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Portfolio Selection with Transaction Costs

by
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Abstract

This paper is concerned with the single-period portfolio that consists of holdings in n risky assets. The goal is to choose the optimal portfolio to maximize the expected value of the end of period wealth in the presence of transaction costs, while satisfying a set of constraints on the portfolio. The case of a portfolio optimization problem with fuzzy transaction costs is also considered.

Key Words: Portfolio optimization, transaction costs, fuzzy portfolio selection model.
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1 Introduction

In this paper we consider an investment portfolio that consists of holdings in n assets. This portfolio is to be adjusted by performing a number of transactions, after which the portfolio will be held over a fixed time period. The investor's goal is to maximize the expected wealth at the end of period, while taking transaction costs into account and satisfying a set of constraints on the portfolio which typically include limits on exposure to risk and bounds on the amount held in each asset. The problem is also considered in a fuzzy context.

Recent years have seen a growing interest in portfolio optimization problem. The paper of Best and Hlouskova [1] deals with the portfolio selection problem of risky assets with a diagonal covariance matrix, upper bounds on all assets and transactions costs. Blog et al. [2] consider the specific optimal selection problem of small portfolios. Kellerer et al. [11] introduce mixed-integer linear programming models dealing with fixed costs and minimum lots and propose heuristic procedures based on the construction and optimal solution of mixed integer subproblems. Konno and Wijayanayake [12] propose a branch and bound algorithm for calculating a globally optimal solution of a portfolio construction/rebalancing problem under concave transaction costs and minimal transaction unit constraints. Schattman [19] develops an iterative heuristic for finding a suboptimal solution for the portfolio problem. Meyer [14] proved the convergence of a class of algorithms that includes the heuristic in this paper. If the portfolio optimization problem is nonlinear, the algorithm presented in Fulga [7] that combines penalty concepts and sequential quadratic programming techniques can be used. The interest in exact penalty methods is due to their ability to handle degenerate problems and inconsistent constraint linearizations, see also Chen and Goldfarb [4], Coope and Price [5], Gould et al. [9]. In recent years, a large amount of work has been devoted to the problem of solving nonlinear programming problems with hypothesis of generalized convexity on the functions involved, see Preda [17], Preda et al. [18], Fulga and Preda [8].

The rest of the paper goes as follows. In the next section we present a single-period portfolio selection problem. Transaction cost functions and portfolio constraints are also described. In Section 3, fixed costs are included and it is shown how to obtain a feasible suboptimal portfolio. Section 4 deals with the portfolio optimization problem with fuzzy transaction costs.

2 The portfolio selection problem

We are concerned with the single-period portfolio that consists of holdings in n risky assets. The portfolio is adjusted at the beginning of the time-period. The goal is to choose the optimal portfolio to maximize the expected value of the end of period wealth in the presence of transaction costs, while satisfying a set of constraints on the portfolio.

2.1 The model

The current holdings in each asset are $w = (w_1, ..., w_n)^T$. The total current wealth is then $\sum_{i=1}^{n} w_i$. The amount of money transacted in each asset i, $i = \overline{1, n}$, is denoted by x_i, with $x_i > 0$ for buying, $x_i < 0$ for selling and $x = (x_1, ..., x_n)^T \in R^n$ is the vector of transactions. After transactions, the adjusted portfolio is $w + x$. Representing the sum of all transaction costs associated with x by $f(x)$, the budget, or self-financing constraint is

$$\sum_{i=1}^{n} x_i + f(x) = 0. \tag{1}$$

The adjusted portfolio $w + x$ is then held for a fixed period of time. At the end of that period, the return on asset i is the random variable $\widetilde{r}_i$, $i = \overline{1, n}$. All random variables are on a given probability space. We assume knowledge of the first and second moments of the joint distribution of $\widetilde{r} = (\widetilde{r}_1, ..., \widetilde{r}_n)$, $E(\widetilde{r}) = r$, $r = (r_1, ..., r_n)^T \in R^n$ and $E\left((\widetilde{r} - r)(\widetilde{r} - r)^T\right) = C$. A riskless asset can be included, in which case the corresponding r_i is equal to its (certain) return, and the ith row and column of C are zero.

The end of period wealth is a random variable, $\widetilde{w} = \widetilde{r}^T(w + x)$, with expected value and variance given by $E(\widetilde{w}) = r^T(w + x)$, $E\left((\widetilde{w} - E(\widetilde{w}))^2\right) = (w+x)^T C(w+x)$. The budget constraint 1 can also be written as an inequality,

$$\sum_{i=1}^{n} x_i + f(x) \leq 0. \tag{2}$$

With some obvious assumptions, solving an expected wealth maximization problem with either form of the budget constraint yields the same result. The inequality form is more appropriate for use with numerical optimization methods. (For example, if f is convex, the inequality constraint 2 defines a convex set, while the equality constraint 1 does not.)

We summarize the portfolio selection problem as

$$\begin{cases} \max r^T(w + x) \\ s.t. \sum_{i=1}^{n} x_i + f(x) \leq 0 \\ w + x \in X, \end{cases} \tag{3}$$

where $r = (r_1, ..., r_n)^T \in R^n$ is the vector of expected returns on each asset, $w = (w_1, ..., w_n)^T \in R^n$ is the vector of current holdings in each asset, $x = (x_1, ..., x_n)^T \in R^n$ is the vector of amounts transacted in each asset, $f : R^n \to R$ is the transaction cost function, $X \subset R^n$ is the set of feasible portfolios.

A related problem is that of minimizing the total transaction costs subject to portfolio constraints. Among all possible transactions that result in portfolios achieving a given expected return and meeting the other portfolio constraints, we would like to perform those transactions that incur the smallest total cost. This problem is written as

$$\begin{cases} \min f(x) \\ r^T(w + x) \geq \underline{r} \\ w + x \in X, \end{cases} \tag{4}$$

where $\underline{r}$ is the desired lower bound on the expected return. In this paper we focus mostly on problem 3, but we will also consider problem 4 in Section 3.

Next, we describe a variety of transaction cost functions f and portfolio constraint sets X.

2.2 Transaction costs

Transaction costs can be used to model a number of costs, such as brokerage fees, bid-ask spreads, taxes, or even fund loads. In this paper, we assume the transaction costs to be separable, i.e., the sum of the transaction costs associated with each trade $f(x) = \sum_{i=1}^{n} f_i(x_i)$, where f_i is the transaction cost function for asset i, $i = \overline{1,n}$.

The simplest model for transaction costs is that there are none, i.e., $f(x) = 0$. In this case the original portfolio is irrelevant, except for its total value. We can make whatever transactions are necessary to arrive at the optimal portfolio.

A better model of realistic transactions costs is a linear one, with the costs for each transaction proportional to the amount traded:

$$f_i(x_i) = \begin{cases} a(x_i)\,|x_i|, & x_i \neq 0 \\ 0, & x_i = 0 \end{cases}, \quad i = \overline{1,n}. \tag{5}$$

where $a(x_i) = \begin{cases} a_i^{buy}, & x_i > 0 \\ -a_i^{sell}, & x_i < 0 \end{cases}$. Here $a_i^{buy} > 0$ and $a_i^{sell} > 0$ are the cost rates associated with buying and selling asset i, $i = \overline{1,n}$. Linear transaction costs can be used, for example, to model the gap between bid and ask prices. Since the linear transaction cost functions f_i are convex, the budget constraint 2 can be handled by convex optimization. Specifically, linear costs can be handled by introducing new variables $u = (u_1, ..., u_n)^T, v = (v_1, ..., v_n)^T \in R^n$, with $u_i \geq 0, v_i \geq 0$ expressing the total transaction as $x_i = u_i - v_i, \forall\, i = \overline{1,n}$.

The transaction cost function is then represented as $f_i(x_i) = a_i^{buy}u_i + a_i^{sell}v_i, \forall$ $i = \overline{1,n}$.

Any piecewise linear convex transaction cost function can be handled in a similar way.

In practice, transaction costs are not convex functions of the amount traded. Indeed, the costs for either buying or selling are likely to be concave. For example, a fixed charge for any nonzero trade is common, and there may be one or more breakpoints above which the transaction costs per share decrease. We will consider a simple model that includes fixed plus linear costs, but our method is readily extended to handle more complex transaction cost functions.In this case, the transaction cost function is given by

$$f_i(x_i) = \begin{cases} a(x_i)\,|x_i| + b(x_i), & x_i \neq 0 \\ 0, & x_i = 0 \end{cases}, \quad i = \overline{1,n}.$$

where $b(x_i) = \begin{cases} b_i^{buy}, & x_i > 0 \\ b_i^{sell}, & x_i < 0 \end{cases}$ and $b_i^{buy} > 0$ and $b_i^{sell} > 0$ are the fixed costs associated with buying and selling asset i, $i = \overline{1,n}$.

Evidently the function f_i is not convex, unless the fixed costs are zero. Therefore, the budget constraint 2 cannot be handled by convex optimization.

Constraints on portfolio diversification can be expressed in terms of linear inequalities, and therefore are readily handled by convex optimization. Individual diversification constraints limit the amount invested in each asset i to a maximum of s_i,

$$w_i + x_i \leq s_i, \; i = \overline{1,n}.$$

Alternatively, we can limit the fraction $0 \leq \alpha_i < 1$ of the total (post transaction) wealth held in each asset:

$$w_i + x_i \leq \alpha_i \sum_{i=1}^{n} (w_i + x_i), \ i = \overline{1, n}.$$

These are linear, and therefore convex, inequality constraints on x. The reversed constraints (such as requiring a minimum position in an asset) are also convex.

More sophisticated diversification constraints limit the amount of the total wealth that can be concentrated in any small group of assets. Suppose, for example, that we require that no more than a fraction of the total wealth be invested in fewer than k assets. Letting $(w+x)^{(j)}$ denote the jth largest component of the vector $w + x$, this constraint can be expressed as

$$\sum_{j=1}^{k} (w+x)^{(j)} \leq \alpha \sum_{i=1}^{n} (w_i + x_i) \tag{6}$$

where α is a given positive factor. To see that the constraint 6 is convex, we can express it as a set of C_n^k linear inequalities, one for each possible combination of k assets chosen from the n assets. This representation is clearly impractical, however, as this number of linear inequalities can be extremely large. The diversification constraint 6 can be far more efficiently represented by $2n + 1$ linear inequalities,

$$\begin{cases} kp + \sum_{i=1}^{n} q_i \leq \alpha \sum_{i=1}^{n} (w_i + x_i) \\ w_i + x_i \leq p + q_i, \ i = \overline{1, n} \\ q_i \geq 0, \ i = \overline{1, n}, \end{cases} \tag{7}$$

where $p \in R$ and $q = (q_1, ..., q_n)^T \in R^n$ are new variables.

2.3 Constraints

In practice, often shortselling constraints are imposed. This type of constraints also leads to linear inequalities. Individual bounds c_i on the maximum amount of shortselling allowed on asset i are

$$w_i + x_i \geq -c_i, \ i = \overline{1, n}.$$

If it is the case of a riskless asset, c_i is a credit line. If shortselling is not permitted, the $c_i = 0$, $i = \overline{1, n}$. If we denote by B the bound on total shortselling, the costraint is

$$\sum_{i=1}^{n} (w_i + x_i)_- \leq B,$$

where $(w_i + x_i)_- = \max\{-(w_i + x_i); 0\}$. This can be rewritten as a set of linear constraints by introducing an auxiliary variable $z = (z_1, ..., z_n)^T \in R^n$,

$$\begin{cases} -(w_i + x_i) \le z_i, \ i = \overline{1,n} \\ z_i \ge 0, \ i = \overline{1,n} \\ \sum\limits_{i=1}^{n} z_i \le B. \end{cases}$$

Another type of constraint that we mention for its practical interest is

$$\sum_{i=1}^{n} (w_i + x_i)_- \le \alpha \sum_{i=1}^{n} (w_i + x_i)_+$$

which limits the total of short positions to a fraction of the total of long positions.

There are also constraints imposed on the variance. The standard deviation of the end of period wealth W is constrained to be less than a chosen value $\overline{\sigma}$ by the (convex) quadratic inequality

$$(w + x)^T C(w + x) \le \overline{\sigma}^2.$$

If any number of convex transaction costs and convex constraints are combined, the resulting problem is convex. Linear transaction costs, as well as all the portfolio constraints described above, are convex programs. Such problems can be globally solved with great efficiency, even for problems with a large number of assets and constraints.

3 Portfolio optimization problems with fixed transaction costs

We assume from now on equal costs for buying and selling, the extension for nonsymmetric costs being straightforward. The transaction cost function is then $f(x) = \sum\limits_{i=1}^{n} f_i(x_i)$, with

$$f_i(x_i) = \begin{cases} a_i |x_i| + b_i, \ x_i \ne 0 \\ 0, \ x_i = 0 \end{cases}, \ i = \overline{1,n}.$$

In the general case, costs of this form lead to a hard combinatorial problem.

The simplest way to obtain an approximate solution is to ignore the fixed costs, and solve with $f_i(x_i) = a_i |x_i|$. If the b_i are very small, this may lead to an acceptable approximation. In general, however, it will generate inefficient solutions with too many transactions. Note that if this approach is taken and the solution is computed disregarding the fixed costs, some margin must be added to the budget constraint to allow for the payment of the fixed costs.

On the other hand, by considering the fixed costs, we discourage trading small amounts of a large number of assets. Thus, we obtain a sparse vector of trades; i.e., one that has many zero entries. This means most of the trading will be concentrated in a few assets, which is a desirable property.

We will next propose a heuristic, which can be used to find approximate solutions (and therefore lower bounds).

We assume that lower and upper bounds for x_i are known i.e., there exist m_i^l and m_i^u such that $m_i^l \leq x_i \leq m_i^u$. We denote by f_i^c the convex envelope of f_i, which is the largest convex function which is lower or equal to f_i in the interval $\left[m_i^l, m_i^u\right]$. For $m_i^l \neq 0$ and $m_i^u \neq 0$, the function f_i^c is given by

$$f_i^c\left(x_i\right) = \left\{ \begin{array}{c} \left(\frac{b_i}{m(x_i)} + a_i\right)\left|x_i\right|, \ x_i \neq 0 \\ 0, \ x_i = 0 \end{array} \right. , \ i = \overline{1, n}.$$

where $m\left(x_i\right) = \left\{ \begin{array}{c} m_i^u, \ x_i > 0 \\ m_i^l, \ x_i < 0 \end{array} \right.$. Using f_i^c for f_i relaxes the budget constraint, in the sense that it enlarges the search set. Consider the portfolio selection problem 3, with f_i^c replaced for f_i,

$$\left\{ \begin{array}{c} \max r^T(w + x) \\ s.t. \displaystyle\sum_{i=1}^{n} x_i + f^c\left(x\right) \leq 0 \\ w + x \in X, \end{array} \right. \tag{8}$$

where $f^c\left(x\right) = \sum\limits_{i=1}^{n} f_i^c\left(x_i\right)$. This corresponds to optimizing the same objective, the expected end of period wealth, subject to the same portfolio constraints, but with a looser budget constraint. Therefore the optimal value of 8 is an upper bound on the optimal value of the unmodified problem 3. Since the problem 8 is convex, we can compute its optimal solution, and hence the upper bound on the optimal value of the original problem 3, very efficiently.

Note that in most cases the optimal transaction vector for the relaxed problem 8 will not be feasible for the original problem 3. The unmodified budget constraint will not be satisfied by the solution of the modified program, except in the very special case when each transaction amount x_i is either m_i^l, m_i^u or 0.

This relaxation can also be used in problem 4, where the goal is to minimize transaction costs. This results in the relaxed problem

$$\left\{ \begin{array}{c} \min f^c\left(x\right) \\ r^T(w + x) \geq \underline{r} \\ w + x \in X. \end{array} \right. \tag{9}$$

Here, compared to the original problem, the relaxed problem has the same feasible set, but a different objective function.

Following the approach in Schattman [19], we describe a heuristic for finding a feasible suboptimal portfolio. The iterative procedure uses a modified transaction cost function ${f_i}^k$ which, like the relaxed cost function, is convex. An iterated reweighting is used. Since each of these modified functions is convex, each iteration consists in solving a convex program. The feasibility of the portfolio is obtained by ensuring that the modified transaction cost function ${f_i}^k$ agrees with the true f_i at the solution transactions x_i^*, $i = \overline{1, n}$.

l orithm Step 1. Initialization

Initialize $k = 0$; α and δ.

Step 2. Solvin the problem (8)
Solve the convex relaxed problem (8) and let $x^0 = \left(x_1^0, ..., x_n^0\right)$ be the solution to this problem.
Step 3. Solvin the modified portfolio selection problem (MPP)
Set $k = k + 1$..
Solve the modified (convex) portfolio selection problem

$$(MPP) \begin{cases} \max r^T (w + x) \\ s.t. \sum_{i=1}^{n} x_i + f_i{}^k (x) \leq 0 \\ w + x \in X, \end{cases}$$

where $f^{\ k}(x) = \sum_{i=1}^{n} f_i{}^k (x_i)$ and

$$f_i{}^k (x_i) = \left(\frac{b_i}{\left|x_i^{k-1}\right| + \alpha} + a_i \right) |x_i| \, , i = \overline{1, n}.$$

The optimal solution of this problem is denoted by $x^k = \left(x_1^k, ..., x_n^k\right)$.
Step 4. Checkin stoppin condition
If $\left\| x^k - x^{k-1} \right\|_\infty < \delta$, set $x^* = x^k$. Stop.
Otherwise, the algorithm proceeds to Step 3.
Remark We note that Meyer [14] established the convergence of a large class of algorithms that includes the proposed algorithm.

4 Portfolio optimization problem with fuzzy transaction costs

In the classical problems of operations research generally, and in the optimization models in particular, the coefficients of the problems are assumed to be exactly known. However in practice this assumption is seldom satisfied by great majority of real-life problems. The modeling of input data inaccuracy can be made by means of the fuzzy set theory. Generally, two types of problems implying fuzzy uncertainty are studied. Fuzzy approaches to solve deterministic problems could be developed and also fuzzy models, implying fuzzy goals and fuzzy coefficients, could be defined and solved.

In [22] two fuzzy portfolio selection models are presented. Models objective are to minimize the downside risk constrained by a given expected return, the rates of returns on securities are approximated as LR-fuzzy numbers of the same shape, and the expected return and risk are evaluated by interval-valued means. The portfolio selection problem is formulated as a linear program when the returns on the assets are of trapezoidal form.

In [10] some fuzzy linear programming methods and techniques from a practical point of view are reviewed. Using a numerical example, some models of fuzzy linear programming are described and advantages and disadvantages of fuzzy mathematical programming approaches are exemplified in the setting of an optimal portfolio selection problem. Some newly developed ideas and techniques in fuzzy mathematical programming are also briefly took into consideration.

In [13] a multistage stochastic soft constraints fuzzy program with recourse in order to capture both uncertainty and imprecision as well as to solve a portfolio management problem is developed.

In this section, the case of a portfolio optimization problem with fuzzy transaction costs is considered.

4.1 Fuzzy model

The model of a portfolio optimization problem with fuzzy transaction costs is formally similar to (8) and it is presented below.

$$\left\{ \begin{array}{c} \max r^T (w + x) \\ s.t. \sum_{i=1}^{n} x_i + \overline{f^c}(x) \leq 0 \\ w + x \in X, \end{array} \right. \tag{10}$$

where function $\overline{f_i^c}$ is defined using fuzzy coefficients $\overline{a_i}$ and $\overline{b_i}$ respectively for each $i = \overline{1,n}$. Moreover, the definition

$$\overline{f_i^c}(x_i) = \left\{ \begin{array}{c} \left(\frac{\overline{b_i}}{m(x_i)} + \overline{a_i} \right) |x_i|, \; x_i \neq 0 \\ 0, \; x_i = 0 \end{array} \right., \; i = \overline{1,n}$$

which describes fuzzy transaction costs has to be interpreted according to extension's principle for aggregating fuzzy quantities. In the following triangular fuzzy numbers will be used in order to describe fuzzy quantities.

4.2 Fuzzy arithmetic

Definitions for triangular fuzzy numbers and the way of applying extension's principle to add two triangular fuzzy numbers are inserted below.

Definition 4.1. *[6] A triangular fuzzy number $\overline{Y}$ is a triplet $\left(y^1, y^2, y^3\right) \in R^3$. The membership function of $\overline{Y}$ is defined in connection with the real numbers y^1, y^2, y^3 as follows:*

$$\overline{Y}(x) = \left\{ \begin{array}{ll} 0, & x \in (-\infty, y^1) \\ \dfrac{x - y^1}{y^2 - y^1}, & x \in [y^1, y^2] \\ \dfrac{x - y^3}{y^2 - y^3}, & x \in (y^2, y^3] \\ 0, & x \in (y^3, \infty) \end{array} \right.$$

$\overline{Y}(x)$ represents a number in $[0,1]$, which is the membership function of $\overline{Y}$ evaluated in x. It can be easily verified that graph $y = \overline{Y}(x)$ of $\overline{Y}$ is a triangle with base on $\left[y^1, y^3\right]$ and vertex at $x = y^2$ for $y^1 < y^2 < y^3$.

The extension principle was formulated by Zadeh [23] in order to extend the known models implying fuzzy elements to the case of fuzzy entities. pplying this principle the following definition of the addition of triangular fuzzy numbers results:

Definition 4.2. *Being given two triangular fuzzy numbers* $\overline{A} = \left(a^1, a^2, a^3\right), \overline{B} = \left(b^1, b^2, b^3\right)$, $a^1, a^2, a^3, b^1, b^2, b^3 \in R$, *we have:*

$$\overline{A} + \overline{B} = \left(a^1 + b^1, a^2 + b^2, a^3 + b^3\right).$$

Multiplying a triangular fuzzy number by a positive real number consists on multiplying each parameter of the fuzzy number by the real number. The presence of these fuzzy numbers in problem's constraints makes Problem (10) to be an optimization problem with deterministic objective function subject to fuzzy inequalities. Fuzzy sets theory has to be used to deal with fuzzy constraints in optimization problems [21].

The inequality between two fuzzy numbers $\overline{M}, \overline{N}$ having their membership functions $\overline{M}(x)$, and $\overline{N}(x)$ respectively, is defined by Kerre and presented by Buckley and Feuring [3]. In the following, we apply this manner of defining an inequality between fuzzy numbers to the triangular fuzzy numbers $\overline{M}$ şi $\overline{0} = (0, 0, 0)$. The inequality $(m_1, m_2, m_3) \leq (0, 0, 0)$ is equivalent (see [16]) to the following system of disjunctive deterministic constraints:

$$m_3 \leq 0$$

or

$$(m_1 \leq 0 \leq m_2) \cap (m_1 (m_1 + m_2 + m_3) \geq m_2 m_3) \tag{11}$$

or

$$(m_2 \leq 0 \leq m_3) \cap (m_3 (m_1 + m_2 + m_3) \leq m_1 m_2).$$

4.3 Solvin method

In Section 2.2 it was assumed that transaction costs are separable. It means that the transaction cost function is the sum of the transaction cost functions associated with each trade. Consequently, transaction cost function $\overline{f}(x)$ is computed as $\sum\limits_{i=1}^{n} \overline{f_i^c}(x_i)$.

In order to compute $\overline{f_i^c}$ we can use fuzzy transaction costs $\overline{a_i} = \left(a_i^1, a_i^2, a_i^3\right)$, $\overline{b_i} = \left(b_i^1, b_i^2, b_i^3\right)$ defined by real parameters $a_i^1, a_i^2, a_i^3, b_i^1, b_i^2, b_i^3 \in R$.

Considering $\overline{f}(x) = (f_1(x), f_2(x), f_3(x))$ we have:

$$\begin{aligned} f_1(x) &= \sum_{i=1}^{n} \left(\frac{b_i^1}{m(x_i)} + a_i^1\right) |x_i|, \ f_2(x) = \sum_{i=1}^{n} \left(\frac{b_i^2}{m(x_i)} + a_i^2\right) |x_i|, \\ f_3(x) &= \sum_{i=1}^{n} \left(\frac{b_i^3}{m(x_i)} + a_i^3\right) |x_i|. \end{aligned}$$

Applying Kerre's method to transform fuzzy inequalities in disjunctive deterministic constraints the following equivalent Problem (12) is obtained.

$$\begin{cases} & \max r^T (w+x) \\ s.t. & R_1 \cup (R_2 \cap R_3) \cup (R_4 \cap R_5) \\ & w+x \in X, \end{cases} \tag{12}$$

where

$$R_1 : \sum_{i=1}^{n} x_i + f_3(x) \leq 0$$

$$R_2 : f_1(x) \leq -\sum_{i=1}^{n} x_i \leq f_2(x)$$

$$\begin{aligned} R_3 : & (\sum_{i=1}^{n} x_i + f_1(x))(3\sum_{i=1}^{n} x_i + f_1(x) + f_2(x) + f_3(x)) \geq \\ & \geq (\sum_{i=1}^{n} x_i + f_2(x))(\sum_{i=1}^{n} x_i + f_3(x)) \end{aligned}$$

$$R_4 : f_2(x) \leq -\sum_{i=1}^{n} x_i \leq f_3(x)$$

$$\begin{aligned} R_5 : & (\sum_{i=1}^{n} x_i + f_3(x))(3\sum_{i=1}^{n} x_i + f_1(x) + f_2(x) + f_3(x)) \leq \\ & \leq (\sum_{i=1}^{n} x_i + f_1(x))(\sum_{i=1}^{n} x_i + f_2(x)) \end{aligned}$$

According to the method described by Patkar and Stancu-Minasian in [15], we shall consider the indicator variables δ^1, δ^2, δ^3 in order to eliminate the disjunctivity and to obtain (13)-(21) which is a system of conjunctive constraints.

$$\sum_{i=1}^{n} x_i + f_3(x) \leq \left(1-\delta^1\right) M, \tag{13}$$

$$f_1(x) + \sum_{i=1}^{n} x_i \leq \left(1-\delta^2\right) M, \tag{14}$$

$$f_2(x) + \sum_{i=1}^{n} x_i \geq \left(1-\delta^2\right) M, \tag{15}$$

$$\begin{aligned} & (\sum_{i=1}^{n} x_i + f_2(x))(\sum_{i=1}^{n} x_i + f_3(x)) - \\ & -(\sum_{i=1}^{n} x_i + f_1(x))(3\sum_{i=1}^{n} x_i + f_1(x) + f_2(x) + f_3(x)) \leq \left(1-\delta^2\right) M, \end{aligned} \tag{16}$$

$$f_2(x) + \sum_{i=1}^{n} x_i \leq \left(1 - \delta^3\right) M, \tag{17}$$

$$f_3(x) + \sum_{i=1}^{n} x_i \geq \left(1 - \delta^3\right) M, \tag{18}$$

$$\begin{aligned}(\sum_{i=1}^{n} x_i + f_3(x))(3\sum_{i=1}^{n} x_i + f_1(x) + f_2(x) + f_3(x)) - \\ -(\sum_{i=1}^{n} x_i + f_1(x))(\sum_{i=1}^{n} x_i + f_2(x)) \leq \left(1 - \delta^3\right) M,\end{aligned} \tag{19}$$

$$\delta^1 + \delta^2 + \delta^3 \geq 1, \delta^1, \delta^2, \delta^3 \in \{0,1\}, \tag{20}$$

$$w + x \in X. \tag{21}$$

M represents an upper bounds for all expressions which appear in constraints.

Computing max $\left(r^T(w+x)\right)$ subject to (13)-(21) will allow us to obtain the solution $x^* = (x_1^*, ..., x_n^*)$, δ^1, δ^2, δ^3. Components of x^* represent the solution of Problem (10).

An algorithm to solving Problem (10) will consist of the following steps:

- **Step 1.** Define fuzzy transaction costs by choosing proper values for fuzzy coefficients $(\overline{a_i})_{i=\overline{1,n}}$ and $\left(\overline{b_i}\right)_{i=\overline{1,n}}$.
- **Step 2.** Apply Kerr's method to transform fuzzy constraints into deterministic disjunctive system of constraints (12).
- **Step 3.** Remove the disjunctivity of constraints system by using boolean variables δ^1, δ^2, δ^3 (as in (13)-(21)).
- **Step 4.** Maximize $r^T(w+x)$ subject to (13)-(21) using classical methods for linear programming problems with quadratic constraints (see [20]).

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