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On the Solvability of Nonlinear Singular Integral Equations with Cauchy Kernel on an Interval

by

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Abstract

This paper concerns the investigation of a class of nonlinear singular integral equations with Cauchy kernel under certain conditions. The technique being based on the application of Schauder's fixed point theorem.

Key Words: Nonlinear singular integral equations, Schauder's fixed point theorem, Sobolev space.
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1 Introduction

The theory of nonlinear singular integral equations has been developed during the last years. Many engineering problems of applied mechanics and applied mathematics character, are reduced to the solution of such types of nonlinear equations. The existence of the solutions of nonlinear singular integral equations with Hilbert and Cauchy kernel and its related Riemann- Hilbert problems have been developed by many authors (see [3,5,10,11,14,15,16]) and others . Other applications may be found in aerodynamics, fluid mechanics and theory of elasticity are found in [8]. The Schauder's fixed point theorem is one of the basic tools to investigate the existence results of many classes of nonlinear singular integral equations (see [1,2,4,6,15,16]).

The goal of this paper is to study of nonlinear singular integral equations of the following form:

$$F(x,u(x)) = [Su](x) + c,\ -1 \leq x \leq 1 \tag{1}$$

where

$$[Su](x) = \frac{1}{\pi}\int_{-1}^{1} \frac{u(y)}{y-x}dy$$

c is a real number satisfying equation (1) and the condition

$$u(-1) = u(1) = 0, \tag{2}$$

the function $F(., u(.))$has a continuous partial derivative

$$\begin{aligned} F_u : [-1,1] \times R \quad &\longrightarrow \quad R,\ -m \le F_u(x, u(x)) \le n, \\ mn < 1, & \end{aligned} \tag{3}$$

and a partial derivative $F_x(., u(.))$ which is continuous with respect to $u \in R$ for almost all x and measurable with respect to $x \in [-1, 1]$ for all $u \in R$.

Furthermore, satisfies the inequality:

$$|F_x(x, u)| \le l(1 - x^2)^{-\delta}, \tag{4}$$

where $l \ge 0$ is a constant and

$$\delta < \frac{3}{4} + (2\pi)^{-1}\ (arctan\ m + arctan\ n). \tag{5}$$

From (3) we have

$$\delta < \frac{3}{4} + (2\pi)^{-1}\ (arctanm + arctann) = \frac{3}{4} + (2\pi)^{-1}\ arctan\frac{m+n}{1-mn} > \frac{3}{4}$$

The related nonlinear Riemann-Hilbert problem of the class (1),(2) has been studied in [15].

In [6,16], existence of solution of equation (1),(2) has been studied for some cases of constantδ: $0 < \delta < \frac{1}{2}$ and $\delta < \frac{3}{4} - (2\pi)^{-1}(arctanm + arctann)$.

In the present paper we show that the existence of solution of equation (1),(2) can be proved when the restriction on the constant δ in (4) can be given by (5).

2 Preliminaries

In this section we introduce some materials for our results.

Definition 2.1 [7,9]

a) Let $L_p[-1, 1], p > 1$ be the Banach space of all measurable functions u, for which $|u|^p$ is summable.

b) We denote by $C_0[-1, 1]$ the Banach space of all continuous functions u on $[-1, 1]$ satisfying the boundary condition $u(\pm 1) = 0$.

c) We denote by $W_0^1[-1, 1]$ the set of all functions u which belong to $C_0[-1, 1]$ and its derivative $v(x) = \acute{u}(x)$ belong to $L_p[-1, 1]$ such that

$$u(x) = \int_{-1}^{x} v(y)dy,\ u(-1) = 0.$$

Lemma 2.1. (De le Vallee-Poussin, [13] , Chapt. VI)
Let $\Psi : [0,\infty] \longrightarrow [0,\infty]$ be monotone increasing , $\lim_{u\longrightarrow\infty} \Psi(u) = \infty$ and V be a set of measurable functions on $[-1,1]$. If there exist a constant M such that

$$\int_{-1}^{1} |\phi(x)|\psi(|\phi(x)|)dx \leq M,\ \phi \in V.$$

Then $V \subset L_1$, and the integral of all $\phi \in V$ are uniformly absolutely continuous.

Lemma 2.2. (Vitali , [13] , Chapt. VI)
Let $\{\phi_n\} \subset L_1$ and $\phi_n \longrightarrow \phi$ in measure . If the integrals of $\{\phi_n\}$ are uniformly absolutely continuous, then $\phi \in L_1$ and

$$\lim_{n\longrightarrow\infty} \int_{-1}^{1} \phi_n(y)dy = \int_{-1}^{1} \phi(y)dy.$$

Lemma 2.3[9].
sequence $\{\phi_n\} \subset L_p$ converges weakly to $\phi_n \in L_p$ if and only if the sequence $\{||\phi_n||_p\}$is bounded and

$$\lim_{n\longrightarrow\infty} \int_{-1}^{x} \phi(y)dy = \int_{-1}^{x} \phi(y)dy,\ -1 \leq x \leq 1.$$

Lemma 2.4 [7].
Let $p > 1,\ \{\phi_n\} \subset L_p$, and $\phi \in L_p$. Then converges to ϕ in L_p if and only if ϕ_n converges weakly to ϕ in L_p and $||\phi_n||_p \longrightarrow ||\phi||_p$.

Lemma 2.5 [16].
Let γ be continuous function on $[-1,1]$, μ be a positive number such that $|\gamma(x)| \leq \mu$, $-1 \leq x \leq 1$. Then

$$||(1-x^2)^{\frac{-1}{2\epsilon}} exp\{\pm(S\gamma)(x)\}||_\epsilon \leq (\pi(cos(\epsilon\mu))^{-1})^{\frac{1}{\epsilon}}$$

for each $\epsilon \in (0, \pi(2\mu)^{-1})$.

Lemma 2.6 [10]. If $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \frac{-1}{q} < \alpha < \frac{1}{p}$, $j = 1, ..., r$,

$$-1 \leq x_1 \leq x_2 \leq ... \leq x_r \leq 1,$$

and

$$(\widetilde{S}\phi)(x) = \prod_{j=1}^{r} |x - x_j|^{-\alpha_j} \{S(\prod_{j=1}^{r} |y - x_j|^{\alpha_j} (\phi(y)))\}(x),$$

then the operator $\widetilde{S}$ is a continuous operator in the space L_p.

3 The Fixed Point Equation

By differentiate equation (1) ([10],Chapt. II , Lemma 6.1) and by using the condition (2) we obtain the following quasi-linear singular integro-differential equation

$$c(x)\nu(x) - (S\nu)(x) = f(x) \tag{6}$$

$$\int_{-1}^{1} \nu(y)dy = 0 \tag{7}$$

where
$c = F_u(., u(.))$, $f = -F_x(., u(.))$ and $\nu(x)$ is the solution of equation (6) such that $u(x) = \int_{-1}^{x} \nu(y)dy$.
The equation (6) , (7) can be written in the equivalent form (see [10]):

$$A\nu = [P(c - i) + Q(c + i)]\nu = f(x),$$

where the operators P and Q are defined by

$$P = \frac{1}{2}(I - iS)\ Q = \frac{1}{2}(I + iS),$$

and I denotes the identity operator.
Putting:

$$B(x) = \frac{c(x) - i}{c(x) - i} = exp(-2\pi\alpha(x)),$$

where $\alpha : [-1, 1] \longrightarrow (0, 1)$ is a continuous function.
The generalized L_p -factorization for the function B is defined by

$$B(x) = B_-(x)(x - i)^{-1}B_+(x),$$

where

$$B_-(x) = \frac{x - i}{1 - x} exp(-i\pi\alpha(x) + \int_{-1}^{1} \frac{\alpha(y)}{y - x} dy) = \frac{(x - i)(c(x) - i)}{(1 - x)\rho(x)} exp(\pi(S\alpha)(x))$$

and

$$B_+(x) = (1 - x)exp(-i\pi\alpha(x) + \int_{-1}^{1} \frac{\alpha(y)}{y - x} dy) = \frac{(1 - x)}{\rho(x)}(c(x) - i)exp(-\pi(S\alpha)(x)),$$

$\rho : [-1, 1] \longrightarrow R$ such that $c(x) + i = \rho(x)\ exp(i\pi(\alpha)(x))$.
Hence the index of the operator A is equal 1 in the space L_p.

From [10] the right inverse of A is given by

$$A^{-1} = \frac{B_+^{-1}}{c(x)+i}[P(c(x)+i)+Q(c(x)-i)]\frac{B_+}{c(x)-i}I = Z^{-1}(c(x)I+S)\frac{Z}{\rho^2}I,$$

where

$$Z(x) = (1-x)\rho(x)\ exp[-\pi(S\alpha)(x)],$$

The solution of the problem (6), (7) is given by the form

$$\nu = A^{-1}f. \tag{8}$$

Integrating the equation (8), we obtain the fixed point equation

$$u(x) = (Tu)(x) = \int_{-1}^{x} H(y,u(y))dy, \tag{9}$$

where

$$H(x,u(x)) = \frac{c(x)}{\rho^2(x)}f(x) + \frac{Z^{-1}}{\pi}\int_{-1}^{1}\frac{Z(y)f(y)}{\rho^2(y)}\frac{dy}{y-x}.$$

We can write

$$H(x,u(x)) = sin[\theta(x)]d(x) + cos[\theta(x)]R(x,u(x)), \tag{10}$$

where

$$\theta(x) = tan^{-1}c(x) = \frac{\pi}{2} - \pi\alpha(x)\ d(x) = cos(\theta(x))f(x)$$

and

$$R(x,u(x)) = (1-x)^{\frac{-1}{2}}exp[(-S\theta)(x)]S\{(1-y^2)^{\frac{1}{2}}d(y)exp[(S\theta)(y)]\}(x).$$

Lemma 3.1. *Let $\{u_n\}$ a sequence converges uniformly to u in $C_0[-1,1]$. Then the function $d \in L_p$ and the sequence of functions $\{d_n sin(\theta_n)\}$ converges to $dsin(\theta)$ in L_p for $1 < p < \frac{1}{\delta}$.*

Proof: From the inequality (4) we have

$$||d||_p = \left(\ \int_{-1}^{1}|d(x)|^p dx\ \right)^{\frac{1}{p}} \leq l\left(\ \int_{-1}^{1}(1-x^2)^{-\delta p}dx\ \right)^{\frac{1}{p}} = k(\delta,p)^{\frac{1}{p}}. \tag{11}$$

Then $d \in L_p$.
We use the notations

$$c = F_u(.,u(.)),\ c_n = F_u(.,u_n(.))$$

and

$$f = F_x(., u(.)),\ f_n = F_x(., u_n(.)).$$

From the conditions of the derivatives of the function $F(., u(.))$, if $u_n \longrightarrow u$ as $n \longrightarrow \infty$ we have

$$c_n \longrightarrow c\ ,\ f_n \longrightarrow f. \tag{12}$$

From here we obtain $d_n sin(\theta_n) \longrightarrow sin(\theta)$and

$$\int_{-1}^{1} |d_n(x) sin(\theta_n(x)) - d(x) sin(\theta(x))|^p dx \longrightarrow 0.$$

Hence the Lemma is valid. □

Let us introduce the parameters

$$2\beta = arctan\ m - arctan\ n$$

and

$$2\lambda = arctan\ m - arctan\ n.$$

From (5) we have

$$|\beta| < \frac{\pi}{4}\ ,\ 0 \le 2\lambda \le \frac{\pi}{2} \tag{13}$$

from here there exist $\frac{3}{2} \le \chi < 2$ such that $0 < 3 - 2\delta\chi$.
Suppose that

$$\eta(x) = \theta(x) - \beta,$$

therefore

$$|\eta(x)| \le \lambda,\ -1 \le x \le 1,$$

$$(S\theta)(x) = (S\eta)(x) + (\frac{\beta}{\pi}) \ln \frac{1-x}{1+x}.$$

The function $R(x, u)$ can be written in the form:

$$R(x, u) = A(x)\ exp[-S\eta)(x)] S\{A^{-1}(y) dy\ exp[-S\eta)(y)]\}(x),$$

where

$$A(x) = (1-x)^{\frac{-1}{2} - \frac{\beta}{\pi}} (1+x)^{\frac{-1}{2} + \frac{\beta}{\pi}}.$$

For $p\chi^{-1} = \frac{3}{2}$, $3\zeta = 4\chi$ and by using Young's inequality [12] we have

$$||R||_p = ||(1-x^2)^{\frac{-1}{2\zeta}} exp[-(S\eta)(x)]||_\zeta$$

$$||A(x)(1-x)^{\frac{-1}{2\zeta}}\ S(A^{-1}(y)d(y)exp[(S\eta)(y)])(x)||_\zeta$$

$$\leq ||h||_\zeta ||\widetilde{S}\psi||_\zeta \tag{14}$$

where

$$h(x) = (1-x^2)^{\frac{-1}{2\zeta}} exp[-(S\eta)(x)],$$

$$(\widetilde{S}\psi)(x) = A(x)(1-x^2)^{\frac{1}{2\zeta}} S[A^{-1}(y)(1-y^2)^{\frac{-1}{2\zeta}}\psi(y)](x),$$

and

$$\psi(x) = (1-x^2)^{\frac{1}{2\zeta}} d(x)\ exp[(S\eta)(x)].$$

From the relation (13) we have $\zeta < \frac{\pi}{2\lambda}$, then from Lemma 2.5 we obtain

$$\leq ||h(x)||_\zeta \leq [(cos(\zeta\lambda))^{-1}]^{\frac{1}{\zeta}} = k(\lambda,\zeta) \tag{15}$$

Lemma 3.2. *The sequence h_n converges to h in the space $L_\zeta[-1,1]$, $\zeta = \frac{4\chi}{3}$.*

Proof: Suppose $e > 0,\ 0 < w < \frac{e-1}{e},\ e\zeta(1+w) < \frac{\pi}{2\lambda}$, the function $\Psi(\phi) = \phi^w$ and

$$E(x) = |h|^\zeta. \tag{16}$$

Then

$$\begin{aligned}
\int_{-1}^{1} |E_n(x)|\Psi(|E_n(x)|)dx &= \int_{-1}^{1}(1-x^2)^{-(\frac{1}{2}+\frac{w}{2})}\ exp[-\zeta(1+w)(S\eta_n)(x)]dx \\
&= \int_{-1}^{1}(1-x^2)^{\frac{1-e}{2e}-\frac{w}{2}}(1-x^2)^{\frac{-1}{2e}} \\
&\quad exp[-\zeta(1+w)(S\eta_n)(x)]dx \qquad (17)\\
&= \{\int_{-1}^{1}(1-x^2)^{(\frac{1-e}{2e}-\frac{w}{2})(\frac{e}{e-1})}dx\}^{\frac{e-1}{e}} \\
&\quad \{\int_{-1}^{1}(1-x^2)^{\frac{-1}{2}} exp[-e\zeta(1+w)(S\eta_n)(x)]dx\}^{\frac{1}{e}} \\
&= \{\int_{-1}^{1}(1-x^2)^{\frac{-1}{2}-\frac{we}{2(e-1)}}dx\}^{\frac{e-1}{e}} \\
&\quad \{\int_{-1}^{1}(1-x^2)^{\frac{-1}{2}} exp[-w_0(S\eta_n)(x)]dx\}^{\frac{1}{e}} \\
&\leq const.\{||(1-x^2)^{\frac{-1}{2w_0}} exp[-(S\eta_n)(x)||\}^{\frac{w_0}{e}}.
\end{aligned}$$

From Lemma 2.5, we have

$$\int_{-1}^{1} |E_n(x)|\Psi(|E_n(x)|)dx \le const.(\pi[cos(w_0\lambda)]^{-1})^{\frac{1}{e}} = k(\lambda, e) \tag{18}$$

where $w_0 = e\zeta(1+w)$ and $k(\lambda, e)$ is a constant depends on λ and e. From here $\{E_n\} \subset L_1$, and the integral of all elements of $\{E_n\}$ are uniformly absolutely continuous. From the relation (12) we have

$$\lim_{n \longrightarrow \infty} \|\eta_n - \eta\| = 0.$$

From the continuity of the operator S in L_p for $p > 1$, then $S\eta_n$ tends to $S\eta$ as $n \longrightarrow \infty$. Since the exponential function is monoton increasing and E_n tends to E in measure then from Lemma 2.2 we obtain $E \in L_1$ and

$$\lim_{n \longrightarrow \infty} \int_{-1}^{1} E_n(Z)dZ = \int_{-1}^{1} E(Z)dZ.$$

Therefore from the equation (16) we have

$$\|h_n\|_\zeta \longrightarrow \|h\|_\zeta \; as \; n \longrightarrow \infty \tag{19}$$

From the inequality (15) and Lemma 2.3 $h_n \subset L_\zeta$ converges weakly to $h \subset L_\zeta$.Hence by using the relation (19) and Lemma 2.4 the sequence h_n converges to h. Hence the lemma is proved. □

Lemma 3.3. *The functions ψ_n, ψ, which are represented above, satisfy*

$$\|\Psi_n - \Psi\|_\zeta \longrightarrow 0$$

Proof: Let $k > 0$, $\frac{4\chi}{9-8\chi\delta} < k > \frac{\pi}{2\lambda}$.
Since

$$\psi(x) = (1-x^2)^{\frac{1}{2\zeta}} d(x) exp[(S\eta)(x)],$$

$$\|\psi(x)\|_\zeta \le \|(1-x^2)^{\frac{-1}{2k}} exp[(S\eta)(x)]\|_k \|(1-x^2)^{\frac{k+\zeta}{2k\zeta}} d(x)\|_{\frac{k\zeta}{k-\zeta}}$$

$$\le lk(\lambda, k)(\int_{-1}^{1} (1-x^2)^{\frac{k+\zeta-2k\zeta\delta}{2(k-\zeta)}} dx)^{\frac{k-\zeta}{k\zeta}} \le \sigma(\lambda, k)$$

where $\sigma(\lambda, k)$ is a constant depends on λ and k. Similar to Lemma 3.2, we can prove that $\psi_n \longrightarrow \psi$ in L_ζ. □

Lemma 3.4. *The operator $\widetilde{S}$ is continuous in the space L_ζ.*

Proof: From the definition of $\widetilde{S}$ we have $\widetilde{S} = r^{-1}Sr$,
where

$$r(x) = (1+x)^{\alpha_1}(1-x)^{\alpha_2},$$
$$\alpha_1 = \frac{1}{2} - \frac{\beta}{\pi} - \frac{1}{2\zeta}\ ,\ \alpha_2 = \frac{1}{2} + \frac{\beta}{\pi} - \frac{1}{2\zeta}.$$

From the inequality (13) we have $\frac{3}{1+2(\frac{|\beta|}{\pi})} > 2$, therefore

$$-1 + \frac{1}{\zeta} < \alpha_i < 1 + \frac{1}{\zeta},\ i = 1, 2.$$

From Lemma 2.6 the operator $\widetilde{S}$ is continuous in the space L_ζ. □

Lemma 3.5. *By applying Lemma 3.3 and Lemma 3.4 we have $\widetilde{S}\psi_n \longrightarrow \widetilde{S}\psi$ in L_ζ.*

4 Existence Theorem.

We define the convex and compact set:

$$K^{0,\mu}_{A_1,A_2} = \{u \in C_0[-1,1] : \|u\|_\infty \leq A_1, |u(x_1) - u(x_2)| \leq A_2|x_1 - x_2|^\mu\},$$

where A_1, A_2 are positive constants and $\mu \in (0,1)$. Now we find the conditions for the image $T(u)$ belongs to the set $K^{0,\mu}_{A_1,A_2}$. From (10),(11),(14),(15) and Lemma 3.3 we can estimate the norm of $H(x, u(x))$ in the space $L_p,\ p = \frac{3\chi}{2},\ 3\zeta = 4\chi$ as follows

$$\begin{aligned} \|H(.,u(.))\|_p &\leq \|d\|_p + \|R(.,u(.))\|_p \\ &\leq \|d\|_p + \|h\|_\zeta \|\widetilde{S}\|_\zeta \|\psi\|_\zeta \\ &\leq k(\delta,p)^{\frac{1}{p}} + k(\lambda,\chi)\|\widetilde{S}\|_\zeta[\sigma(\lambda,k)] = A_2, \end{aligned} \tag{20}$$

where $\|\widetilde{S}\|_p$ denotes to the norm of the operator $\widetilde{S}$ in L_ζ.
Since

$$\|T(u)\|_{C_0} \leq (\int_{-1}^{1} dx)^{\frac{1}{q}} \|H(.,u(.))\|_p = A_2(2)^{\frac{1}{q}}, \tag{21}$$

$$|T(u)(x_1) - T(u)(x_2)| \leq (\int_{x_1}^{x_2} dy)^{\frac{1}{q}} \|H(.,u(.))\|_p \leq A_2|x_1 - x_2|^{\frac{1}{q}}. \tag{22}$$

If we choose $A_1 = A_2(2)^{\frac{1}{q}}$ and $\mu = \frac{1}{q}$, then all the transformed functions $T(u)$ belong to the set $K^{0,u}_{A_1,A_2}$. This means that the functions $T(u)$ are uniformly bounded and equicontinuous, [4,9] . Therefore the following lemma is valid.

Lemma 4.1. *Let the function $F(., u(.)$ has a continuous partial derivative*

$$F_u : [-1, 1] \times R \longrightarrow R \quad , \quad -m \leq F_u(x, u(x)) \leq n, \ mn < 1,$$

and a partial derivative $F_x(., u(.))$ continuous with respect to $u \in R$ for almost all x and measurable on $[-1, 1]$ for all $u \in R$, satisfies the inequality

$$|F_x(x, u)| \leq l(1 - x^2)^{-\sigma}$$

$$\sigma < \frac{3}{4} + (2\pi)^{-1}(arctan\ m + arctan\ n).$$

If we choose $2\beta = arctan\ m - arctan\ n, \ 2\lambda = arctan\ m + arctan\ n,$

$$\theta(x) = tan^{-1} F_u(x, u(x)) = \frac{\pi}{2} - \pi\alpha(x)$$

and

$$\eta(x) = \theta(x) - \beta, \ |\eta(x)| \leq \lambda, \ -1 \leq x \leq 1$$

where

$$|\beta| < \frac{\pi}{4}, \ 0 \leq 2\lambda \leq \frac{\pi}{2}.$$

Then for any $3/2 \leq \chi < 2$ such that, $0 < 3 - 2\delta\chi, \ \frac{4\chi}{9 - 8\chi\delta} < \frac{\pi}{2\lambda},$

$$\chi \leq min(\frac{3}{2\delta}, \frac{3\pi}{8\lambda}, \frac{9/4}{1 + 2(|\beta|/\pi)}),$$

$$\frac{4}{3} e\chi(1 + w) < \frac{\pi}{2\lambda}, \ 0 < w < \frac{e - 1}{e}, \ e > 0$$

and by choosing $A_1 = A_2(2)^{1/q}, \ \mu = 1/q$, the transformed points Tu belong to the set $k^{0,\mu}_{A_1,A_2}$.

Lemma 4.2. *The operator T which transforms the set $k^{0,\mu}_{A_1,A_2}$ into itself is continuous.*

Proof: Let $\{u_n\}_{n=1}^{\infty}$ be a sequences of elements of the set $k^{0,\mu}_{A_1,A_2}$ which converges uniformly to the element $u \in k^{0,\mu}_{A_1,A_2}$. The assertion is proved if we can show that

$$\lim_{n \longrightarrow \infty} \|H(., u(.)) - H(., u(.))\|_p = 0.$$

From equation (10) we have

$$\|H(., u_n(.)) - H(., u(.))\|_p \leq \|sin(\theta_n)d_n - sin(\theta)d\|_p +$$

$$+ \|cos(\theta_n)R(., u_n(.)) - cos(\theta)R(., u(.))\|_p.$$

From Lemma 3.1 the first norm of the right hand side of the above estimation tends to zero, also from the relation (12), Lemma 3.2 and Lemma 3.5 the second norm tends to zero . Therefore the operator T is continuous. □

From the preceding Lemmas and Arzela's theorem [4,9] the image of the set $k^{0,\mu}_{A_1,A_2}$ is compact , therefore we can use Schauder's fixed point theorem . Hence the operator T has at least one fixed point. Thus we can state the following theorem.

Theorem 4.3. *If the conditions of Lemmas 4.1 and 4.2 are satisfied , then the problem (1), (2) has at least one solution in the set* $W^1_0[-1,1]$.

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