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On the metric dimension of the Jahangir graph *

by
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Abstract

Let G be a connected graph and $d(x,y)$ be the distance between the vertices x and y. A subset of vertices $W = \{w_1, \ldots, w_k\}$ is called a resolving set for G if for every two distinct vertices $x, y \in V(G)$ there is a vertex $w_i \in W$ such that $d(x, w_i) \neq d(y, w_i)$. A resolving set containing a minimum number of vertices is called a metric basis for G and the number of vertices in a metric basis is its metric dimension $dim(G)$.
Let J_{2n} be the graph obtained from the wheel W_{2n} by alternately deleting n spokes. In this note it is shown that $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ for every $n \geq 4$.

Key Words: Metric dimension, basis, resolving set, Jahangir graph.
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1 Introduction

If G is a connected graph, the *distance* $d(u,v)$ between two vertices u and v is the length of a shortest path between them. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$. If $d(x,t) \neq d(y,t)$ we shall say that vertex t *distinguishes* vertices x and y. If distinct vertices of G have distinct representations with respect to W, then W is called a *resolving* set for G [1]. A resolving set of minimum cardinality is called a *basis* for G and this cardinality is the *metric dimension* of G, denoted $dim(G)$.

The concepts of resolving set and metric basis have previously appeared in the literature (see [1], [2], [5]–[8]). These concepts have some applications in chemistry for representing chemical compounds [2] and to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [6]. Note that the problem of determining whether $dim(G) < k$ is an NP-complete problem [4]. We observe that for a given ordered set $W = \{w_1, \ldots, w_k\}$ of vertices

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of a graph G, the i-th component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $d(x|W) \neq d(y|W)$ for each pair of distinct vertices $x, y \in V(G)\backslash W$.

The wheel $W_n = C_n + K_1$ for $n \geq 3$. In [1] it was shown that $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for every $n \geq 7$.

The Jahangir graph J_{2n} is defined as follows: consider an even cycle C_{2n} : $v_1, v_2, \ldots, v_{2n}, v_1$, where $n \geq 2$ and a new vertex v adjacent to n vertices of $C_{2n} : v_2, v_4, \ldots, v_{2n}$. J_{2n} has order $2n+1$ and size $3n$ and can be obtained from the wheel W_{2n} by alternately deleting n spokes. Note that the figure J_{16} appears in Jahangir's mausoleum, built around 1637 A.D. at 5 km. north-west of Lahore, Pakistan across the river Ravi. This graph is sometimes referred to as the gear graph G_n [3].
In the next section we will found a formula for the metric dimension of J_{2n}.

2 The metric dimension of J_{2n}

The vertices of C_{2n} in the graph J_{2n} are of two kinds: vertices of degree two and three, respectively. The vertices of degree two will be referred to as minor vertices and vertices of degree three to as major vertices. It follows that J_{2n} is a bipartite graph, with one bipartition class consisting of the central vertex v together with all the minor vertices, and the other bipartition class consisting of the major vertices. It is not difficult to see that $dim(J_4) = 3$ (a basis consists of two minor vertices and one major vertex of C_4), $dim(J_6) = 3$ (two minor vertices and the central vertex v), $dim(J_8) = 2$ (two minor vertices u_1 and u_2 such that $d(u_1, u_2) = 2$) and $dim(J_{10}) = 3$ (three minor vertices u_1, u_2 and u_3 satisfying $d(u_1, u_2) = d(u_2, u_3) = 2$ and $d(u_1, u_3) = 4$).

Lemma 2.1. *For $n \geq 6$ the central vertex v does not belong to any basis of J_{2n}.*

Proof: Assume the result is false, and let B be a basis of J_{2n} that contains v. Since $B\backslash\{v\}$ is not a basis, there exist vertices u and u' such that $d(u, x) = d(u', x)$ for every $x \in B\backslash\{v\}$. Clearly $B = \{v\}$ is not a basis, so $B\backslash\{v\} \neq \emptyset$ and thus u and u' must belong to the same bipartition class. If neither $u = v$ nor $u' = v$, then $d(u, v) = d(u', v)$ and B is not a basis of J_{2n}. So we can assume that $u' = v$, and without loss of generality that u is the minor vertex v_7. In this case, we note that since $d(w, v) \leq 2$ for all $w \in V(J_{2n})$ and $d(v, x) = d(u, x)$ for each $x \in B\backslash\{v\}$, $B\backslash\{v\} \subseteq \{v_i | 1 \leq d(v_7, v_i) \leq 2\} = \{v_5, v_6, v_8, v_9\}$. In this case $d(v_2, x) = d(v_{12}, x)$ for any $x \in B \subseteq \{v, v_5, v_6, v_8, v_9\}$, a contradiction.

□

If B is a basis of J_{2n} then it contains $r \geq 2$ vertices on C_{2n} for $n \geq 6$ and we can order the vertices of $B = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ so that $i_1 < i_2 < \ldots < i_r$. We shall say that the pairs of vertices $\{v_{i_a}, v_{i_{a+1}}\}$ for $1 \leq a \leq r-1$ and $\{v_{i_r}, v_{i_1}\}$ are pairs of neighboring vertices. Given such an ordering, as in [1] we will define the gap G_a for $1 \le a \le r-1$ as the set of vertices $\{v_j | i_a < j < i_{a+1}\}$ and $G_r = \{v_j | 1 \le j < i_1 \text{ or } i_r < j \le 2n\}$. Thus we have r gaps, some of which may be empty. We will say that gaps G_a and G_b are neighboring gaps when $|a-b| = 1$ or $r-1$. A gap determined by neighboring vertices v_i and v_j will be called an $\alpha - \beta$ gap with $\alpha \le \beta$ when $d(v_i) = \alpha$ and $d(v_j) = \beta$ or when $d(v_i) = \beta$ and $d(v_j) = \alpha$. Hence we have three kinds of gaps: 2–2, 2–3 and 3–3.

Lemma 2.2. *If B is a basis of J_{2n} $(n \ge 6)$, then every $2-2$, $2-3$ and $3-3$ gap of B contains at most 5, 4 and 3 vertices, respectively.*

Proof: Suppose that there is a 2–2 gap of B containing seven consecutive vertices $x_1, x_2, \ldots, x_7$ of C_{2n} such that $d(x_1) = d(x_7) = 3$. In this case $r(x_3|B) = r(x_5|B)$, a contradiction. If there is a 2–3 gap containing six consecutive vertices of C_{2n}: $x_1, x_2, \ldots, x_6$ such that $d(x_1) = 3$ and $d(x_6) = 2$ it follows that $r(x_3|B) = r(x_5|B)$ and the existence of a 3–3 gap of B containing five consecutive vertices of C_{2n}, namely $x_1, x_2, \ldots, x_5$ such that $d(x_1) = d(x_5) = 2$ would imply $r(x_2|B) = r(x_4|B)$, a contradiction.

□

The 2–2, 2–3 and 3–3 gaps containing 5, 4 and 3 vertices, respectively, will be referred to as major gaps; the remaining ones are called minor gaps. In the proof of lemmas 2.3–2.5 the major vertices will be labeled by a star.

Lemma 2.3. *If B is a basis of J_{2n} $(n \ge 6)$, then it contains at most one major gap.*

Proof: Suppose that B contains two distinct major gaps:

- 3–3 and 3–3: x_1, x_2^*, x_3 and y_1, y_2^*, y_3; in this case $r(x_2^*|B) = r(y_2^*|B)$;
- 3–3 and 2–2: x_1, x_2^*, x_3 and $y_1^*, y_2, y_3^*, y_4, y_5^*$; we have $r(x_2^*|B) = r(y_3^*|B)$;
- 3–3 and 2–3: x_1, x_2^*, x_3 and y_1^*, y_2, y_3^*, y_4; it follows $r(x_2^*|B) = r(y_3^*|B)$;
- 2–2 and 2–2: $x_1^*, x_2, x_3^*, x_4, x_5^*$ and $y_1^*, y_2, y_3^*, y_4, y_5^*$; one deduces $r(x_3^*|B) = r(y_3^*|B)$;
- 2–2 and 2–3: $x_1^*, x_2, x_3^*, x_4, x_5^*$ and y_1^*, y_2, y_3^*, y_4; we have $r(x_3^*|B) = r(y_3^*|B)$;
- 2–3 and 2–3: x_1^*, x_2, x_3^*, x_4 and y_1^*, y_2, y_3^*, y_4; we have again $r(x_3^*|B) = r(y_3^*|B)$, which contradicts the hypothesis.

□

The following two lemmas show that any two neighboring gaps contain together at most four vertices, unless one gap is a major gap; in this case two neighboring gaps contain at most six vertices.

Lemma 2.4. *Let $n \geq 6$ and B be a basis of J_{2n}. Then any two neighboring gaps, one of which being a major gap, contain together at most six vertices.*

Proof: If the major gap is a 3–3 gap (with three vertices), then by Lemma 2.3 its neighboring gap may be a minor 2–2 or 2–3 gap including at most three vertices, which concludes the proof in this case.
If the major gap is a 2–2 gap (with five vertices), it is necessary to show that its neighboring gap cannot be a minor 2–2 gap with three vertices, nor a minor 2–3 gap with two vertices. If a major 2–2 gap has a neighboring 2–2 gap with three vertices, we have the following path consisting of consecutive vertices of C_{2n}: $x_1^*, x_2, x_3^*, x_4, x_5^*, x_6, x_7^*, x_8, x_9^*$, where $x_4 \in B$. In this case $r(x_3^*|B) = r(x_5^*|B)$, a contradiction. A similar conclusion holds if a major 2–2 gap has a neighboring 2–3 gap with two vertices. If the major gap is a 2–3 gap (with four vertices), by Lemma 2.3 it is sufficient to show that its neighboring gap cannot be a minor 2–2 gap with three vertices. If this case holds, we consider the following path: $x_1^*, x_2, x_3^*, x_4, x_5^*, x_6, x_7^*, x_8$, where $x_4 \in B$. In this case $r(x_3^*|B) = r(x_5^*|B)$.
□

Lemma 2.5. *If B is a basis of J_{2n} $(n \geq 6)$, then any two minor neighboring gaps contain together at most four vertices.*

Proof: Since by Lemma 2.2 any minor 2–2, 2–3 and 3–3 gap contains 3, 2 and 1 vertex, respectively, it is sufficient to show that the following cases: 1) a 2–2 gap with three vertices has a neighboring 2–2 gap with three vertices, and 2) a 2–2 gap with three vertices has a neighboring 2–3 gap with two vertices cannot occur.
If case 1) holds then there exists the path: $x_1^*, x_2, x_3^*, x_4, x_5^*, x_6, x_7^*$, where $x_4 \in B$; in this case $r(x_3^*|B) = r(x_5^*|B)$; if case 2) holds, then the path is: $x_1^*, x_2, x_3^*, x_4, x_5^*$, x_6 $(x_4 \in B)$, which implies $r(x_3^*|B) = r(x_5^*|B)$, a contradiction. □

Theorem 2.6. *If $n \geq 4$, then $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$.*

Proof: We have seen that $dim(J_8) = 2$ and $dim(J_{10}) = 3$. Let $n \geq 6$. By Lemma 2.1 the central vertex v does not belong to any basis B of J_{2n}.
First we prove that $dim(J_{2n}) \leq \lfloor \frac{2n}{3} \rfloor$ by constructing a resolving set S in J_{2n} with $\lfloor \frac{2n}{3} \rfloor$ vertices.
We consider three cases according to the residue class modulo 3 to which n belongs.

Case 1. $n \equiv 0 \pmod 3$. Let $2n = 3k$, where k is even, $k \geq 4$, and $\lfloor \frac{2n}{3} \rfloor = k$. In this case $S = \{v_{6i+1}, v_{6i+3} : 1 \leq i \leq k/2 - 1\} \cup \{v_1, v_{2n-1}\}$.

Case 2. $n \equiv 1 \pmod 3$. Let $2n = 3k + 2$, where k is even, $k \geq 4$, $\lfloor \frac{2n}{3} \rfloor = k$ and $S = \{v_{6i+1}, v_{6i+3} : 1 \leq i \leq k/2 - 1\} \cup \{v_1, v_{2n-1}\}$.

Case 3. $n \equiv 2 \pmod 3$. Then $2n = 3k + 1$, where k is odd, $k \geq 5$ and $\lfloor \frac{2n}{3} \rfloor = k$. We define $S = \{v_{6i+1}, v_{6i+3} : 1 \leq i \leq (k-1)/2\} \cup \{v_1\}$.

The set S contains only minor vertices, there is a unique 2–2 major gap and all other gaps are 2–2 minor gaps alternately containing one and three vertices. All vertices contained in a 2–2 minor gap with one vertex are major vertices.
The set S is a resolving set of J_{2n} since any two major or any two minor vertices, respectively, lying in different gaps are separated by at least one vertex in the set of three or four vertices of S determining these two gaps (neighboring or not). This property also holds for vertices lying in the same gap. Note that $r(v|S) = (2, 2, \ldots, 2)$ and the representation of every vertex x of J_{2n}, $x \neq v$, with respect to S is different from $r(v|S)$.

Now we show that $dim(J_{2n}) \geq \lfloor \frac{2n}{3} \rfloor$. Let B be a basis of J_{2n} and $|B| = r$. Then B induces r gaps on C_{2n} which we denote by $G_1, \ldots, G_r$ such that G_i and G_{i+1} for every $1 \leq i \leq r-1$ and also G_1 and G_r are neighboring gaps.
By Lemma 2.3 at most one of them, say G_1, is a major gap. By Lemmas 2.4 and 2.5 we can write $|G_1| + |G_2| \leq 6$, $|G_r| + |G_1| \leq 6$ and $|G_i| + |G_{i+1}| \leq 4$ for every $i = 2, \ldots, r-1$. By summing these inequalities, we get

$$2(2n - r) = 2\sum_{i=1}^{r} |G_i| \leq 4r + 4;$$

whence $r \geq (2n-2)/3$. Since r is an integer, for each $n \equiv 0, 1$ or $2 \pmod 3$ we obtain $r \geq \lfloor \frac{2n}{3} \rfloor$, which concludes the proof.

$\square$

References

[1] P. S. Buczkowski, G. Chartrand, C. Poisson, P. Zhang, On k-dimensional graphs and their bases, *Periodica Math. Hung.* 46(1)(2003), 9-15.

[2] G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.* 105(2000), 99-113.

[3] J. A. Gallian, Dynamic survey DS6: Graph Labeling, *Electronic J. Combinatorics*, DS6, 1-58, Jan. 3, 2007.

[4] M. R. Garey, D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, New York, 1979.

[5] F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combin.* 2(1976), 191-195.

[6] R. A. Melter, I. Tomescu, Metric bases in digital geometry, *Computer Vision, Graphics, and Image Processing* 25(1984), 113-121.

[7] P. J. Slater, Leaves of trees, *Congress. Numer.* 14(1975), 549-559.

[8] P. J. Slater, Dominating and reference sets in graphs, *J. Math. Phys. Sci.* 22(1988), 445-455.

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