as above. $\square$

As a consequence of the above result we get:

Theorem 3.32. *([4]) $\mathcal{F}_n(A)$ is a complete sublattice of $(\mathcal{F}(A), \subseteq)$.*

For any normal filter H of A we associate a binary relation $\equiv_H$ on A by defining $x \equiv_H y$ iff $x \to y, y \to x \in H$ iff $x \rightsquigarrow y, y \rightsquigarrow x \in H$.

Proposition 3.33. *([6]) For a given normal filter H of A the relation $\equiv_H$ is a congruence relation on A.*

For any $x \in A$, let x/H be the equivalence class $x/\equiv_H$ and $A/H = \{x/H \mid x \in A\}$. A/H becomes a pseudo-MTL algebra with the natural operations induced from those of A. If $x, y \in A$, then $x/H \leq y/H$ *iff* $x \to y \in H$ *iff* $x \rightsquigarrow y \in H$.

Definition 3.34. *A proper (normal) filter of A is called maximal (normal) filter or (normal) ultrafilter if it is not contained in any other proper (normal) filter of A.*

Denote:

$$Max(A) = \{F \mid F \,\text{is maximal filter of}\, A\}$$

and

$$Max_n(A) = \{F \mid F \,\text{is maximal normal filter of}\, A\}.$$

Clearly, $Max_n(A) \subseteq Max(A)$.

Example 3.35. *Let's consider the pseudo-MTL algebra A from Example 2.3. It is obvious that $H_1 = \{c, 1\}$ is a maximal filter of A, but $H_2 = \{1\}$ is not a maximal filter.*

Theorem 3.36. *([4]) If F is a proper filter of A, then the following are equivalent:*
(1) $F \in Max(A)$;
(2) *For any $x \notin F$ there is $f \in F$, $n, m \in \mathbb{N}$, $n, m \geq 1$ such that $(f \odot x^n)^m = 0$.*

Theorem 3.37. *([6]) If H is a proper normal filter of A, then the following are equivalent:*
(a) $H \in Max_n(A)$;
(b) *For any $x \in A$, $x \notin H$ iff $(x^n)^- \in H$ for some $n \in \mathbb{N}$;*
(c) *For any $x \in A$, $x \notin H$ iff $(x^n)^\sim \in H$ for some $n \in \mathbb{N}$.*

Proposition 3.38. *([6]) If H is a proper normal filter of A, then the following are equivalent:*
(a) $H \in Max_n(A)$;
(b) *A/H is locally finite.*

Proposition 3.39. *Let F be a maximal filter of a pseudo-MTL algebra A and $x, y \in A$. Then:*
(1) *$y \notin F$ and $y \odot x = x$ implies $x = 0$;*
(2) *$y \notin F$ and $x \odot y = x$ implies $x = 0$.*

Proof: Let's consider $y \in A \setminus F$ such that $y \odot x = x$.
(1) Assume $x \in A$, $x > 0$ and consider $E = \{z \in A \mid z \odot x = x\}$. First we prove that E is a proper filter. Obviously, $1, y \in E$ and $0 \notin E$. Consider $z \in A$ such that $y \to z \in E$, so $(y \to z) \odot x = x$. Since $(y \to z) \odot y \odot x = (y \to z) \odot x = x$, it follows that

$$x = [(y \to z) \odot y] \odot x \leq (y \wedge z) \odot x \leq (y \odot x) \wedge (z \odot x) = x \wedge (z \odot x) = z \odot x \leq x$$

Thus $z \odot x = x$, hence $z \in E$. Therefore E is a proper filter. Since $y \in E$ and F is maximal, it follows that $y \in F$, a contradiction. Thus, $x = 0$.
(2) Similarly as in (1). $\square$

Let A be a pseudo-MTL algebra and F a filter of A. We define two binary relations on A by:

$$x \equiv_{L(F)} y \Leftrightarrow (x \to y) \wedge (y \to x) \in F \text{ and } \mathrm{x} \equiv_{\mathrm{R(F)}} \mathrm{y} \Leftrightarrow (\mathrm{x} \rightsquigarrow \mathrm{y}) \wedge (\mathrm{y} \rightsquigarrow \mathrm{x}) \in \mathrm{F}.$$

Proposition 3.40. *([5]) For a given filter F, the relations $\equiv_{L(F)}$ and $\equiv_{R(F)}$ are equivalence relations on A.*

We also define two order relations $\leq_{L(F)}$ on $A/L(F)$ and $\leq_{R(F)}$ on $A/R(F)$ by:

$$x/L(F) \leq_{L(F)} y/L(F) \Leftrightarrow x \to y \in F \text{ and } \mathrm{x/R(F)} \leq_{\mathrm{R(F)}} \mathrm{y/R(F)} \Leftrightarrow \mathrm{x} \rightsquigarrow \mathrm{y} \in \mathrm{F}.$$

Definition 3.41. *A proper (normal) filter P of A is called (normal) prime filter if for all $x, y \in A$, $x \vee y \in P$ implies $x \in P$ or $y \in P$.*

The set of all prime filters of A will be denoted by $Spec(A)$. We also denote by $Spec_n(A)$ the set of all prime normal filters of A. Clearly, $Spec_n(A) \subseteq Spec(A)$.

Example 3.42. *Let's consider $A = \{0, a, b, c, 1\}$ with $0 < a < b, c < 1$, but b, c are incomparable. Consider also the operations $\odot, \to, \rightsquigarrow$ given by the following tables:*

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	a	0	a
b	0	0	b	0	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	c	1	c	1
c	b	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

.

Then $\mathcal{A} = (A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ is a pseudo-MTL algebra. Because $(b \to a) \odot b = c \odot b = a$ and $b \odot (b \rightsquigarrow a) = b \odot c = 0$, it follows that $(b \to a) \odot b \neq b \odot (b \rightsquigarrow a)$, so $\mathcal{A}$ is not a pseudo-BL algebra. Clearly, A is not a pseudo-MTL chain.
Obviously, $P = \{c, 1\}$ is a prime filter of A. Consider the filter $F = \{1\}$ of A. Since $b \vee c = 1 \in F$, but $b, c \notin F$, it follows that F is not a prime filter of A.

Proposition 3.43. *([5]) If P is a proper filter of A, then the following properties are equivalent:*
(a) P is prime;
(b) For all $x, y \in A$, $x \to y \in P$ or $y \to x \in P$;
(c) For all $x, y \in A$, $x \rightsquigarrow y \in P$ or $y \rightsquigarrow x \in P$;
(d) $A/L(P)$ is a chain;
(e) $A/R(P)$ is a chain.

Corollary 3.44. *([5]) If P is a prime filter and Q is a proper filter such that $P \subseteq Q$, then Q is a prime filter.*

Proposition 3.45. *If P is a prime filter of A, then $x \odot y \in P$ implies $x^2 \in P$ or $y^2 \in P$.*

Proof: Since $x \odot y \in P$ and $x \odot y \leq x \wedge y \leq x, y$ we have $x, y \in P$. Hence, $x \odot y \in P$. Because P is a prime filter, for all $x, y \in A$, we have $x \to y \in P$ or $y \to x \in P$.
Consider $x \to y \in P$. Applying (c_{23}) for $z = y$, we get

$$(x \odot y) \wedge y^2 \geq (x \wedge y) \odot y \geq (x \to y) \odot x \odot y.$$

Thus, $y^2 \geq (x \odot y) \wedge y^2 \geq (x \to y) \odot (x \odot y)$. Since $x \to y, x \odot y \in P$ we have $(x \to y) \odot (x \odot y) \in P$, hence $y^2 \in P$.
If $y \to x \in P$, applying (c_{23}) for $z = x$, we get

$$x^2 \wedge (y \odot x) \geq (x \wedge y) \odot x \geq (y \to x) \odot y \odot x.$$

Thus, $x^2 \geq x^2 \wedge (y \odot x) \geq (y \to x) \odot (y \odot x)$. Since $y \to x, y \odot x \in P$ we have $(y \to x) \odot (y \odot x) \in P$, hence $x^2 \in P$. $\square$

Theorem 3.46. *([5]) (Prime filter theorem) Let F be a of A and let S be an ideal of the lattice $\mathcal{L}(A)$ such that $F \cap S = \emptyset$. Then there exists a prime filter P of A such that $F \subseteq P$ and $P \cap S = \emptyset$.*

Corollary 3.47. *([5]) Let F be a filter of A and $a \in A \setminus F$. Then there is a prime filter P of A such that $F \subseteq P$ and $a \notin P$.*

Corollary 3.48. *([5]) If $x \in A$, $x \neq 1$, then there is a prime filter P such that $x \notin P$.*

Corollary 3.49. *([5]) Every proper filter F is the intersection of those prime filters which contain F. In particular, $\cap Spec(A) = \{1\}$.*

Corollary 3.50. *([5]) $Max(A) \subseteq Spec(A)$.*

Proposition 3.51. *([5]) Any proper filter can be extended to a maximal filter.*

Proposition 3.52. *([5]) The set of proper filters including a prime filter P of A is a chain.*

Proposition 3.53. *([5]) A pseudo-MTL algebra A is a chain if and only if any proper filter of A is prime.*

Example 3.54. (1) *Consider the pseudo-MTL chain A from Example 3.24. It is obvious that: $\mathcal{F}(A) = \{\{1\}, \{c, 1\}, \{a, b, c, 1\}, A\}$, $\mathcal{F}_n(A) = \{\{1\}, \{a, b, c, 1\}, A\}$, $Max(A) = \{\{a, b, c, 1\}\}$, $Max_n(A) = \{\{a, b, c, 1\}\}$, $Spec(A) = \{\{1\}, \{c, 1\}, \{a, b, c, 1\}\}$, $Spec_n(A) = \{\{1\}, \{a, b, c, 1\}\}$. Obviously, in this case we have $Max_n(A) = Max(A)$.*
(2) *For the pseudo-MTL algebra A from Example 3.42 we have: $\mathcal{F}(A) = \{\{1\}, \{b, 1\}, \{c, 1\}, A\}$, $\mathcal{F}_n(A) = \{\{1\}, A\}$, $Max(A) = \{\{b, 1\}, \{c, 1\}\}$, $Max_n(A) = \emptyset$, $Spec(A) = \{\{b, 1\}, \{c, 1\}\}$, $Spec_n(A) = \emptyset$. The filter $\{1\}$ is not prime and A is not a chain. This fact is in accordance with the assertion of Proposition 3.53.*

Definition 3.55. *An element $p < 1$ of a bounded lattice $(A, \wedge, \vee, 0, 1)$ is called meet-irreducible if $p = x \wedge y$ implies $p = x$ or $p = y$.*

Theorem 3.56. *([4]) If P is a proper filter of A, then the following are equivalent:*
(a) *P is prime;*
(b) *P is meet-irreducible in the lattice $\mathcal{F}(A)$;*
(c) *If $x, y \in A$ such that $x \vee y = 1$, then $x \in P$ or $y \in P$;*
(d) *For all $x, y \in A \setminus P$ there is $z \in A \setminus P$ such that $x \leq z$ and $y \leq z$;*
(e) *If $x, y \in A$ and $[x) \wedge [y) \subseteq P$, then $x \in P$ or $y \in P$.*

Proposition 3.57. *Any locally finite pseudo-MTL algebra A is a chain.*

Proof: Let $x, y \in A$ such that $x \vee y = 1$. Applying (c_{54}), we get:

$$1 = x \vee y = [(x \to y) \rightsquigarrow y] \wedge [(y \rightsquigarrow x) \to x] \leq (x \to y) \rightsquigarrow y,$$

so $(x \to y) \rightsquigarrow y = 1$, that is $x \to y \leq y$. Taking in consideration that $y \leq x \to y$, we get $x \to y = y$. Let's suppose that $x \neq 1$. Since A is locally finite, there is $n \in \mathbb{N}$ such that $x^n = 0$. We have:

$$y = x \to y = x \to (x \to y) = x^2 \to y = \cdots = x^n \to y = 0 \to y = 1.$$

Thus, $x \vee y = 1$ iff $x = 1$ or $y = 1$. But, for all $x, y \in A$ we have $(x \to y) \vee (y \to x) = 1$, so, applying the above result we get $x \to y = 1$ or $y \to x = 1$. Hence, $x \leq y$ or $y \leq x$. We conclude that A is a chain. $\square$

4 Good pseudo-MTL algebras

Definition 4.1. *A pseudo-MTL algebra A is called good if $x^{-\sim} = x^{\sim -}$ for any $x \in A$.*

Example 4.2. (1) *The pseudo-MTL chains from Examples 2.3 and 3.24 are good;*
(2) *The pseudo-MTL chain from Example 2.12 and the pseudo-MTL algebra from Example 3.42 are not good.*

Proposition 4.3. *If $F = A \setminus \{0\}$ is a maximal filter of a pseudo-MTL algebra A, then A is good.*

Proof: Obviously $(0^-)^\sim = (0^\sim)^- = 0$. Assume $x > 0$, that is $x \in F$. If $x^-, x^\sim \in F$ it follows that $x^- \odot x$, $x \odot x^\sim \in F$, that is $0 \in F$, a contradiction. Thus $x^- = x^\sim = 0$, hence $(x^-)^\sim = (x^\sim)^- = 1$. Therefore, A is a good pseudo-MTL algebra. $\square$

Proposition 4.4. *In any good pseudo-MTL algebra we have* $(x^{\sim} \odot y^{\sim})^{-} = (x^{-} \odot y^{-})^{\sim}$.

Proof: Applying $(c_{34}), (c_{49}), (c_{50})$ we have:

$$\begin{aligned}(x^{\sim} \odot y^{\sim})^{-} &= x^{\sim} \to y^{\sim -} = x^{\sim} \to y^{-\sim} = y^{-\sim -} \rightsquigarrow x^{\sim -} \\ &= y^{-} \rightsquigarrow x^{\sim -} = y^{-} \rightsquigarrow x^{-\sim} = (x^{-} \odot y^{-})^{\sim}.\end{aligned}$$

(In the last equality we also applied (c_{34})). $\square$

Proposition 4.5. *In any good pseudo-MTL algebra* A *we have* $x^{-\sim} \odot y^{-\sim} \leq (x \odot y)^{-\sim}$.

Proof: Because A is good, by (c_{10}), we have:

$$\begin{aligned}(x \odot y)^{-\sim} &= (x \odot y)^{\sim -} \geq (x \odot y)^{\sim -} \wedge x^{\sim -} \geq x^{\sim -} \odot (x^{\sim -} \rightsquigarrow (x \odot y)^{\sim -}) \\ &= x^{\sim -} \odot (x^{\sim -} \rightsquigarrow (x \odot y)^{-\sim}) = x^{\sim -} \odot (x^{\sim -} \rightsquigarrow (x \to y^{-})^{\sim}).\end{aligned}$$

But, applying (c_{34}) and (c_1) we have:

$$\begin{aligned}x^{\sim -} \rightsquigarrow (x \to y^{-})^{\sim} &= x^{\sim -} \rightsquigarrow ((x \to y^{-}) \rightsquigarrow 0) = (x^{\sim -} \to y^{-}) \odot x^{\sim -} \rightsquigarrow 0 \\ &= ((x^{\sim -} \to y^{-}) \odot x^{\sim -})^{\sim} \geq (x^{\sim -} \wedge y^{-})^{\sim} \geq x^{\sim -\sim} \vee y^{-\sim} \\ &= x^{\sim} \vee y^{-\sim}.\end{aligned}$$

By (c_{34}) we have $(x^{\sim -} \to y^{-}) \odot x^{\sim -} \leq (x^{\sim -} \wedge y^{-})$, so
$((x^{\sim -} \to y^{-}) \odot x^{\sim -})^{\sim} \geq (x^{\sim -} \wedge y^{-})^{\sim})$.
It follows that

$$\begin{aligned}(x \odot y)^{-\sim} &\geq x^{\sim -} \odot (x^{\sim} \vee y^{-\sim}) = (x^{\sim -} \odot x^{\sim}) \vee (x^{\sim -} \odot y^{-\sim}) \\ &= 0 \vee (x^{\sim -} \odot y^{-\sim}) = x^{\sim -} \odot y^{\sim -} = x^{-\sim} \odot y^{-\sim}.\end{aligned}$$

(we applied (c_{42}) and (c_{31})). $\square$

If A is a good pseudo-MTL algebra, then we will denote

$$M(A) = \{x \in A \mid x^{-\sim} = x^{\sim -} = x\}.$$

Proposition 4.6. *([12]) Let* A *be a good pseudo-MTL algebra. Then:*
(1) $0, 1 \in M(A)$;
(2) $x^{-}, x^{\sim} \in M(A)$ *for all* $x \in A$;
(3) *if* $x, y \in M(A)$, *then* $x \to y = y^{-} \rightsquigarrow x^{-}$ *and* $x \rightsquigarrow y = y^{\sim} \to x^{\sim}$;
(4) *if* $x, y \in M(A)$, *then* $(x^{-} \odot y^{-})^{\sim} = (x^{\sim} \odot y^{\sim})^{-} = x^{-} \rightsquigarrow y = y^{\sim} \to x$.

Definition 4.7. *([10]) If* A *is a good pseudo-MTL algebra we say that two elements* $x, y \in A$ *are orthogonals, denoted* $x \perp y$, *if* $x^{-\sim} \leq y^{\sim}$.

Remark 4.8. *([10]) If A is a good pseudo-MTL algebra, then the following are equivalent:*
(a) $x \perp y$;
(b) $y^{-\sim} \leq x^{-}$;
(c) $y^{-\sim} \odot x^{-\sim} = 0$.

Remark 4.9. *([10]) Let A be a good pseudo-MTL algebra. For all $x, y \in A$ we have:*
(1) *if $x \leq y$, then $x \perp y^{\sim}$ and $y^{-} \perp x$;*
(2) *$x \perp x^{\sim}$ and $x^{-} \perp x$.*

5 Local pseudo-MTL algebras

Definition 5.1. *A pseudo-MTL algebra is called* local *if it has a unique ultrafilter.*

If A is a pseudo-MTL algebra, we will denote by:

$$D(A) = \{x \in A \mid ord(x) = \infty\}$$

$$D(A)^* = \{x \in A \mid ord(x) < \infty\}.$$

Obviously, $D(A) \cap D(A)^* = \emptyset$ and $D(A) \cup D(A)^* = A$.
We also can remark that $1 \in D(A)$ and $0 \in D(A)^*$.

Let A be a pseudo-MTL algebra and F a filter of A. We will use the following notations:

$$F^*_{-} = \{x \in A \mid x \leq y^{-} \text{ for some } y \in F\};$$

$$F^*_{\sim} = \{x \in A \mid x \leq y^{\sim} \text{ for some } y \in F\}.$$

Remark 5.2. *([12]) Let A be a pseudo-MTL algebra. Then:*
(1) $F^*_{-} = \{x \in A \mid y \odot x = 0$ *for some* $y \in F\}$;
(2) $F^*_{\sim} = \{x \in A \mid x \odot y = 0$ *for some* $y \in F\}$;
(3) $F^*_{-} = \{x \in A \mid x^{\sim} \in F\}$;
(4) $F^*_{\sim} = \{x \in A \mid x^{-} \in F\}$.

Proposition 5.3. *([12]) Let A be a local pseudo-MTL algebra. Then:*
(1) *any proper filter of A is included in the unique maximal filter of A;*
(2) *A_0^{-} and $A_0^{\sim}$ are included in the unique maximal filter of A.*

Proposition 5.4. *([12]) Let A be a pseudo-MTL algebra. Then the following are equivalent:*
(a) *$D(A)$ is a filter of A;*
(b) *$D(A)$ is a proper filter of A;*
(c) *A is local ;*
(d) *$D(A)$ is the unique ultrafilter of A;*
(e) *for all $x, y \in A$, $ord(x \odot y) < \infty$ implies $ord(x) < \infty$ or $ord(y) < \infty$.*

Corollary 5.5. *([12]) If A is a local pseudo-MTL algebra, then:*
(1) *for any $x \in A$, $ord(x) < \infty$ or ($ord(x^-) < \infty$ and $ord(x^\sim) < \infty$);*
(2) *$D(A)^*_- \subseteq D(A)^*$ and $D(A)^*_\sim \subseteq D(A)^*$;*
(3) *$D(A) \cap D(A)^*_- = D(A) \cap D(A)^*_\sim = \emptyset$.*

Proposition 5.6. *([12]) Any pseudo-MTL chain is a local pseudo-MTL algebra.*

Example 5.7. (1) *The good pseudo-MTL chains from Examples 2.3, 2.12 and 3.24 are local;*
(2) *The pseudo-MTL algebra A from Example 3.42 is not local. Indeed, $D(A) = \{b, c, 1\}$ is not a filter of A ($b \odot c = 0 \notin A$).*

Open problem 1. *Find an example of local pseudo-MTL algebra which is not a chain.*

Definition 5.8. *([12]) A pseudo-MTL algebra A is called perfect if it satisfies the following conditions :*
(1) *A is a local good pseudo-MTL algebra;*
(2) *for any $x \in A$, $ord(x) < \infty$ iff $ord(x^-) = \infty$ iff $ord(x^\sim) = \infty$.*

Proposition 5.9. *([12]) Let A be a local good pseudo-MTL algebra. Then the following are equivalent:*
(a) *A is perfect;*
(b) *for any $x \in A$, $ord(x) < \infty$ implies $ord(x^-) = \infty$;*
(c) *for any $x \in A$, $ord(x) < \infty$ implies $ord(x^\sim) = \infty$;*
(d) *$D(A)^*_- = D(A)^*$;*
(e) *$D(A)^*_\sim = D(A)^*$.*

Example 5.10. (1) *Consider the good pseudo-MTL chain A from Example 3.24. By Proposition 5.6 it is local. One can easily check that $ord(x) < \infty$ iff $ord(x^-) = \infty$ iff $ord(x^\sim) = \infty$ for all $x \in A$. Thus, A is a perfect pseudo-MTL chain.*
(2) *The pseudo-MTL chain from Example 2.3 is not perfect ($ord(a) = 2 < \infty$, but $ord(a^-) = ord(b) = 2 < \infty$).*
(3) *The pseudo-MTL chain from Example 2.12 and the pseudo-MTL algebra from Example 3.42 are not perfect, since they are not good.*

Open problem 2. *Find an example of perfect pseudo-MTL algebra which is not a chain.*

Corollary 5.11. *If A is a perfect pseudo-MTL algebra, then*

$$D(A)^* = \{x^- \mid x \in D(A)\} = \{x^\sim \mid x \in D(A)\}.$$

Definition 5.12. *Let A be a pseudo-MTL algebra. The intersection of all maximal filters of A is called the radical of A and it is denoted by $Rad(A)$. The intersection of all maximal normal filters of A is called the normal radical of A and it is denoted by $Rad_n(A)$. It is obvious that $Rad(A)$ and $Rad_n(A)$ are filters of A and $Rad(A) \subseteq Rad_n(A)$.*

Example 5.13. (1) *In the case of the pseudo-MTL algebra A from Example 3.42 we have $Max(A) = \{\{b,1\},\{c,1\}\}$. It follows than $Rad(A) = \{1\}$. Note that A is not a chain.*
(2) *In the case of the pseudo-MTL chain A from Example 3.24 we have $Max(A) = Max_n(A) = \{\{a,b,c,1\}\}$. Hence, $Rad(A) = Rad_n(A) = \{\{a,b,c,1\}\}$.*

Proposition 5.14. *If A is a local pseudo-MTL algebra, then $Rad(A) = D(A)$.*

Proof: By Proposition 5.4 it follows that $D(A)$ is the unique maximal filter of A, so $Rad(A) = D(A)$. $\square$

Remark 5.15. *If A is a local pseudo-MTL algebra and $x \in Rad(A)^*$, $y \in A$ such that $y \leq x$, then $y \in Rad(A)^*$.*

Proposition 5.16. *([4]) For any $x, y \in Rad(A)$, $x^- \odot y^- = x^\sim \odot y^\sim = 0$.*

Corollary 5.17. *Let A be a perfect pseudo-MTL algebra. If $x \in Rad(A)$ and $y \in Rad(A)^*$, then $x^- \leq y^-$ and $x^\sim \leq y^\sim$.*

Proof: Since $x, y^\sim \in Rad(A)$, by Proposition 5.16 we get $x^- \odot y^{\sim -} = 0$. Because $y \leq y^{-\sim}$, we have $x^- \odot y \leq x^- \odot y^{\sim -} = 0$, so $x^- \odot y = 0$. Hence, $x^- \leq y^-$. Similarly, $x^\sim \leq y^\sim$. $\square$

Proposition 5.18. *If A is a perfect pseudo-MTL algebra and $x, y \in Rad(A)^*$, then $x \perp y$ and $y \perp x$.*

Proof: Since $x, y \in Rad(A)^*$, it follows that $y^-, x^- \in Rad(A)$. Hence, $y^{-\sim} \odot x^{-\sim} = 0$. By Proposition 4.8($c$) we get $x \perp y$. Similarly, $y \perp x$. $\square$

Theorem 5.19. *If A is a perfect pseudo-MTL algebra, then $Rad(A)$ is a normal filter of A.*

Proof: We have to prove that $x \to y \in Rad(A)$ iff $x \rightsquigarrow y \in Rad(A)$ for all $x, y \in A$. Consider $x, y \in A$ such that $x \to y \in Rad(A)$ and suppose $x \rightsquigarrow y \notin Rad(A)$. From $y \leq y^{-\sim}$ we get $x \to y \leq x \to y^{-\sim}$ (by (c_{33}) and (c_{13})). Since $Rad(A)$ is a filter of A, it follows that $x \to y^{-\sim} \in Rad(A)$, that is $(x \odot y^\sim)^- \in Rad(A)$ (by (c_{34})).
Hence, $x \odot y^\sim \in Rad(A)^*$.
On the other hand, from $x \rightsquigarrow y \notin Rad(A)$, it follows that $x \rightsquigarrow y \in Rad(A)^*$.
Since $x \leq x^{-\sim}$, by (c_{14}) we get $x^{-\sim} \rightsquigarrow y \leq x \rightsquigarrow y$, so $x^{-\sim} \rightsquigarrow y \in Rad(A)^*$ (by Remark 5.15). By (c_{37}) we have $x^\sim \leq x^{\sim -} \rightsquigarrow y$, so $x^\sim \in Rad(A)^*$, that is $x \in Rad(A)$. But $y \leq x \rightsquigarrow y$, so $y \in Rad(A)^*$, that is $y^\sim \in Rad(A)$. Since $Rad(A)$ is a filter of A and $x, y^\sim \in Rad(A)$, we get $x \odot y^\sim \in Rad(A)$ which is a contradiction. Thus, $x \to y \in Rad(A)$ implies $x \rightsquigarrow y \in Rad(A)$.
Similarly, $x \rightsquigarrow y \in Rad(A)$ implies $x \to y \in Rad(A)$ and we conclude that $Rad(A)$ is a normal filter of A. $\square$

Remark 5.20. *If the pseudo-MTL algebra A is not perfect, then the above result is not always valid, as we can see in the following example.*
Let's consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\odot, \rightarrow, \rightsquigarrow$ given by the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	a	a	a
b	0	0	b	b	b
c	0	0	b	b	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	0	a	1	1	1
c	0	a	c	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	a	1	1	1	1
b	a	a	1	1	1
c	a	a	c	1	1
1	0	a	b	c	1

.

One can easily check that the pseudo-MTL chain A is not perfect and $H = \{b, c, 1\}$ is the unique maximal filter of A, so $Rad(A) = \{b, c, 1\}$. Since $a \rightarrow 0 = c \in H$ and $a \rightsquigarrow 0 = a \notin H$, it follows that $Rad(A)$ is not a normal filter of A.

Remark 5.21. *There exist pseudo-MTL algebras which are not perfect, but $Rad(A)$ is a normal filter, so it is not a necessary condition. Indeed, let's consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\odot, \rightarrow, \rightsquigarrow$ given by the following tables:*

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	a	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

.

One can easily check that the pseudo-MTL chain A is not perfect and $H = \{b, c, 1\}$ is the unique maximal filter of A, so $Rad(A) = \{b, c, 1\}$. One can easily check that $Rad(A)$ is also a normal filter of A.

6 Archimedean pseudo-MTL algebras

We will introduce the notion of *Archimedean* pseudo-MTL algebra in the same way as in the case of pseudo-BL algebras (see [4]).

Proposition 6.1. *In any pseudo-MTL algebra the following are equivalent:*
(a) $x^n \geq x^- \vee x^\sim$ for any $n \in \mathbb{N}$ implies $x = 1$;
(b) $x^n \geq y^- \vee y^\sim$ for any $n \in \mathbb{N}$ implies $x \vee y = 1$;
(c) $x^n \geq y^- \vee y^\sim$ for any $n \in \mathbb{N}$ implies $x \rightarrow y = x \rightsquigarrow y = y$.

Proof: $(a) \Rightarrow (b)$ Take $x, y \in A$ such that $x^n \geq y^- \vee y^\sim$ for any $n \in \mathbb{N}$.
By (c_{46}) and by the hypothesis we have:

$$(x \vee y)^- = x^- \vee y^- \leq y^- \leq y^- \vee y^\sim \leq x^n \leq (x \vee y)^n \text{ and}$$

$$(x \vee y)^{\sim} = x^{\sim} \vee y^{\sim} \leq y^{\sim} \leq y^{-} \vee y^{\sim} \leq x^n \leq (x \vee y)^n,$$

hence $(x \vee y)^n \geq (x \vee y)^{-} \vee (x \vee y)^{\sim}$ for any $n \in \mathbb{N}$. Thus, by hypothesis we get $x \vee y = 1$.
$(b) \Rightarrow (c)$ By (c_{54}) and (c_{55}), for all $x, y \in A$ we have:

$$(x \vee y) = [(x \to y) \rightsquigarrow y] \wedge [(y \rightsquigarrow x) \to x]$$

$$(x \vee y) = [(x \rightsquigarrow y) \to y] \wedge [(y \to x) \rightsquigarrow x].$$

Since $x \vee y = 1$, it follows that:

$$[(x \to y) \rightsquigarrow y] \wedge [(y \rightsquigarrow x) \to x] = 1,$$

$$[(x \rightsquigarrow y) \to y] \wedge [(y \to x) \rightsquigarrow x] = 1,$$

hence $(x \to y) \rightsquigarrow y = 1$ and $(x \rightsquigarrow y) \to y = 1$. From $(x \to y) \rightsquigarrow y = 1$ we have $x \to y \leq y$ and taking in consideration that $y \leq x \to y$, we obtain $x \to y = y$. Similarly, $x \rightsquigarrow y = y$.
$(c) \Rightarrow (a)$ Consider $x \in A$ such that $x^n \geq x^{-} \vee x^{\sim}$ for any $n \in \mathbb{N}$. Taking $y = x$ in (c), we get $x \to x = x$, hence $x = 1$. $\square$

Definition 6.2. *A pseudo-MTL algebra is called Archimedean if one of the equivalent conditions from the above proposition is satisfied.*

Definition 6.3. *An element $x \in A$ is called* Archimedean *if there is $n \in \mathbb{N}, n \geq 1$ such that $x^n \in B(A)$. A pseudo-MTL algebra A is called* hyperarchimedean *if all its elements are Archimedean.*

Proposition 6.4. *Any locally finite pseudo-MTL algebra is hyperarchimedean.*

Proof: Let A be a locally finite pseudo-MTL algebra and $x \in A$. Hence, there exists $n \in \mathbb{N}$ such that $x^n = 0 \in B(A)$. It follows that any element x of A is Archimedean, so A is hyperarchimedean. $\square$

Corollary 6.5. *Any hyperarchimedean pseudo-MTL algebra is Archimedean.*

Proof: Let A be a hyperarchimedean pseudo-MTL algebra and $x \in A$ such that $x^n \geq x^{-} \vee x^{\sim}$ for any $n \in \mathbb{N}$. Since A is hyperarchimedean, there exists $m \in \mathbb{N}$, $m \geq 1$ such that $x^m \in B(A)$. According to Proposition 2.17 if follows that $x = 1$, so A is Archimedean. $\square$

Corollary 6.6. *Any locally finite pseudo-MTL algebra is Archimedean.*

Proof: It follows from Proposition 6.4 and Corollary 6.5. $\square$

Theorem 6.7. *([4]) For a pseudo-MTL algebra A, the following are equivalent:*
(a) *A is hyperarchimedean;*
(b) *For any normal filter H, the quotient pseudo-MTL algebra A/H is an Archimedean pseudo-MTL algebra.*

Example 6.8. (1) *The pseudo-MTL chains from Examples 2.3 and 3.24 are neither Archimedean, nor hyperarchimedean (for example, in the first case, $c^n = c \geq c^- \vee c^\sim = 0 \vee 0 = 0$ for all $n \geq 2$);*
(2) *The pseudo-MTL chain from Example 2.12 is locally finite, so it is hyperarchimedean and Archimedean;*
(3) *The pseudo-MTL algebra A from Example 3.42 is Archimedean, but it is not hyperarchimedean. Indeed:*

$$0^n = 0 \ngeq 0^- \vee 0^\sim = 1 \vee 1 = 1,\ n \geq 1$$

$$a^n = 0 \ngeq a^- \vee a^\sim = b \vee c = 1,\ n \geq 2$$

$$b^n = b \ngeq b^- \vee b^\sim = 0 \vee c = c,\ n \geq 1$$

$$c^n = c \ngeq c^- \vee c^\sim = b \vee 0 = b,\ n \geq 1$$

$$1^n = 1 \geq 1^- \vee 1^\sim = 0 \vee 0 = 0,\ n \geq 1.$$

We conclude that, if $x^n \geq x^- \vee x^\sim$ for all $n \in \mathbb{N}$, $n \geq 1$, then $x = 1$. Hence, A is an Archimedean pseudo-MTL algebra. Since $a^2 = 0 \in B(A)$, it follows that a is an Archimedean element. By contrary, $b^n = b \notin B(A)$ for all $n \in \mathbb{N}$, $n \geq 1$, so b is not Archimedean element. Thus, A is not hyperarchimedean.

Remark 6.9. *By Examples 6.8*(2),(3) *we proved that, generally, an Archimedean pseudo-MTL algebra is not commutative (i.e. a MTL algebra).*

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