Bull. Math. Soc. Sci. Math. Roumanie
Tome 50(98) No. 4, 2007, 331–341

# Existence and multiplicity of radial solutions for an elliptic boundary value problem on an annulus

by
Mihai Mihăilescu*and Ionel Roventa†

**Abstract**

This paper deals with the study of the existence and multiplicity of radial solutions for the problem $-\Delta u(x) = f(u(x))$ when $x \in \Omega$ and $u(x) = 0$ when $x \in \partial\Omega$, where $\Omega = \{x \in \mathbb{R}^N;\ a < |x| < b\}$ with $0 < a < b$ is an annulus in $\mathbb{R}^N$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. We use as main tools Schaeffer's fixed point theorem and Leggett-Williams fixed point theorem in order to obtain radial solutions for the above problem.

**Key Words**: Boundary value problem, radial solution, fixed point theorem, Sturm-Liouville equation.
**2000 Mathematics Subject Classification**: Primary: 35G15, Secondary: 35G30, 34B05, 34B15, 47H10 .

## 1 Introduction

In this paper we are concerned with the study of the boundary value problems

$$\begin{cases} -\Delta u(x) = f(u(x)), & \text{if} \quad x \in \Omega \\ u(x) = 0, & \text{if} \quad x \in \partial\Omega\,. \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is the annulus $\Omega = \{x \in \mathbb{R}^N;\ a < |x| < b\}$ with $0 < a < b$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies certain properties.

The study of such type of equations is motivated by the fact that they serve as models for some phenomena which arise in fluid mechanics, such as the exothermic chemical reactions or autocatalytic reactions (see [11], Section 5.11.1). More exactly, if we denote by $-\nabla T$ the heat flux, then temperature $T$ satisfies

$$\nabla^2 T + f(T) = 0$$

*Supported by the Romanian Grant CNCSIS PNII-79/2007 "Procese Neliniare Degenerate si Singulare".
†Partially supported by CNCSIS Grant 589/2007.

according to the notations in [11].

We also point out the fact that problems of type (1) are connected with the classical boundary value theory of Bernstein [4] (see also the studies of Granas, Gunther and Lee [7] for some extensions to nonlinear problems).

The existence and uniqueness of positive radial solutions for equations of type (1) were obtained in [3], [9], [12]. Wang [12] proved that if $f : (0,\infty) \to (0,\infty)$ satisfies

$$\lim_{z\to 0} \frac{f(z)}{z} = \infty \quad \text{and} \quad \lim_{z\to\infty} \frac{f(z)}{z} = 0$$

then problem (1) has a positive radial solution in $\Omega$. That result was extended for the systems of elliptic equations by Ma [10].

Finally, we remember that in the particular case when

$$f(u(x)) = |u(x)|^p \operatorname{sgn}(u(x))$$

equation (1) becomes

$$\begin{cases} -\Delta u(x) = |u(x)|^p \operatorname{sgn}(u(x)), & \text{if} \quad x \in \Omega \\ u(x) = 0, & \text{if} \quad x \in \partial\Omega\,. \end{cases} \tag{2}$$

and is called Lane-Emden or Emden-Fowler equation. Using adequate variational techniques Ambrosetti and Rabinowitz proved in [2] the existence of infinitely many solutions for equation (2) for $p \in (1, (N+2)/(N-2))$. Similar results were obtained by Ambrosetti and Badiale in [1] for $p \in (0,1)$.

In this paper we impose certain conditions on $f$ in order to prove the existence and multiplicity of radial solutions of problem (1). Our idea is to characterize the solutions of the boundary value problem (1) as fixed points of some operators. With that end in view we will use as main tools two classical fixed point theorems. The first one is Schaeffer's fixed point theorem (see Theorem 4.7 in [5]):

**Theorem 1.** *(Schaeffer) Let $X$ be a real normed linear space and let $h : X \to X$ be a completely continuous map. Suppose that $h$ satisfies the Leray-Schauder boundary condition, i.e. there exists $r > 0$ such that $\|x\| = r$ implies $h(x) \neq \lambda x$ for all $\lambda > 1$. Then $h$ has a fixed point.*

**Remark 1.** *We point out the fact that since $\|\lambda \cdot x\| = \lambda \cdot \|x\|$ for any $x \in X$ and any $\lambda > 0$ it follows that $h$ satisfies the Leray-Schauder boundary condition if $\|x\| = r$ implies $\|h(x)\| \leq r$.*

The second fixed point theorem that will be used in our study is due to Leggett and Williams [8].

**Theorem 2.** *(Leggett-Williams) Let $E$ be a real Banach space, $P \subset E$ a cone and $\alpha : P \to [0,\infty)$ a nonnegative, continuous, concave functional. For any $i$, $j$ positive numbers we define*

$$P_i = \{x \in P;\ \|x\| < i\}$$

$$P(\alpha, i, j) = \{x \in P;\ i \leq \alpha(x),\ \|x\| \leq j\}.$$

*Assume that there exists $\theta \geq 0$ such that $\alpha(x) \leq \|x\|$ for all $x \in \overline{P}_\theta$ and let $S : \overline{P}_\theta \to \overline{P}_\theta$ be a completely continuous operator. Suppose that there exists $0 < \gamma < q < r \leq \theta$ such that*

*(i) $\{x \in P(\alpha, q, r);\ \alpha(x) > q\} \neq \emptyset$ and $\alpha(Sx) > q$ for $x \in P(\alpha, q, r)$;*

*(ii) $\|Sx\| < \gamma$ for $\|x\| \leq \gamma$;*

*(iii) $\alpha(Sx) > q$ for $x \in P(\alpha, q, \theta)$ with $\|Sx\| > r$.*

*Then $S$ has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{P}_\theta$ satisfying $\|x_1\| < \gamma$, $q < \alpha(x_2)$, $\|x_3\| > \gamma$ and $\alpha(x_3) < q$.*

## 2 Main result

We are interested in finding radial solutions for problem (1), i.e. $u(x) = u(|x|)$. Therefore, we can write equation (1) in the form

$$\begin{cases} -u^{''}(t) - \dfrac{N-1}{t} u^{'}(t) = f(u(t)), & \text{if} \quad t \in (a, b) \\ u(a) = u(b) = 0, \end{cases} \tag{3}$$

or, equivalently

$$\begin{cases} -(t^{N-1} u^{'}(t))^{'} = f(u(t)) t^{N-1}, & \text{if} \quad t \in (a, b) \\ u(a) = u(b) = 0. \end{cases} \tag{4}$$

We point out that problem (4) is a Sturm-Liouville type problem. For that problem we introduce the corresponding Green function by

$$G(t, s) = C \cdot \begin{cases} \left[1 - \left(\dfrac{a}{s}\right)^{N-2}\right] \cdot \left[\left(\dfrac{b}{t}\right)^{N-2} - 1\right], & \text{if} \quad a \leq t \leq s \leq b \\ \left[1 - \left(\dfrac{a}{t}\right)^{N-2}\right] \cdot \left[\left(\dfrac{b}{s}\right)^{N-2} - 1\right], & \text{if} \quad a \leq s \leq t \leq b, \end{cases}$$

where $C = \frac{1}{(N-2)(b^{N-2} - a^{N-2})}$, see e.g. [6], Chapter 5.

The main results of this paper are given by the following theorems:

**Theorem 3.** *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists $\Lambda > 0$ such that $z \cdot f(z) < 0$ for any $|z| > \Lambda$. Then problem (1) has a radial solution.*

**Theorem 4.** *Assume that $f : [0, \infty) \to \mathbb{R}$ is a continuous function and that there exist $a_1,\ b_1 \in (a, b)$ with $a_1 < b_1$ such that function $f$ satisfies the following properties:*

*(F1) there exist two real numbers $\delta$ and $\xi$, $0 < \delta < \xi$ such that*

$$f(z) < \frac{\delta}{M}, \quad \text{if} \quad 0 \leq z \leq \delta,$$

$$f(z) > \frac{\xi}{m}, \quad \text{if } \xi \le z \le \frac{\xi}{c},$$

*where* $M = \max_{t\in[a,b]} \int_a^b G(t,s)s^{N-1}\,ds$, $m = \min_{t\in[a_1,b_1]} \int_a^b G(t,s)s^{N-1}\,ds$ *and*

$$c = \frac{\min\left\{\left(\frac{b}{b_1}\right)^{N-2} - 1\ ,\ 1 - \left(\frac{a}{a_1}\right)^{N-2}\right\}}{\max\left\{\left(\frac{b}{a}\right)^{N-2} - 1\ ,\ 1 - \left(\frac{a}{b}\right)^{N-2}\right\}};$$

*(F2) for $M$ defined in (F1) we have*

$$\limsup_{z\to\infty} \frac{f(z)}{z} < \frac{1}{M}.$$

*Then problem (1) has at least three radial solutions.*

**Example 1.** *We point out an example of function $f$ which satisfies conditions (F1) and (F2)*

$$f(z) = \begin{cases} \delta/2M, & \text{if} \quad z \in [0,\delta] \\ (\delta/2M)\frac{z-\xi}{\delta-\xi} + (2\xi/m)\frac{z-\delta}{\xi-\delta}, & \text{if} \quad z \in [\delta,\xi] \\ 2\xi/m, & \text{if} \quad z \in [\xi,\xi/c] \\ \ln(z+1-\xi/c) + 2\xi/m, & \text{if} \quad z \in [\xi/c,\infty) \end{cases}$$

*where the notations are in accord to those used in Theorem 4.*

**Remark 2.** *We point out the fact that Theorem 4 still remain valid if we replace condition (F2) by the condition imposed in the hypotheses of Theorem 3, i.e. there exists $\Lambda > 0$ such that $f(z) < 0$ for any $z > \Lambda$. Thus, in that case, the existence of a radial solution for problem (1) is already verified via Theorem 3.*

## 3 Proof of Theorem 3

Let $X = C_0^2[a,b] := \{u \in C^2[a,b];\ u(a) = u(b) = 0\}$. The set $X$ endowed with the norm $\|u\|_2 = \|u\| + \|u'\| + \|u''\|$, where $\|v\| = \max_{t\in[a,b]} |v(t)|$, becomes a real normed linear space.

We consider the operator $T : C_0^2[a,b] \to C_0^2[a,b]$ defined by

$$Tu(t) = \int_a^b G(t,s)\cdot f(u(s))\cdot s^{N-1}\,ds,$$

for any $u \in C_0^2[a,b]$. That operator is well defined and completely continuous (see e.g. Chapter 6 in [5]).

We point out that $u \in X$ is a solution of equation (4) if and only if

$$u = Tu,$$

see e.g. Chapter 5 in [6]. Thus, we deduce that the fixed points of $T$ are radial solutions of equation (1). We show that we can apply Theorem 1 in order to obtain a fixed point for operator $T$.

We have to prove that there exists $r > 0$ such that if $u \in C_0^2[a, b]$ with $\|u\|_2 = r$ then $Tu \neq \lambda u$, for all $\lambda > 1$. By Remark 1, it is enough to show that if $Tu = \lambda u$ for some $\lambda > 1$ then $\|u\|_2 \leq r$. We will verify an *a priori* estimate. Suppose that we have $u \in C_0^2[a, b]$ such that $\lambda u = Tu$, for some $\lambda > 1$. Then $u$ is a solution of the equation

$$\begin{cases} -(t^{N-1}u^{'}(t))^{'} = \frac{1}{\lambda} f(u(t))t^{N-1}, & \text{if} \quad t \in (a, b) \\ u(a) = u(b) = 0, \end{cases} \tag{5}$$

for some $\lambda > 1$.

We are looking for three positive constant $\Lambda_0, \Lambda_1, \Lambda_2$ such that $\|u\| < \Lambda_0$, $\|u^{'}\| < \Lambda_1$ and $\|u^{''}\| < \Lambda_2$.

It is clear that $|u(t)|$ can not attain its maximum at $t = a$ or $t = b$, since we must have $u \equiv 0$, which contradict the assumptions on function $f$. Thus, the maximum is attained in a point $t_0 \in (a, b)$. The function $\frac{1}{2}u^2(t)$ has also the maximum at $t = t_0$. It follows that

$$\frac{d}{dt}\left(\frac{1}{2}u^2(t)\right)|_{t_0} = 0 \quad \text{and} \quad \frac{d^2}{dt^2}\left(\frac{1}{2}u^2(t)\right)|_{t_0} \leq 0.$$

On the other hand, since $u$ verifies (7) we obtain

$$\frac{d^2}{dt^2}\left(\frac{1}{2}u^2(t)\right)|_{t_0} = u'(t_0)^2 + u(t_0)u''(t_0) = -\frac{1}{\lambda} \cdot u(t_0) \cdot f(u(t_0)),$$

which implies $u(t_0) \cdot f(u(t_0)) \leq 0$. It follows that $|u(t_0)| \leq \Lambda$ and we may consider $\Lambda_0 = \Lambda$.

In order to find the bound $\Lambda_1$ for $\|u^{'}\|$ we divide the interval $[a, b]$ into a finite number of subintervals $[m, n]$ such that $u^{'}$ is of constant nowhere-zero sign on $[m, n]$ and at least one of $u^{'}(m)$ and $u^{'}(n)$ is zero. There are four cases depending on whether $u^{'}(t) > 0$ or $u^{'}(t) < 0$ on $[m, n]$ and whether $u^{'}(m) = 0$ or $u^{'}(n) = 0$. We consider the case $u^{'}(t) > 0$ on $(m, n)$ and $u^{'}(n) = 0$, the other three cases can be treated similarly. Since $u$ is a solution of (7) we get

$$\begin{aligned} |u^{''}(t)| &= \left|\frac{1}{\lambda} \cdot f(u(t)) + \frac{N-1}{t} \cdot u^{'}(t)\right| \\ &\leq \frac{1}{\lambda} \cdot |f(u(t))| + \frac{N-1}{t} \cdot |u^{'}(t)| \\ &\leq \frac{1}{\lambda} \cdot |f(u(t))| + \frac{1}{2} \cdot \left(\frac{N-1}{a}\right)^2 + \frac{1}{2} \cdot (u^{'}(t))^2, \end{aligned}$$

for any $t \in (a,b)$. Taking into account that $u$ is continuous on $[a,b]$ and $f$ is continuous on $\mathbb{R}$ we can found a positive constant $C>0$ such that

$$|u^{''}(t)| < C + \frac{1}{2}\cdot(u^{'}(t))^2, \quad \forall\, t \in (a,b).$$

The above inequality implies

$$-u^{''}(t) < C + \frac{1}{2}\cdot(u^{'}(t))^2, \quad \forall\, t \in (a,b),$$

or,

$$\frac{u^{''}(t)\cdot u^{'}(t)}{C+\frac{1}{2}\cdot(u^{'}(t))^2} > -u^{'}(t), \quad \forall\, t \in (a,b).$$

Integrating the above inequality from $m$ to $n$ we obtain

$$\ln(C) - \ln\left(C + \frac{1}{2}\cdot(u^{'}(m))^2\right) \geq u(m) - u(n),$$

or,

$$\ln\left(C + \frac{1}{2}\cdot(u^{'}(m))^2\right) \leq \ln(C) + 2\cdot M.$$

We deduce that there exists $\Lambda_1 > 0$ such that $\|u^{'}\| \leq \Lambda_1$.

Finally, since $u^{''}(t) = -\frac{1}{\lambda}\cdot f(u(t)) - \frac{N-1}{t}\cdot u^{'}(t)$ for any $t \in (a,b)$ and $-\Lambda_0 \leq u(t) \leq \Lambda_0$, $-\Lambda_1 \leq u^{'}(t) \leq \Lambda_1$ for any $t \in [a,b]$ we infer the existence of a positive constant $\Lambda_2 > 0$ such that $\|u^{''}\| \leq \Lambda_2$.

The *a priori* estimate proved above implies that $T$ satisfies the hypotheses of Schaeffer's fixed point theorem. Thus, $T$ has a fixed point. We conclude that problem (1) has a radial solution.

The proof of Theorem 3 is complete. $\square$

**Remark 3.** *We point out the fact that similar ar uments as those used in the proof of Theorem 3 enable us to show the existence of a radial solution for problems involvin more complicated nonlinearities than problem (1). Consider the problem*

$$\begin{cases} -\Delta u(x) = g(u(x), |\nabla u(x)|), & \text{if} \quad x \in \Omega \\ u(x) = 0, & \text{if} \quad x \in \partial\Omega\,. \end{cases} \tag{6}$$

*where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is the annulus $\Omega = \{x \in \mathbb{R}^N;\ a < |x| < b\}$ with $0 < a < b$ and $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $g = g(z,w)$, is a continuous function satisfyin the properties:*

*(G1) there exists $\Lambda > 0$ such that*

$$z \cdot g(z,0) < 0, \quad \forall\, |z| > \Lambda;$$

*(G2) there exist $A$, $B > 0$ such that*

$$|g(z,w)| < A \cdot w^2 + B, \quad \forall\, w \in \mathbb{R},\ |z| \le \Lambda.$$

*Then problem (6) has a radial solution.*

## 4 Proof of Theorem 4

Let $E = C([a,b])$ be the set of all real continuous functions defined on $[a,b]$. The set $E$ endowed with the norm $\|u\| = \max_{t\in[a,b]} |u(t)|$ becomes a real Banach space. We define the cone

$$P = \left\{ u \in E;\ u(t) \ge 0,\ t \in [a,b],\ \min_{t\in[a_1,b_1]} u(t) \ge c\|u\| \right\}.$$

For any $u \in P$ we introduce the operator

$$Su(t) = \int_a^b G(t,s) \cdot f(u(s)) \cdot s^{N-1}\, ds,$$

and the functional $\alpha : P \to \mathbb{R}$,

$$\alpha(u) = \min_{t\in[a_1,b_1]} u(t).$$

We point out that $S(P) \subset P$. Indeed, for any $u \in P$ we have

$$\begin{aligned}
&\min_{t\in[a_1,b_1]} Su(t) = \\
= \quad & \frac{1}{(N-2)(b^{N-2}-a^{N-2})} \cdot \min_{t\in[a_1,b_1]} \{ \int_a^t \left[1-\left(\frac{a}{s}\right)^{N-2}\right] \cdot \left[\left(\frac{b}{t}\right)^{N-2} - 1\right] \\
& f(u(s)) s^{N-1}\, ds + \int_t^b \left[1-\left(\frac{a}{t}\right)^{N-2}\right] \cdot \left[\left(\frac{b}{s}\right)^{N-2} - 1\right] f(u(s)) s^{N-1}\, ds \} \\
\ge \quad & \frac{1}{(N-2)(b^{N-2}-a^{N-2})} \cdot \min_{t\in[a_1,b_1]} \{ \int_a^t \left[1-\left(\frac{a}{s}\right)^{N-2}\right] \cdot \left[\left(\frac{b}{b_1}\right)^{N-2} - 1\right] \\
& f(u(s)) s^{N-1}\, ds + \int_t^b \left[1-\left(\frac{a}{a_1}\right)^{N-2}\right] \cdot \left[\left(\frac{b}{s}\right)^{N-2} - 1\right] f(u(s)) s^{N-1}\, ds \}
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\min\left\{\left(\dfrac{b}{b_1}\right)^{N-2}-1\ ,\ 1-\left(\dfrac{a}{a_1}\right)^{N-2}\right\}}{(N-2)(b^{N-2}-a^{N-2})}\cdot \\
&\quad \cdot \min_{t\in[a_1,b_1]}\{\int_a^t \left[1-\left(\frac{a}{s}\right)^{N-2}\right] f(u(s))s^{N-1}\,ds + \\
&\quad + \int_t^b \left[\left(\frac{b}{s}\right)^{N-2}-1\right] f(u(s))s^{N-1}\,ds\} \\
&\geq \frac{\min\left\{\left(\dfrac{b}{b_1}\right)^{N-2}-1\ ,\ 1-\left(\dfrac{a}{a_1}\right)^{N-2}\right\}}{(N-2)(b^{N-2}-a^{N-2})}\cdot \\
&\quad \min_{t\in[a_1,b_1]}\{\int_a^t \frac{(\frac{b}{s})^{N-2}-1}{(\frac{b}{a})^{N-2}-1}\cdot\left[1-\left(\frac{a}{s}\right)^{N-2}\right] f(u(s))s^{N-1}\,ds + \\
&\quad \int_t^b \frac{1-(\frac{a}{s})^{N-2}}{1-(\frac{a}{b})^{N-2}}\left[\left(\frac{b}{s}\right)^{N-2}-1\right] f(u(s))s^{N-1}\,ds\} \\
&\geq \frac{c}{(N-2)(b^{N-2}-a^{N-2})}\cdot \\
&\quad \cdot\int_a^b \left[1-\left(\frac{a}{s}\right)^{N-2}\right]\cdot\left[\left(\frac{b}{s}\right)^{N-2}-1\right] f(u(s))s^{N-1}\,ds \\
&= \frac{c}{(N-2)(b^{N-2}-a^{N-2})}\cdot\max_{t\in[a,b]}\{\int_a^t \left[1-\left(\frac{a}{s}\right)^{N-2}\right]\cdot\left[\left(\frac{b}{s}\right)^{N-2}-1\right] \\
&\quad f(u(s))s^{N-1}\,ds + \int_t^b \left[1-\left(\frac{a}{s}\right)^{N-2}\right]\cdot\left[\left(\frac{b}{s}\right)^{N-2}-1\right] f(u(s))s^{N-1}\,ds\} \\
&\geq \frac{c}{(N-2)(b^{N-2}-a^{N-2})}\cdot \\
&\quad \cdot\max_{t\in[a,b]}\{\int_a^t \left[1-\left(\frac{a}{s}\right)^{N-2}\right]\cdot\left[\left(\frac{b}{t}\right)^{N-2}-1\right] f(u(s))s^{N-1}\,ds + \\
&\quad \int_t^b \left[1-\left(\frac{a}{t}\right)^{N-2}\right]\cdot\left[\left(\frac{b}{s}\right)^{N-2}-1\right] f(u(s))s^{N-1}\,ds\} \\
&= c\max_{t\in[a,b]} Su(t) = c\|Su\|.
\end{aligned}
$$

In other words, we find

$$
\min_{t\in[a_1,b_1]} Su(t) \geq c\|Su\|, \quad \forall\, u\in P. \tag{7}
$$

Thus, we get that $S : P \to P$ is well defined. Moreover, it is easy to show that $S$ is completely continuous.

On the other hand, functional $\alpha$ is nonnegative, continuous, concave and satisfies $\alpha(u) \leq \|u\|$ for any $u \in P$.

We point out that $u \in E$ is a solution of equation (4) if and only if

$$u = Su,$$

see e.g. Chapter 5 in [6]. Thus, we deduce that the fixed points of operator $S$ are radial solutions of equation (1).

We show that we can apply Theorem 2 in order to obtain three fixed points for operator $S$.

First, we prove that there exists a real number $q$, such that $q > \xi/c$ and $S : \overline{P}_q \to P_q$. Indeed, since by (F2) we have $\limsup_{z\to\infty} \frac{f(z)}{z} < 1/M$ it follows that there exist $\tau > 0$ and $d < 1/M$ such that

$$f(z) < d \cdot z, \quad \forall\, z > \tau.$$

Consider $\psi = \max_{z\in[0,\tau]} f(z)$. Then

$$f(z) < d \cdot z + \psi \quad \forall\, z \geq 0.$$

Taking

$$q > \max\left\{\frac{\psi/M}{1 - d/M}\ ,\ \frac{\xi}{c}\right\}$$

we find that for any $u \in \overline{P}_q$ the following inequalities hold

$$\begin{aligned}
\|Su\| &= \max_{t\in[a,b]} \int_a^b G(t,s) \cdot f(u(s)) \cdot s^{N-1}\, ds \\
&\leq \int_a^b G(t,s) \cdot (d \cdot u(s) + \psi) \cdot s^{N-1}\, ds \\
&\leq \int_a^b G(t,s) \cdot (d \cdot \|u\| + \psi) \cdot s^{N-1}\, ds \\
&\leq \frac{d \cdot q + \psi}{M} < q.
\end{aligned}$$

Thus, we found $q > \xi/c$ such that $S(\overline{P}_q) \subset P_q$.

Second, we point out that the constant function

$$\tfrac{\xi+\xi/c}{2} \in \{u \in P(\alpha, \xi, \xi/c);\ \alpha(u) > \xi\}.$$

Moreover, $\alpha(Su) > \xi$, for any $u \in P(\alpha, \xi, \xi/c)$. Indeed, for any $u \in P(\alpha, \xi, \xi/c)$ we have

$$\frac{\xi}{c} \geq \|u\| \geq u(z) \geq \min_{t\in[a_1,b_1]} u(t) = \alpha(u) \geq \xi, \quad \forall\, z \in [a_1, b_1].$$

That fact combined with condition (F1) implies

$$\alpha(Su) = \min_{t\in[a_1,b_1]} \int_a^b G(t,s)\cdot f(u(s))\cdot s^{N-1}\, ds > \frac{\xi}{m}\cdot \min_{t\in[a_1,b_1]} \int_a^b G(t,s)\cdot s^{N-1}\, ds = \xi.$$

Thus, we have verified condition (i) from Theorem 2.

Next, we consider $u \in E$ with $\|u\| < \delta$. Then condition (F1) yields

$$\|Su\| = \max_{t\in[a,b]} \int_a^b G(t,s)\cdot f(u(s))\cdot s^{N-1}\, ds < \frac{\delta}{M}\cdot \max_{t\in[a,b]} \int_a^b G(t,s)\cdot s^{N-1}\, ds = \delta.$$

Thus, we have verified condition (ii) from Theorem 2.

Finally, we consider $u \in P(\alpha, \xi, q)$ with $\|Su\| > \xi/c$. Then it is clear that $\xi \leq \min_{t\in[a_1,b_1]} u(t)$, $\|u\| < q$ and $\|Su\| > \xi/c$. Using these facts and relation (7) we deduce that

$$\alpha(Su) = \min_{t\in[a_1,b_1]} Su(t) \geq c\cdot \|Su\| > c\cdot \frac{\xi}{c} = \xi$$

and thus, we have verified condition (iii) from Theorem 2.

We remark that all the hypotheses of Theorem 2 are satisfied and thus operator $S$ has three fixed points, i.e. problem (1) has three radial solutions.

The proof of Theorem 4 is complete. $\square$

## References

[1] A. Ambrosetti and M. Badiale, The dual variational principle and elliptic problems with discontinuous nonlinearities, *J. Math. Anal. Appl.* **140** (1989), 363-373.

[2] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory, *J. Funct. Anal.* **14** (1973), 349-381.

[3] C. Bandle, C. V. Coffman and M. Marcus, Nonlinear elliptic problems in the annulus, *J. Differential Equations* **69** (1978), 322-345.

[4] S. Bernstein, Sur les équations du calcul des variations, *Ann. Sci. Ecole Norm. Sup.* **29** (1912), 431-485.

[5] R. F. Brown, *A topological Introduction to Nonlinear Analysis*, Birkhäuser, Boston, Basel, Berlin, 1993.

[6] Y. Egorov and V. Kondratiev, *On Spectral Theory of Elliptic Operators*, Birkhäuser, Basel, Boston, Berlin, 1996.

[7] A. Granas, R. Gunther and J. Lee, On a theorem of S. Bernstein, *Pacific J. Math.* **74** (1978), 67-82.

[8] R. W. Leggett and L. R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana University Math. J.* **28** (1979), 373-388.

[9] P. L. Lions, On the existence of positive solutions of semilinear elliptic equations, *SIAM Rev.* **24** (1982), 441-467.

[10] R. Ma, Existence of positive radial solutions for elliptic systems, *J. Math. Anal. Appl.* **201** (1996), 375-386.

[11] J. Ockedon, S. Howison, A. Lacey and A. Movchan, *Applied Partial Differential Equations*, Oxford University Press, 2003.

[12] H. Wang, On the existence of positive radial solutions for semilinear elliptic equations in the annulus, *J. Differential Equations* **109** (1994), 1-8.

Received: 10.09.2007.

Department of Mathematics, University of Craiova,
200585 Craiova, Romania
and
Department of Mathematics,
Central European University,
1051 Budapest, Hungary.
E-mail: `mmihailes@yahoo.com`

Department of Mathematics, University of Craiova,
200585 Craiova, Romania
E-mail: `roventaionel@yahoo.com`