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On Cauchy's bound for zeros of a polynomial

by
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Abstract

In this paper we have improved Cauchy's bound for zeros of a polynomial

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n.$$

In many cases, our result gives better bounds than those obtainable by most of the other known results.

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1 Introduction and statement of results

Let

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n$$

be a polynomial of degree n, $(n \geq 2)$. Then we have the following classical result, due to Cauchy [3], on the location of zeros of a polynomial.

Theorem A. *All the zeros of the polynomial* $p(z)$ *lie in the disc*

$$|z| < 1 + A, \tag{1.1}$$

where

$$A = \max_{1 \leq j \leq n} |a_j|.$$

Joyal et al. [6] improved Cauchy's bound (1.1) and obtained

Theorem B. *All the zeros of the polynomial* $p(z)$ *lie in the disc*

$$|z| \leq (1/2)\{1 + |a_1| + \sqrt{(1 - |a_1|)^2 + 4\delta}\}, \tag{1.2}$$

where

$$\delta = \max_{2 \leq j \leq n} |a_j|.$$

In this paper, we have obtained a result, which gives a refinement of the bound (1.2), due to Joyal et al.. More precisely, we have proved

Theorem 1. *Let*

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n; a_j \neq 0 \textit{ for at least one } j, 2 \leq j \leq n, \tag{1.3}$$

be a polynomial of degree n, with

$$\beta_2^{(\gamma)} \geq \beta_3^{(\gamma)} \geq \ldots \geq \beta_n^{(\gamma)}, \tag{1.4}$$

being the ordered non-negative numbers

$$|b_j^{(\gamma)}| = |a_j \gamma^{-j}|; \gamma > 0, j = 2, 3, \ldots, n, \tag{1.5}$$

and

$$\beta_1^{(\gamma)} = |a_1 \gamma^{-1}|. \tag{1.6}$$

Then all the zeros of the polynomial $p(z)$ *lie in the disc*

$$|z| \leq (\gamma/2)\{\alpha_\gamma + \beta_1^{(\gamma)} + \sqrt{(\alpha_\gamma - \beta_1^{(\gamma)})^2 + 4\sigma_\gamma}\}, \tag{1.7}$$

where

$$0 \leq \alpha_\gamma = \max_{2 \leq j \leq n-1} (\beta_{j+1}^{(\gamma)} / \beta_j^{(\gamma)}) \leq 1, \tag{1.8}$$

(maximum being taken over all j *such that* $\beta_j^{(\gamma)} \neq 0$*),*

$$\sigma_\gamma = \left\{ \begin{array}{ll} \beta_2^{(\gamma)} - (\delta_2^{(\gamma)}/(\alpha_\gamma + t_\gamma)) - (\delta_3^{(\gamma)}/(\alpha_\gamma + t_\gamma)^2) - \ldots - & \\ (\delta_n^{(\gamma)}/(\alpha_\gamma + t_\gamma)^{n-1}), & t_\gamma > 0 \\ & \\ \beta_2^{(\gamma)}, & t_\gamma \leq 0 \end{array} \right\}, \tag{1.9}$$

$$\delta_j^{(\gamma)} = \alpha_\gamma \beta_j^{(\gamma)} - \beta_{j+1}^{(\gamma)}, j = 2, 3, 4, \ldots, n; \beta_{n+1}^{(\gamma)} = 0, \tag{1.10}$$

$$t_\gamma = (1/2) \left\{ \alpha_\gamma + \beta_1^{(\gamma)} + \sqrt{(\alpha_\gamma - \beta_1^{(\gamma)})^2 + 4\beta_2^{(\gamma)}} \right\} - \alpha_\gamma. \tag{1.11}$$

By taking

$$\gamma = 1$$

in Theorem 1, we get

***Corollary* 1.** *All the zeros of the polynomial*

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n; a_j \neq 0 \textit{ for at least one } j, 2 \leq j \leq n,$$

lie in

$$|z| \leq (1/2) \left\{ \alpha_1 + \beta_1^{(1)} + \sqrt{(\alpha_1 - \beta_1^{(1)})^2 + 4\sigma_1} \right\}. \tag{1.12}$$

Remark . Bound (1.12) is a refinement of bound (1.2), due to Joyal et al. as

$$\begin{aligned}(1/2)\{\alpha_1+\beta_1^{(1)}+\sqrt{(\alpha_1-\beta_1^{(1)})^2+4\sigma_1}\} \quad &\leq \\ (1/2)\{\alpha_1+\beta_1^{(1)}+\sqrt{(\alpha_1-\beta_1^{(1)})^2+4\beta_2^{(1)}}\}, &\text{(by (1.8))}, \\ \leq (1/2)\{1+\beta_1^{(1)}+\sqrt{(1-\beta_1^{(1)})^2+4\beta_2^{(1)}}\}, &\text{(by (1.7))}.\end{aligned}$$

Further, in many cases Corollary 1 gives better bounds than those given by many other known results, as can be seen by the example:

$$\begin{aligned} p(z)=z^5+a_2z^3+a_5; |a_2|=|a_5|=2, \\ \text{Datt \& Govil's bound [4]} > 2, \\ \text{Our earlier bound [5]} > 1.823 \\ \text{Aziz \& Zargar's bound [1]} > 1.9, \\ \text{Boese \& Luther's bound [2]} \approx 2.991, \\ \text{Ẑilović et al.'s bound [8]} = \sqrt{5}, \\ \text{Sun \& Hsieh's bound [7]} = 2, \\ \text{Our bound(1.12)} \approx 1.82289 \end{aligned}$$

By considering all possible values of γ in Theorem 1 we get another refinement, of bound (1.2), as

***Corollary* 2.** *All the zeros of the polynomial*

$$p(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+\ldots+a_n; a_j\neq 0 \textit{ for at least one } j, 2\leq j\leq n,$$

lie in

$$|z|\leq \inf_{\gamma>0}((\gamma/2)\{\alpha_\gamma+\beta_1^{(\gamma)}+\sqrt{(\alpha_\gamma-\beta_1^{(\gamma)})^2+4\sigma_\gamma}\}). \tag{1.13}$$

2 Lemma

For the proof of Theorem 1, we require the following lemma.

Lemma 1. *Let*

$$p(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+\ldots+a_n; a_j\neq 0 \textit{ for at least one } j, 2\leq j\leq n, \tag{2.1}$$

be a polynomial, with

$$\eta_2\geq\eta_3\geq\ldots\geq\eta_n, \tag{2.2}$$

being the ordered non-negative numbers

$$|a_j|, j=2,3,\ldots n, \tag{2.3}$$

and

$$\eta_1 = |a_1|. \tag{2.4}$$

Then all the zeros of the polynomial $p(z)$ *lie in the disc*

$$|z| \leq (1/2)\{\beta + \eta_1 + \sqrt{(\beta - \eta_1)^2 + 4\eta_2}\}, \tag{2.5}$$

where

$$0 \leq \beta = \max_{2 \leq j \leq n-1} (\eta_{j+1}/\eta_j) \leq 1, \tag{2.6}$$

(maximum being taken over all j *such that* $\eta_j \neq 0$*).*

Proof: We have for $|z| > \beta$

$$\begin{aligned}
|p(z)| &\geq |z|^n - \eta_1|z|^{n-1} - |a_2||z|^{n-2} - \ldots - |a_n|, \text{(by (2.4))}, \\
&\geq |z|^n - \eta_1|z|^{n-1} - \eta_2|z|^{n-2} - \\
&\quad \eta_3|z|^{n-3} - \ldots - \eta_n, \text{(by (2.2) and (2.3))}, \\
&\geq |z|^n - \eta_1|z|^{n-1} - \eta_2|z|^{n-2} - \eta_2\beta|z|^{n-3} - \\
&\quad \eta_2\beta^2|z|^{n-4} - \ldots - \eta_2\beta^{n-2}, \text{(by (2.6))}, \\
&= |z|^{n-1}\{|z| - \eta_1 - (\eta_2/|z|)(\sum_{j=0}^{n-2}(\beta/|z|)^j)\}, \\
&\geq |z|^{n-1}\{|z| - \eta_1 - (\eta_2/|z|)(\sum_{j=0}^{\infty}(\beta/|z|)^j)\}, \\
&= |z|^{n-1}\{|z|^2 - (\beta + \eta_1)|z| + \beta\eta_1 - \eta_2\}/(|z| - \beta), \\
&> 0,
\end{aligned}$$

if

$$|z| > (1/2)(\beta + \eta_1 + \sqrt{(\beta - \eta_1)^2 + 4\eta_2}),$$

and Lemma 1 follows. □

3 Proof of the Theorem 1

By Lemma 1, all the zeros of the polynomial $p(\gamma z)$ will lie in

$$|z| \leq (1/2)(\alpha_\gamma + \beta_1^{(\gamma)} + \sqrt{(\alpha_\gamma - \beta_1^{(\gamma)})^2 + 4\beta_2^{(\gamma)}}),$$

and therefore all the zeros of the polynomial $p(z)$ will lie in

$$|z| \leq (\gamma/2)(\alpha_\gamma + \beta_1^{(\gamma)} + \sqrt{(\alpha_\gamma - \beta_1^{(\gamma)})^2 + 4\beta_2^{(\gamma)}}). \tag{3.1}$$

Further, let

$$p(\zeta) = 0. \tag{3.2}$$

Then

$$|\zeta| = \gamma r, \tag{3.3}$$

for certain non-negative real number r, with two possibilities:

$$(i) r > \alpha_\gamma \;\; (ii) r \leq \alpha_\gamma. \tag{3.4}$$

For the possibility (ii), Theorem 1 follows trivially. Accordingly we now think with the help of possibility (i). Then by (3.3), (3.4) and (3.1), we get

$$\gamma\alpha_\gamma < \gamma r \leq (\gamma/2)(\alpha_\gamma + \beta_1^{(\gamma)} + \sqrt{(\alpha_\gamma - \beta_1^{(\gamma)})^2 + 4\beta_2^{(\gamma)}}). \tag{3.5}$$

(Further, if t_γ, (as given by (1.11)), is non-positive, then by (3.5), we will get

$$\gamma\alpha_\gamma < \gamma r \leq \gamma\alpha_\gamma,$$

an absurdity, thereby implying that possibility (i) can not arise with non-positive t_γ). Therefore from now onwards, we will assume that t_γ is positive. Further by (3.5) and (1.11), we get

$$\alpha_\gamma < r \leq \alpha_\gamma + t_\gamma. \tag{3.6}$$

Now by (3.2) and (1.3), we have

$$|\zeta|^n \leq |a_1||\zeta|^{n-1} + |a_2||\zeta|^{n-2} + \ldots + |a_n|,$$

i.e

$$\begin{aligned} r^n &\leq \beta_1^{(\gamma)} r^{n-1} + |b_2^{(\gamma)}| r^{n-2} + |b_3^{(\gamma)}| r^{n-3} + \ldots + |b_n^{(\gamma)}|, \text{(by (3.3), (1.6) \& (1.5))}, \\ &\leq \beta_1^{(\gamma)} r^{n-1} + \beta_2^{(\gamma)} r^{n-2} + \beta_3^{(\gamma)} r^{n-3} + \ldots + \beta_n^{(\gamma)}, \text{(by (1.5) \& (1.4))}, \end{aligned}$$

i.e.

$$\begin{aligned} (r - \beta_1^{(\gamma)}) \leq (\beta_2^{(\gamma)}/r) + (\beta_3^{(\gamma)}/r^2) + \ldots + (\beta_n^{(\gamma)}/r^{n-1}), \\ (= \text{a +ve number, by (1.3)}), \end{aligned} \tag{3.7}$$

i.e.

$$\begin{aligned} (r - \beta_1^{(\gamma)})(r - \alpha_\gamma) &\leq \beta_2^{(\gamma)} - (\delta_2^{(\gamma)}/r) - (\delta_3^{(\gamma)}/r^2) - \ldots - \\ &\quad (\delta_n^{(\gamma)}/r^{n-1}), \text{(by (1.10))} && (3.8) \\ &= \text{a +ve number, (by (1.8), (3.4) \& (3.7))}, && (3.9) \\ &\leq \beta_2^{(\gamma)} - (\delta_2^{(\gamma)}/(\alpha_\gamma + t_\gamma)) - (\delta_3^{(\gamma)}/(\alpha_\gamma + t_\gamma)^2) - \\ &\qquad \ldots - (\delta_n^{(\gamma)}/(\alpha_\gamma + t_\gamma)^{n-1}), \text{(by (3.6))}, \\ &= \sigma_\gamma, \text{(by (1.9))} && (3.10) \\ &= \text{a +ve number, (by (3.9))} && (3.11) \end{aligned}$$

i.e.

$$r^2 - r(\alpha_\gamma + \beta_1^{(\gamma)}) + \alpha_\gamma\beta_1^{(\gamma)} - \sigma_\gamma \leq 0,$$

i.e.

$$(r-\zeta_1)(r-\zeta_2)\leq 0, \tag{3.12}$$

where

$$\zeta_1 = (1/2)(\alpha_\gamma+\beta_1^{(\gamma)}+\sqrt{(\alpha_\gamma-\beta_1^{(\gamma)})^2+4\sigma_\gamma}), \tag{3.13}$$

$$\zeta_2 = (1/2)(\alpha_\gamma+\beta_1^{(\gamma)}-\sqrt{(\alpha_\gamma-\beta_1^{(\gamma)})^2+4\sigma_\gamma}). \tag{3.14}$$

Hence by (3.10), (3.11), (3.13), (3.14) and (3.12), we have

$$r\leq (1/2)(\alpha_\gamma+\beta_1^{(\gamma)}+\sqrt{(\alpha_\gamma-\beta_1^{(\gamma)})^2+4\sigma_\gamma}),$$

and accordingly

$$|\zeta|\leq (\gamma/2)(\alpha_\gamma+\beta_1^{(\gamma)}+\sqrt{(\alpha_\gamma-\beta_1^{(\gamma)})^2+4\sigma_\gamma}), \text{(by (3.3))},$$

thereby proving Theorem 1, for the possibility (i) under consideration. This completes the proof of Theorem 1.

Remark . One observes that bound in Theorem 1 can be repeatedly improved e.g. by using (1.7), instead of Lemma 1 and then repeating the proof of Theorem 1, thereby leading to the conclusion that all the zeros of $p(z)$ will lie in

$$|z|\leq (\gamma/2)(\alpha_\gamma+\beta_1^{(\gamma)}+\sqrt{(\alpha_\gamma-\beta_1^{(\gamma)})^2+4\sigma'_\gamma}),$$

where

$$\sigma'_\gamma=\begin{cases}\beta_2^{(\gamma)}-(\delta_2^{(\gamma)}/(\alpha_\gamma+t'_\gamma))-(\delta_3^{(\gamma)}/(\alpha_\gamma+t'_\gamma)^2)-\ldots- & \\ (\delta_n^{(\gamma)}/(\alpha_\gamma+t'_\gamma)^{n-1}), & t'_\gamma>0,\\ \sigma_\gamma, & t'_\gamma\leq 0,\end{cases}$$

$$t'_\gamma=(1/2)\{\alpha_\gamma+\beta_1^{(\gamma)}+\sqrt{(\alpha_\gamma-\beta_1^{(\gamma)})^2+4\sigma_\gamma}\}-\alpha_\gamma,$$

with

$$\sigma'_\gamma\leq\sigma_\gamma.$$

References

[1] Aziz, A. and Zargar, B. A., On the zeros of polynomials, *Proc. Ind. Acad. Sci. (Math. Sci.)* 106 (1996), 127–132.

[2] Boese, F. G. and Luther, W. J., A note on classical bound for the moduli of all zeros of polynomial, *IEEE Trans. Automat. Contr.* 34 (1989), 998–1001.

[3] Cauchy, A. L., Exercises de mathematique, *Oeuvres* 9 (1829), 122.

[4] Datt, B. and Govil, N. K., On the location of zeros of polynomial, *J. Approx. Theory* 24 (1978), 78–82.

[5] Jain, V. K., On Cauchy's bound for zeros of a polynomial, *Turk. J. Math.* 30 (2006), 95–100.

[6] Joyal, A., Labelle, G. and Rahman, Q. I., On the location of zeros of polynomials, *Canad. Math. Bull.* 10 (1967), 53–63.

[7] Sun, Y. J. and Hsieh, J. G., A note on circular bound of polynomial zeros, *IEEE Trans. Circuits Syst. I* 43 (1996), 476–478.

[8] Ẑilović,M. S., Roytman, L. M., Combettes, P. L. and Swamy, M. N. S., A bound for zeros of polynomials, *ibid* 39 (1992), 476–478.

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