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Arbitrary Lexsegment ideals with linear quotients and their minimal free resolutions

by
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Abstract

We determine all arbitrary lexsegment ideals with linear quotients and we describe their minimal free resolutions, as well as, their Hilbert series, Betti numbers and projective dimension. We also give some algorithms useful to study monomial ideals with linear quotients.

Key Words: Lexsegment ideals, linear quotients, mapping cone, minimal resolution.
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Introduction

Let $R = k[x_1, \ldots, x_n]$ be the ring of polynomials in n variables over a field k. We view R as a graded algebra, where $\deg(x_i) = 1$. Let us denote by M_d the set of all monomials of degree d of R. We order the monomials lexicographically so that $x_1 > x_2 > \ldots > x_n$. Now a lexsegment of degree d is a subset of M_d of the form $L(u,v) = \{w \in M_d : u \geq w \geq v\}$, for some u, $v \in M_d$, with $u \geq v$. An ideal generated by a lexsegment is called a lexsegment ideals. Lexsegment ideals in this generality have first been introduced by Hulett and Martin [8]. In the theory of Hilbert functions or in extremal combinatorics usually one considers initial lexsegment ideals. These are ideals which are spanned by an initial lexsegment $L^i(v) = \{w \in M_d : w \geq v\}$. The initial lexsegments have the nice property that their shadows are again initial lexsegments, a fact which is not true for arbitrary lexsegments. The shadow of a set S of monomials is the set $\mathrm{Shad}(S) = \{vx_i : v \in S, i = 1, \ldots, n\}$. We define the $i-$th shadow recursively by $\mathrm{Shad}^i(S) = \mathrm{Shad}^i(\mathrm{Shad}^{i-1}(S))$. Hulett and Martin call a lexsegment L completely lexsegment if all the iterated shadows of L are again lexsegments, that is, if for each i the set $\mathrm{Shad}^i(L)$ is a lexsegment. Arbitrary lexsegment ideals were studied in [1] and [4]. In particular, in [1] all lexsegment ideals with linear

resolution are determined. However their explicit minimal resolutions are only known when the lexsegment ideals are initial or final. In fact if I is an initial lexsegment ideal, I is stable and it has a resolution of Eliahou-Kervaire type [6]. The same is true if I is a final lexsegment ideal, because it is stable with respect to the order of the variables $x_1 > x_2 > \ldots > x_n$.
In the present paper we look at the class of lexsegment ideals with linear quotients. Let $I = (L(u, v)) = (u = u_1, u_2, \ldots, u_{n-1}, u_n = v)$, with $u_1 > \ldots > u_n$, $I_j = (u_1, \ldots, u_j)$. We say that I has linear quotients with respect to the lexicographic order if the colon ideals $I_{j-1} : u_j$ are generated by subsets of the variables $x_1, \ldots, x_n$. The aim of this paper is to consider classes of monomial ideals with linear quotients since this hypothesis is useful to construct a free resolution as iterated mapping cone [5], [7]. In [7] Herzog and Takayama generalize the theorem of Eliahou-Kervaire for stable ideals. Their construction is based on decomposition function and provides a minimal free resolution for the class of monomial ideals with linear quotients whose decomposition function satisfies a certain additional property which they call regular.
In the first section we recall the procedure determined in [1] characterizing all lexsegment ideals with linear resolution which we will use throughout the paper. In the second section we characterize the class of lexsegment ideals with linear quotients with respect to the lexicographic order giving conditions on generators. In the third section we describe the resolutions of these ideals. In particular, we show that all lexsegment ideals with linear quotients have a regular decomposition function, so that the Herzog-Takayama result explicitly describes their resolutions. As an application, we compute the Betti numbers, Hilbert series and the projective dimension of these ideals. In the fourth section we give some algorithms, designed and implemented with CoCoA, to test if an ideal has linear quotients, to compute the decomposition function and to test its regularity.

1 Lexsegment ideals with linear resolution

We quote some results obtained in [1] and [4] which we will use in the next section. In the following all completely lexsegment ideals are characterized:

Theorem 1.1. *Let $u = x_1^{a_1} \ldots x_n^{a_n}$ and $v = x_1^{b_1} \ldots x_n^{b_n}$ be monomials of degree d in R and let $u \geq v$ and $v \neq x_n^d$, and let I be the ideal generated by $L(u, v)$. Then I is completely lexsegment if and only if $a_1 \neq 0$ and one of the following conditions holds:*

(a) $u = x_1^a x_2^{d-a}$ *and* $v = x_1^a x_n^{d-a}$*, for some* a*,* $0 < a \leq d$*;*

(b) $a_1 \neq b_1$*, and for every* $w < v$ *there exists an index* $i > 1$ *such that* $x_i | w$ *and* $x_1 w / x_i \leq u$*.*

Proof: See [4]. $\square$

Let w be a monomial in the polynomial ring $R = k[x_1, \ldots, x_n]$, let $\max(w) = M(w)$ denote the largest index of the variables dividing w and $w' = w/x_{M(w)}$. The next theorem gives us the sufficient and necessary conditions for completely lexsegment ideals with linear resolution.

Theorem 1.2. *Let $u = x_1^{a_1} \ldots x_n^{a_n}$ and $v = x_1^{b_1} \ldots x_n^{b_n}$ be monomials of degree d and let $I = (L(u,v))$ be a completely lexsegment ideal. Then I has linear resolution if and only if one of the following conditions holds:*

(a) $u = x_1^a x_2^{d-a}$ *and* $v = x_1^a x_n^{d-a}$*, for some a, $0 < a \leq d$;*

(b) $b_1 < a_1 - 1$;

(c) $b_1 = a_1 - 1$ *and for the largest $w < v$, $w \in M_d$ one has $x_1 w' \leq u$.*

Proof: See [1]. □

The next theorem treats an essential case in the characterization of the lexsegment ideals with linear resolution.

Theorem 1.3. *Let $u = x_1^{a_1} \ldots x_n^{a_n}$, $v = x_2^{b_2} \ldots x_n^{b_n}$ be two monomials of degree d in $k[x_1, \ldots, x_n]$, with $a_1 \neq 0$. Suppose that the ideal I generated by $L(u,v)$ is not completely lexsegment. Then I has linear resolution if and only if u and v are of the form*

$$u = x_1 x_{l+1}^{a_{l+1}} \ldots x_n^{a_n},\ v = x_l x_n^{d-1}$$

for some l, $2 \leq l < n$.

Proof: See [1]. □

Now, we are able to give the procedure described in [1] by Aramova, De Negri and Herzog which determines all lexsegment ideals with linear resolution. Let $I = (L(u,v))$ be a lexsegment ideal with $u = x_1^{a_1} \ldots x_n^{a_n}$, $v = x_1^{b_1} \ldots x_n^{b_n}$.

- If $u = v$, then I has linear resolution. In the next steps we may therefore assume that $u > v$.

- If I is completely lexsegment, see Theorem 1.2.

- If I is not completely lexsegment, we let $m \geq 1$ be such that $a_i = b_i$ for $i = 1, \ldots, m-1$ and $a_m > b_m$. Set $f = x_1^{a_1} \ldots x_{m-1}^{a_{m-1}} x_m^{b_m}$, and let $\tilde{I}$ be the ideal in $k[x_m, \ldots, x_n]$ spanned by $L(u/f, v/f)$. It is clear that I has linear resolution if and only if $\tilde{I}$ has linear resolution.

- If $\tilde{I}$ is completely lexsegment, see 1.2, and if $\tilde{I}$ is not completely lexsegment, see 1.3.

2 Lexsegment ideals with linear quotients

Let k be a field, $R = k[x_1, \dots, x_n]$ be the polynomial ring in n indeterminates, and $I \subset R$ a monomial ideal. The unique minimal set of monomial generators of I will be denoted by $G(I)$. The ideal I is said to have *linear quotients* if for some order of the elements $u_1, \dots, u_m$ of $G(I)$ and $j = 1, \dots, m$ the colon ideals $(u_1, \dots, u_{j-1}) : u_j$ are generated by a subset of $\{x_1, \dots, x_n\}$. We define

$$\mathrm{set}(u_j) = \{k \in [n] : x_k \in (u_1, \dots, u_{j-1}) : u_j\}, \text{ for } j = 1, \dots, m.$$

According to Eliahou-Kervaire [6] a monomial ideal I in R is *stable* if for every monomial $w \in I$ and for every index $i < M(w)$, the monomial $x_i w / x_{M(w)}$ again belongs to I. In [7] J. Herzog and Y. Takayama proved that stable monomial ideals have linear quotients with respect to the reverse lexicographical order. We are interested in the lexsegment ideals with linear quotients with respect to the lexicographic order. Let $u = x_1^{a_1} \dots x_n^{a_n}$ and $v = x_1^{b_1} \dots x_n^{b_n}$ be monomials of the same degree. We say that $v > u$ in the lexicographic order if there exists an integer i such that $b_k = a_k$ for $k = 1, \dots, i-1$ and $b_i > a_i$. We set $\nu_i(u) = a_i$ for $i = 1, \dots, n$. In the following we prove that stable monomial ideals have linear quotients with respect to the lexicographic order.

Proposition 2.1. *Let I be a stable monomial ideal. Let $G(I) = \{u_1, \dots, u_m\}$, where $u_1 > u_2 > \dots > u_m$ in the lexicographic order with regard to $x_1 > x_2 > \dots > x_n$. Then I has linear quotients for this order of generators.*

Proof: Let $u \in G(I)$ and let J be the ideal generated by all $v \in G(I)$ with $v > u$ in the lexicographic order. Then $J : u = (v/[v, u] : v \in J)$, where $[v, u]$ denotes the greatest common divisor of u and v. Thus in order to prove that $J : u$ is generated by monomials of degree 1, we have to show that for each $v > u$ there exists $x_j \in J : u$ such that x_j divides $v/[v, u]$. In fact, let $u = x_1^{a_1} \dots x_n^{a_n}$ and $v = x_1^{b_1} \dots x_n^{b_n}$. Since $v > u$, there exists an integer i such that $b_k = a_k$ for $k = 1, \dots, i-1$ and $b_i > a_i$. Obviously $i < M(u)$. Since I is stable then $\overline{u} = x_i u' \in I$. Since $i < M(u)$ we see that $\overline{u} \in J$ and from the equation $x_{M(u)} \overline{u} = x_i u$ we deduce that $x_i \in J : u$. Finally since $\nu_i(v/[u, v]) = b_i - min\{a_i, b_i\} = b_i - a_i > 0$ we obtain that x_j divides $v/[v, u]$. □

In the theory of monomial ideals, there is the following hierarchy of ideals: initial lexsegment ideals ⇒ strongly stable monomial ideals ⇒ stable monomial ideals. Then, an immediate consequence of 2.1 is the following:

Corollary 2.2. *Let $I = (L^i(u))$ be an initial lexsegment ideal. Then I has linear quotients with respect to the lexicographic order.*

Proposition 2.3. *Let I be a stable ideal, $G(I) = \{u_1, \dots, u_m\}$, with $u_1 > u_2 > \dots > u_m$ in the lexicographic order. Then*

$$\mathrm{set}(u_j) = \{1, \dots, M(u_j) - 1\}$$

for all $u \in G(I)$.

Proof: Let $u_j \in G(I)$, $k \in \operatorname{set}(u_j)$. Then $x_k u_j \in (u_1, \ldots, u_{j-1})$, and this means $x_k u_j = x_l u_i$, with $u_i > u_j$, $u_i \in G(I)$. Since $u_i > u_j$ then $k < l$ and since $M(x_k u_j) = M(x_l u_i)$, then $k < M(u_j)$. Then $\operatorname{set}(u_j) \subseteq \{1, \ldots, M(u_j) - 1\}$.
Now let $k \in \{1, \ldots, M(u_j) - 1\}$. Then from $k < M(u_j)$ it follows that $x_k u'_j > u_j$. Moreover since I is stable $x_k u'_j \in I$. Then $x_k u'_j \in (u_1, \ldots, u_{j-1})$ and this means $k \in \operatorname{set}(u_j)$. Then $\{1, \ldots, M(u_j) - 1\} \subseteq \operatorname{set}(u_j)$ and the assert follows. □

In the general case, final lexsegment ideals do not have a linear quotients.

Example 2.4. Let $R = k[x_1, x_2, x_3, x_4]$ and $I = (L^f(x_2^2 x_4))$. Then $I = (x_2^2 x_4, x_2 x_3^2, x_2 x_4^2, x_3^3, x_3^2 x_4, x_3 x_4^2, x_4^3)$. The colon ideal $(x_2^2 x_4) : x_2 x_3^2 = (x_2 x_4)$ is not generated by a subset of $\{x_1, x_2, x_3, x_4\}$.

Lemma 2.5. *If I is an ideal of R with linear quotients, such that all generators of I have the same degree, then I has a linear resolution.*

Proof: The assertion follows from Lemma 2.16 in [9]. □

In the following we characterize all final lexsegment ideals with linear quotients in the lexicographic order.

Theorem 2.6. *Let u be a monomial of degree d in $k[x_1, \ldots, x_n]$. The following conditions are equivalent:*

(1) $J = (L^f(u))$ has linear quotients.

(2) $(L(u, v))$ has linear resolution for all $v < u$.

(3) $u = x_i^a x_{i+1}^{d-a}$ for some $1 \le i < n$, $0 \le a \le d$.

Proof: We prove that (1) ⇒ (2), (2) ⇒ (3) and (3) ⇒ (1).

(1) ⇒ (2). Obviously if J has linear quotients respect to the lexicographic order then $(L(u, v))$ has linear quotients for all $v < u$. It follows from Lemma 2.5 $(L(u, v))$ has linear resolution for all $v < u$.

(2) ⇒ (3). We prove that if $u \neq x_i^a x_{i+1}^{d-a}$, for all $1 \le i < n$, for all $0 \le a \le d$, then there exists a monomial $v < u$ such that $(L(u, v))$ does not have a linear resolution. Suppose that $u = x_1^{a_1} \ldots x_n^{a_n}$. Let us consider $i = m(u) = min\{i : i \in \operatorname{supp}(u)\}$. Note that i is the first index such that $a_i \neq 0$. We may consider the monomial $v = x_i^{a_i - 1} x_{i+1}^{d - a_i + 1}$, $v < u$. If $I = (L(u, v))$ is completely lexsegment, then from 1.1 $a_1 \neq 0$. Then $v = x_1^{a_1 - 1} x_2^{d - a_1 + 1}$. We consider the monomial $\overline{w} = x_1^{a_1 - 1} x_2^{d - a_1} x_3$, $\overline{w} < v$ and we show that condition (c) of Theorem 1.2 is not satisfied. Since $a_1 \neq 0$ and $u \neq x_1^{a_1} x_2^{d - a_1}$, then $x_1 \overline{w}' = x_1^{a_1} x_2^{d - a_1} > u$. Then $(L(u, v))$ does not have a linear resolution. Now suppose that I is not completely lexsegment and consider the ideal $\tilde{I} = (L(u/f, v/f))$, with $f = x_1^{a_1 - 1}$ in $k[x_1, \ldots, x_n]$. In order to show that I does not have a linear resolution we

show that $\tilde{I}$ does not have a linear resolution. Suppose that $\tilde{I}$ is completely lexsegment and take the largest monomial $\overline{w} < v/f$, $\overline{w} = x_2^{d-a_1}x_3$. It follows from $u \neq x_1^{a_1}x_2^{d-a_1}$ that $u/f \neq x_1x_2^{d-a_1}$ and this implies $a_2 < d - a_1$. Then $x_1\overline{w}' > x_1x_2^{d-a_1} > u/f$ and it follows from Theorem 1.2, condition (c) that $\tilde{I}$ does not have a linear resolution. Now suppose that $\tilde{I}$ is not completely lexsegment. Then it follows from Theorem 1.3 that $\tilde{I} = (L(x_1x_2^{a_2}\dots x_n^{a_n}, x_2^{d-a_1+1}))$ does not have a linear resolution. Then I does not have a linear resolution.

(3) $\Rightarrow$ (1). We prove that if $u = x_i^a x_{i+1}^{d-a}$ then J has linear quotients. Let $w \in G(J)$, i.e. $w \leq u$, and let I be the ideal generated by all $v \in G(J)$ with $v > w$ (in the lexicographic order). Then $I : w = (v/[v,w] : v \in I)$. Let $w = x_1^{b_1}\cdots x_n^{b_n}$, $v = x_1^{a_1}\cdots x_n^{a_n}$. Since $u = x_i^a x_{i+1}^{d-a} \geq v = x_1^{a_1}\cdots x_n^{a_n} > w = x_1^{b_1}\cdots x_n^{b_n}$ then for all $k < i$, $a_k = b_k = 0$ then v and w are of the following form:

$$v = x_i^{a_i}\cdots x_n^{a_n}, w = x_i^{b_i}\dots x_n^{b_n}.$$

Moreover since $v > w$, in the lexicographic order, there exists an index j such that $a_k = b_k$ for every $k = i, \dots, j-1$ and $a_j > b_j$.
From the inequality $u = x_i^a x_{i+1}^{d-a} > w = x_i^{b_i}\cdots x_n^{b_n}$ we obtain $a \geq b_i$. We distinguish two cases:
First Case: $a > b_i$. We prove that in this case the following inequalities hold:

$$\overline{u} = x_j w' \leq x_i w' \leq x_i^a x_{i+1}^{d-a}. \tag{1}$$

The first inequality of (1) follows from $j \geq i$. In order to prove the second inequality we note that since $w < u$ then $M(w) \geq i+1$. If $M(w) > i+1$ then $x_iw' < u$. If $M(w) = i+1$ and $a = b_i + 1$ then $x_iw' = x_i^{b_i+1}x_{i+1}^{d-a+1}/x_{i+1} = u$. If $M(w) = i+1$ and $a > b_{i+1}$ then $x_iw' < u$.
From (1) it follows $\overline{u} \in J$ and since $j < M(w)$ then $\overline{u} > w$, i.e., $\overline{u} \in I$. Then it follows from equation $x_jw = \overline{u}x_{M(w)}$ that $x_j \in I : w$. Finally, since $\nu_j(v/[v,w]) = a_j - min\{a_j, b_j\} = a_j - b_j > 0$, we have that x_j divides $v/[v,u]$.

Second Case: $a = b_i$. If $a = b_i$ then $d - a > b_{i+1}$ and $a_i = a = b_i$. Since $b_{i+1} < d - a$ and $j \geq i+1$ then $\hat{u} = x_jw' \leq x_{i+1}w' \leq u$ and this means that $\hat{u} \in J$. Moreover since $M(w) > j$ then $\hat{u} \in I$. From equality $x_{M(w)}\hat{u} = x_jw$ it follows that $x_j \in I : w$. Finally, since $\nu_j(v/[v,w]) = a_j - min\{a_j, b_j\} = a_j - b_j > 0$ then x_j divides $v/[v,w]$. $\square$

Besides characterizing final lexsegment ideals with linear quotients Theorem 2.6 also determines a large class of arbitrary lexsegment ideals with linear quotients, as shows the following:

Corollary 2.7. *If $u = x_i^a x_{i+1}^{d-a}$ for some $1 \leq i \leq n$, $0 \leq a \leq d$, then $I = (L(u,v))$ has linear quotients with respect to lexicographic order for every monomial $v \leq u$.*

In the following we characterize all arbitrary lexsegment ideals with linear quotients in the lexicographic order.

Theorem 2.8. *Let $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials of degree d in $k[x_1, \ldots, x_n]$. Let $k = \max(u) - 2$. The following conditions are equivalent:*

(1) $I = (L(u, v))$ *has linear quotients.*

(2) $(L(u, w))$ *has linear resolution for all* $u > w \geq v$

(3) $v \geq x_1^{a_1} \cdots x_k^{a_k} x_n^{d-(a_1+a_2+\ldots+a_k)}$.

Proof: We prove that (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Obviously if I has linear quotients respect to the lexicographic order then $(L(u, w))$ has linear quotients for all $u > w \geq v$. It follows from Lemma 2.5 $(L(u, w))$ has linear resolution for all $u > w \geq v$.

(2) $\Rightarrow$ (3). We prove that if $v < x_1^{a_1} \cdots x_k^{a_k} x_n^{d-(a_1+a_2+\ldots+a_k)}$, then there exists a monomial $u > w \geq v$ such that $(L(u, w))$ does not have a linear resolution. Let $j = \max\{i \in \mathrm{supp}(u), 1 \leq i \leq k : a_i \neq 0\}$. Then u is of the form $u = x_1^{a_1} \cdots x_j^{a_j} x_{k+1}^{a_{k+1}} x_{k+2}^{a_{k+2}}$. Take $w = v = x_1^{a_1} \cdots x_j^{a_j-1} x_{j+1}^{d-(a_1+\ldots+a_j-1)}$. We first prove that $(L(u, w))$ is not completely lexsegment. In fact, suppose that $(L(u, v))$ is completely lexsegment, then from Theorem 1.1 it follows that $a_1 \neq 0$ and since $a_1 = b_1$ then $(L(u, v))$ is completely lexsegment if and only if condition (a) of 1.1 is satisfied, i.e., if and only if $u = x_1^a x_2^{d-a}$ and $v = x_1^a x_n^{d-a}$. But this is a contradiction because $v < x_1^{a_1} \cdots x_k^{a_k} x_n^{d-(a_1+a_2+\ldots+a_k)}$. Then $(L(u, w))$ is not completely lexsegment. Now we prove that $(L(u, w))$ does not have a linear resolution by applying the procedure described in section 1. Take $f = x_1^{a_1} \cdots x_{j-1}^{a_{j-1}}$. The ideal $(L(u, w))$ has a linear resolution in $k[x_1, \ldots x_n]$ if and only if the ideal $(L(u/f, v/f))$ has linear resolution in $k[x_j, \ldots, x_n]$. Note that $(L(u/f, v/f)) = (L(x_j x_{k+1}^{a_{k+1}} x_{k+2}^{a_{k+2}}, x_{j+1}^{d-(a_1+\ldots+a_j-1)}))$. It follows from 1.3 that $(L(u/f, v/f))$ does not have a linear resolution in $k[x_j, \ldots, x_n]$.

(3) $\Rightarrow$ (1). We prove that if $v \geq x_1^{a_1} \cdots x_k^{a_k} x_n^{d-(a_1+a_2+\ldots+a_k)}$ then I has linear quotients. Let $w \in G(I)$, i.e, $u \geq w \geq v$ and let J be the ideal generated by all $z \in G(I)$ with $z > w$ (in the lexicographic order). Then $J : w = (z/[z, w] : z \in J)$. Let $w = x_1^{c_1} \cdots x_n^{c_n}$, $z = x_1^{r_1} \cdots x_n^{r_n}$. Since $u = x_1^{a_1} \cdots x_k^{a_k} x_{k+1}^{a_{k+1}} x_{k+2}^{a_{k+2}} \geq z > w \geq x_1^{a_1} \cdots x_k^{a_k} x_n^{d-a_1+\ldots+a_k}$, then for all $l = 1, \ldots, k$ we have $a_l = r_l = c_l$. Moreover since $z > w$ there exists an index $j \in \mathrm{supp}(z)$ such that $r_l = c_l$, $\forall l = 1, \ldots, j-1$ and $r_l > c_l$. From the inequality $u > w$ we obtain $a_{k+1} \geq c_{k+1}$. We distinguish two cases:

First case. $a_{k+1} > c_{k+1}$. We prove that in this case the following inequalities hold:

$$v \leq \overline{w} = x_j w' \leq x_{k+1} w' \leq u. \tag{2}$$

We note that $j \geq k+1$. Then $x_j w' \leq x_{k+1} w'$. It follows from $j < \max(w)$ that $x_j w' > w \geq v$ and then we obtain the inequality $x_j w' \geq v$. Finally the inequality

$x_{k+1}w' \leq u$ follows from $a_{k+1} > c_{k+1}$.
From (2) it follows that $\overline{w} \in I$. Moreover since $\overline{w} > w$ then $\overline{w} \in J$. Then it follows from equation

$$x_j w = \overline{w} x_{M(w)}$$

that $x_j \in J : w$. Finally since $\nu_j(z/[z,w]) = r_j - min\{r_j, c_j\} = r_j - c_j > 0$ then x_j divides $z/[z,w]$.

Second Case. $a_{k+1} = c_{k+1}$. If $a_{k+1} = c_{k+1}$ then $a_{k+2} > c_{k+2}$. Moreover since $j \geq k+2$ then $v \leq \hat{w} = x_j w' \leq x_{k+2} w' \leq u$ and this means that $\hat{w} \in I$. Moreover since $M(w) > j$ then $\hat{w} > w$ and then $\hat{w} \in J$. From equality $x_{M(w)}\hat{w} = x_j w$ it follows that $x_j \in J : w$. Finally, since $\nu_j(z/[z,w]) = r_j - min\{c_j, r_j\} = r_j - c_j > 0$ then x_j divides $z/[z,w]$. □

In the following we give an alternative proof of Corollary 2.7 using Theorem 2.8.

Another immediate consequence of 2.8 gives us information on the first syzygy module of lexsegment ideals. In fact a monomial ideal I has linear quotients if and only if the first syzygy module of I has a quadratic Gröbner basis [7]. Then we have the following:

Corollary 2.9. *Let $u = x_1^{a_1} \cdots x_n^{a_n}$, $k = M(u) - 2$, $I = (L(u,v))$ a lexsegment ideal. Then the first syzygy module of I has a quadratic Gröbner basis if and only if $v \geq x_1^{a_1} \cdots x_k^{a_k} x_n^{d-(a_1+a_2+\ldots+a_k)}$.*

3 The minimal free resolution of lexsegment ideals with linear quotients

In the general case, the problem of determining the resolutions of arbitrary lexsegment ideals is open. Partial results are contained in [8], where the basis of Koszul homology and a formula for the lower degree Betti number are computed, but a description of the maps is not known. In particular it holds the following:

Theorem 3.1. *Let $I = (L(u,v))$ be a lexsegment ideal in R. Then for $i \geq 2$ the lower degree Betti number is:*

$$\beta_{i,d+i-1}(R/I) = \sum_{w \in G(I)} \sum_{j=1}^{M(w)-i+1} \binom{M(w) - j - s_j(w) + l_j(w) - 1}{i-2} \qquad (3)$$

where, for $w \in G(I)$ $l_j(w)$ and $s_j(w)$ are defined as follows: if there exists a pair of integers l and s such that $j < l \leq s \leq M(w)$ and $\frac{x_i w' x_s}{x_l} \in G(I)$, then l_j is the largest l satisfying this condition, s_j is the smallest s; if a pair of this kind does not exist then $l_j(w) = j$ and $s_j(w) = M(w)$.

Proof: See [8]. □

This section contains results describing the resolutions of all lexsegment ideals with linear quotients. This description is obtained from Herzog-Takayama's results, based on iterated mapping cones [7]. Recall that the mapping cone of φ is the complex $C(\varphi)$ with $C(\varphi)_i = B_i \oplus A_{i-1}$ for all i, and chain map d with $d_i : C(\varphi)_i \to C(\varphi)_{i-1}$, $d_i(b,a) = (\varphi(a) + \partial(b), -\partial(a))$. We recall the following:

Lemma 3.2. *Suppose* $\deg u_1 \leq \deg u_2 \leq \ldots \leq \deg u_m$. *Then the iterated mapping cone F, derived from the sequence $u_1, \ldots, u_m$, is a minimal graded free resolution of R/I, and for all $i > 0$, the symbols*

$$f(\sigma; u) \text{ with } u \in G(I), \sigma \subset \operatorname{set}(u), |\sigma| = i - 1$$

form a homogeneous basis of the $R-$module F_i. Moreover $\deg(f(\sigma; u)) = |\sigma| + \deg(u)$.

Proof: See [7], Lemma 1.5. □

Let $M(I)$ be the set of all monomials of I. The map $g : M(I) \to G(I)$ is defined as follows: we set $g(u) = u_j$ if j is the smallest integer such that $u \in I_j$ and it is called *decomposition function* of I.

Definition 3.3. We say that the decomposition function $g : M(I) \to G(I)$ is *regular*, if $\operatorname{set}(g(x_s u)) \subset \operatorname{set}(u)$ for all $s \in \operatorname{set}(u)$ and $u \in G(I)$.

We note that the decomposition function for an arbitrary monomial ideal is not always regular. For example, consider $I = (x_1x_3, x_1x_5, x_5^2)$. Then, with respect to the lexicographic order, I has linear quotients. It can be verified that $\operatorname{set}(x_5^2) = \{x_1\}$, and $\operatorname{set}(g(x_1x_5^2)) = \{x_3\}$.
We will show that lexsegment ideals with linear quotients always have a regular decomposition function.
In the following propositions we describe the set and the decomposition function of a generator of I, in the case that I is a lexsegment ideal with linear quotients.

Proposition 3.4. *Let* $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_1^{b_1} \cdots x_n^{b_n}$, $I = (L(u,v))$ *with* $G(I) = L(u,v) = \{u_1 = u, u_2, \cdots, u_m = v\}$. *Let* $w \in G(I)$, $w = x_1^{d_1} \cdots x_n^{d_n}$. *Suppose I has linear quotients. Then we have:*

$$\operatorname{set}(w) = \{l, \ldots, M(w) - 1\}$$

being l the smallest integer such that $a_l > d_l$.

Proof: Let $r \in \operatorname{set}(w)$. We will show $l \leq r \leq M(w) - 1$. We have $w = u_j$, $1 \leq j \leq m$. Then

$$x_r w = x_s u_i, \tag{4}$$

$u_i > w$. This means $M(x_r w) = M(x_s u_i)$ and $s > r$. Then $r < M(w)$.
Let $u_i = x_1^{c_1} \cdots x_n^{c_n}$. It follows from 2.8 that

$$u = x_1^{a_1} \cdots x_k^{a_k} x_{k+1}^{a_{k+1}} x_{k+2}^{a_{k+2}} \geq u_i > w \geq v \geq x_1^{a_1} \cdots x_k^{a_k} x_n^{d-(a_1+\ldots+a_k)},$$

with $k = M(u) - 2$ and then $r \geq k+1$. With the notations of 2.8 we have $w = x_1^{a_1} \cdots x_k^{a_k} x_{k+1}^{d_{k+1}} x_{k+2}^{d_{k+2}} \cdots x_n^{d_n}$, $u_i = x_1^{a_1} \cdots x_k^{a_k} x_{k+1}^{c_{k+1}} x_{k+2}^{c_{k+2}} \cdots x_n^{c_n}$. It follows that

$$x_{k+1}^{a_{k+1}} x_{k+2}^{a_{k+2}} \geq x_{k+1}^{c_{k+1}} x_{k+2}^{c_{k+2}} \cdots x_n^{c_n} > x_{k+1}^{d_{k+1}} x_{k+2}^{d_{k+2}} \cdots x_n^{d_n} \geq x_n^{d-(a_1+\ldots+a_k)}. \tag{5}$$

We have $a_l \geq c_l > d_l$. Note that $k+1 \leq l \leq k+2$. If $l = k+1$ then $r \geq k+1 = l$. Now suppose $l = k+2$. Since $r \geq k+1$, it suffices to prove that $r \neq k+1$. Let $r = k+1$. From (4) $u_i = x_{k+1}w/x_s$ and then

$$x_{k+1}^{c_{k+1}} x_{k+2}^{c_{k+2}} \cdots x_n^{c_n} = x_{k+1} x_{k+1}^{d_{k+1}} x_{k+2}^{d_{k+2}} \cdots x_n^{d_n}/x_s. \tag{6}$$

By substituting (6) in the first inequality of (5) we obtain:

$$x_{k+1}^{a_{k+1}} x_{k+2}^{a_{k+2}} \geq x_{k+1} x_{k+1}^{d_{k+1}} x_{k+2}^{d_{k+2}} \cdots x_n^{d_n}/x_s.$$

But $l = k+2$, then $a_{k+1} = d_{k+1}$ and we obtain:

$$x_{k+2}^{a_{k+2}} \geq x_{k+1} x_{k+2}^{d_{k+2}} \cdots x_n^{d_n}/x_s,$$

with $s > r = k+1$. But this is a contradiction. Then $r \geq l$. Now suppose $r \in \{l, \ldots, M(w)-1\}$. We show that $x_r w = x_t u_i$, for some $1 \leq t \leq n$, $u \geq u_i > w$. Consider $u_i = x_r w'$. Since $r < M(w)$ we have $u_i > w$. Since $r \geq l$ we have $u_i \leq u$. We can write: $x_r w = x_r w' x_{M(w)} = u_i x_{M(w)}$ and then $\{l, \ldots, M(w)-1\} \subset \mathrm{set}(w)$. $\square$

Proposition 3.5. *Let $u = x_1^{a_1} \cdots x_n^{a_n}$, $v = x_1^{b_1} \cdots x_n^{b_n}$, $I = (L(u,v))$ with $G(I) = L(u,v) = \{u_1 = u, u_2, \ldots, u_m = v\}$. Let $w \in G(I)$, $w = x_1^{d_1} \cdots x_n^{d_n}$, g be the decomposition function of I. Suppose I has linear quotients. Then we have:*

$$g(x_s w) = x_s w', \quad \textit{for all } w \in G(I), s \in \mathrm{set}(w).$$

Proof: We have $w = u_j$, with $u \geq u_j \geq v$. From Proposition 3.4 it follows that $l \leq s \leq M(w) - 1$, with l the first integer such that $a_l > d_l$. Then $x_s u_j' \in (u_1, \ldots, u_{j-1})$. Now suppose that there exists $u_i > w$ such that $x_s u_j = x_t u_i$, for some $1 \leq t \leq n$. We prove $u_i \geq x_s u_j'$. But this is obvious because $u_i = x_s u_j/x_t \geq x_s u_j/x_{M(u_j)}$, being $M(u_j) \geq t > s$. $\square$

Proposition 3.6. *Let $u = x_1^{a_1} \ldots x_n^{a_n}$, $v = x_1^{b_1} \cdots x_n^{b_n}$, $I = (L(u,v))$ with $G(I) = L(u,v) = \{u_1 = u, u_2, \ldots, u_m = v\}$. Let $w \in G(I)$, $w = x_1^{d_1} \cdots x_n^{d_n}$. Suppose I has linear quotients. Then I has a regular decomposition function.*

Proof: Let $w \in G(I)$, $s \in \mathrm{set}(w)$. We prove that $\mathrm{set}(g(x_s w)) \subset \mathrm{set}(w)$. It follows from 3.4 that

$$set(w) = \{l, \ldots, M(w)-1\}, \ \text{ with } l \text{ the first integer such that } a_l > d_l.$$

Moreover from 3.5

$$\mathrm{set}(g(x_s w)) = \mathrm{set}(x_s w')$$

and again from 3.4 we obtain

$$\mathrm{set}(g(x_s w')) = \{t, \ldots, M(x_s w') - 1\}.$$

Since $s \in \mathrm{set}(w)$, then $l \leq s \leq M(w')$. In order to prove that $\mathrm{set}(g(x_s w)) \subset \mathrm{set}(w)$ it suffices to prove $t \geq l$ and $M(x_s w') \leq M(w)$.
Suppose $x_s w' = x_1^{c_1} \ldots x_n^{c_n}$, then t is the first integer such that $a_t > c_t$ and since $s \geq l$ then $t \geq l$. Moreover since $s \leq M(w')$ we have $M(x_s w') = M(w') < M(w)$ and the assertion follows. $\square$

Now we are able to give the following:

Theorem 3.7. *Let I be a lexsegment ideal with linear quotients, and F the graded minimal free resolution of R/I. Then the chain map ∂ of F is given by*

$$\partial(f(\sigma; u)) = -\sum_{t\in\sigma}(-1)^{\alpha(\sigma,t)} x_t f(\sigma \setminus t; u) + xM(u) \sum_{t\in\sigma} f(\sigma \setminus t; x_t u'), \text{ if } \sigma \neq \emptyset, \text{ and}$$

$$\partial(f(\emptyset; u)) = u, \text{ otherwise} .$$

Here $\alpha(\sigma, t) = |\{s \in \sigma : s < t\}|$.

Proof: Note that from 3.6 I has a regular decomposition function. Then the assertion follows from Theorem 1.12 in [7] and from the description of the regular decomposition function given in 3.5. $\square$

n immediate consequence of 3.2 and 3.6 is the following:

Corollary 3.8. *Let I be a lexsegment ideal with linear quotients, $G(I) = L(u, v)$ its minimal system of generators, $\mathbb{F}$. the minimal free resolution of R/I. Denote, by i_w, the smallest integer such that $a_{i_w} > d_{i_w}$, if $w \neq u$ and $i_u = M(u)$, where $w = x_1^{d_1} \ldots x_n^{d_n}$, $u = x_1^{a_1} \ldots x_n^{a_n}$. Then:*
(a) $\beta_i(R/I) = \sum_{w\in G(I)} \begin{pmatrix} M(w) - i_w \\ i - 1 \end{pmatrix}$,
(b) $\mathrm{proj\,dim}(R/I) = \max\{M(w) - i_w + 1, w \in G(I)\}$,
(c) $H_{F_i}(t) = \frac{1}{(1-t)^n} \sum_{w\in G(I)} \begin{pmatrix} M(w) - i_w \\ i - 1 \end{pmatrix} t^{i+d-1}$.

Proof: The assertions (a) and (b) follow from Lemma 3.2 and from Proposition 3.6. The assertion (c) follows from (a) and from the computation of the Hilbert series through free resolutions (see [2] pp. 153, Lemma 4.1.13). $\square$

Remark 3.9. Let $v \in M_d$, $I = (L^i(v))$ be an initial lexsegment ideal. We can write $I = (L(u,v))$, with $u = x_1^d$. In this case $i_w = 1$ for all $w \in G(I)$. From Corollary 3.8 we obtain:

$$\beta_i(A/I) = \sum_{w \in G(I)} \binom{M(w)-1}{i-1},$$

that is the formula for Betti numbers of stable ideals (see [6]).

Remark 3.10. If $i \geq 2$ the formula obtained in Corollary 3.8 can be deduced by Theorem 3.1. In fact let I be an arbitrary lexsegment ideal with linear quotients. Then, it follows from 2.5 that I has a linear resolution. Then it follows from formula (3) that:

$$\beta_i(R/I) = \sum_{w \in G(I)} \sum_{j=1}^{M(w)-i+1} \binom{M(w)-j-s_j(w)+l_j(w)-1}{i-2}, \tag{7}$$

But, in this case, we obtain:

$$l_j(w) = s_j(w), j = i_w. \tag{8}$$

By substituting (8) in (7) we obtain the formula obtained in Corollary 3.8, (a).

We conclude this section with a remark on arbitrary lexsegment ideals without linear quotients.
We first quote a result of Herzog which computes a bound for Betti numbers of the sum of a finite number of ideals.

Corollary 3.11. *Let I and J monomial ideals in R. Then:*

$$\beta_i(R/(I+J)) \leq \textstyle\sum_{j=0}^{i} \beta_j(R/I)\beta_{i-j}(R/J).$$

Proof: See [10], Corollary 3.1. □

Remark 3.12. Let I be an arbitrary lexsegment ideal. If I does not have linear quotients, it follows from Theorem 2.8 that I can be written as sum of lexsegment ideals with linear quotients, then, in such cases, by using Corollary 3.8 and Corollary 3.11 we can obtain a bound for Betti numbers of I as the following example shows.

Example 3.13. Let $R = k[x_1, \ldots, x_6]$, $I = (L(u,v))$, $u = x_1^3x_2x_4$, $v = x_1^3x_3x_5$. It follows from Theorem 2.8 that I does not have linear quotients, because $v < x_1^3x_2x_6$. We can write $I = I_1 + I_2$, with $I_1 = (L(x_1^3x_2x_4, x_1^3x_2x_6))$, $I_2 = (L(x_1^3x_3^2, x_1^3x_3x_5))$. By applying Theorem 2.8 again we have that I_1 and I_2 have linear quotients. By applying Corollary 3.8 we obtain:

$$\beta_0(R/I_1) = 1,\ \beta_1(R/I_1) = 3,\ \beta_2(R/I_1) = 3\ , \beta_3(R/I_1) = 1,$$
$$\beta_0(R/I_2) = 1,\ \beta_1(R/I_2) = 3,\ \beta_2(R/I_2) = 3\ , \beta_3(R/I_2) = 1.$$

and from Corollary 3.11 we obtain:

$$\beta_0(R/I) = 1,\ \beta_1(R/I) = 6,\ \beta_2(R/I) \leq 15,\ \beta_3(R/I) \leq 20,\ \beta_4(R/I) \leq 13.$$

4 Algorithms to test the regularity of the decomposition function

This section contains some algorithms, designed with CoCoA [3], which have been used to characterize all lexsegment ideals with linear quotients and to test that such ideals have a regular decomposition function.
The following algorithm allows us to establish if an ideal I has linear quotients.

Algorithm 4.1.

```
--Test: Has I linear quotients?
  Define IsIdealQL(I);
   GensI:=Gens(I);
    For T := 2 To Len(GensI) Do
    Icolon:=Ideal(First(GensI,T-1)):Ideal(GensI[T]);
     If Not IsSubSet(Gens(Icolon),Indets()) Then Return FALSE;
     EndIf;
    EndFor;
   Return TRUE;
  EndDefine;
```

The following algorithm returns the value of the decomposition function. This version of Algorithm 4.2, faster than the preliminary version, has been written by professor A. Bigatti.

Algorithm 4.2.

```
Define FirstReducer(U,G);
 For I := 1 To Len(G) Do
   If Type(U/G[I])<>RATFUN Then Return I; EndIf;
 EndFor;
Error("the monomial does not belong to the ideal"); EndDefine;
```

The previous functions are used by the following:

Algorithm 4.3.

```
 -- Is the decomposition function of I regular?
Define IsDecompRegular(I);
 If Not IsIdealQL(I) Then Error("I does not have linear quotients");
 EndIf;
GensI:=Gens(I);
  For T := 2 To Len(GensI) Do
  L:=Ideal(First(GensI,T-1)):Ideal(GensI[T]);
```

```
  Minimalize(L);
  GensL:=Gens(L);
  K:=GensL*GensI[T];
       For S:=1 To Len(K) Do
       Decomp:=FirstReducer(K[S],GensI);
       M:=Min(Decomp);
       W:=Ideal(First(GensI,M-1)):Ideal(GensI[M]);
       GensW:=Gens(W);
           If GensW<>[0] And Not IsSubSet(GensW,GensL)
           Then Return FALSE;
           EndIf;
       EndFor;
  EndFor;
       Return TRUE;
EndDefine;
```

The Algorithm 4.1 allows us to establish if an ideal I has linear quotients, but it does not return the set of the generators. In order to obtain a description of the set one can use the following modified version of 4.1.

Algorithm 4.4.

```
Define Quozients(I); GensI:=Gens(I);
 For T := 2 To N Do
   Ideal(First(I,T-1)):Ideal(I[T]);
 EndFor;
EndDefine;
```

We conclude this section giving the following open problem:

Problem 4.5. *Determining other classes of monomial ideals with linear quotients and admitting a regular decomposition function.*

The previous considerations suggest that these classes enlarge the class of monomial ideals whose resolution is known. Moreover, excluding the large classes of matroidal ideals, stable ideals and lexsegment ideals with linear quotients, no other class with this property is known.

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