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Stanley decompositions, pretty clean filtrations and reductions modulo regular elements

by
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Abstract

We study the behavior of Stanley decompositions and of pretty clean filtrations under reduction modulo a regular element.

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Introduction

Let K be a field and $S = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring in n variables over the field K. Let $I \subset S$ be a monomial ideal, and let $u \in S$ be a monomial such that u is regular on S/I. The purpose of this paper is to investigate how the Stanley depth and the property of S/I to be pretty clean behaves when we pass from S/I to $S/(I, u)$, and vice versa.

We denote by $I^c \subset S$ the K-linear subspace of S generated by all monomials which do not belong to I. Then $S = I \bigoplus I^c$ and $S/I \cong I^c$ as K-linear spaces.

If $u \in S$ is a monomial and $Z \subset \{x_1, \ldots, x_n\}$, the K-subspace $uK[Z]$ whose basis consists of all monomials uv, with $v \in K[Z]$, is called a *Stanley space* of dimension $|Z|$. A decomposition $\mathcal{D}$ of I^c as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of S/I. The minimal dimension of a Stanley space in $\mathcal{D}$ is called the *Stanley depth* of $\mathcal{D}$ and it is denoted by $\operatorname{sdepth}(\mathcal{D})$. We set

$$\operatorname{sdepth}(S/I) := \max\{\operatorname{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } S/I\}$$

and call this number the *Stanley depth* of S/I.

Stanley [4, Conjecture 5.1] made a conjecture on general Stanley decompositions of $\mathbb{Z}^n$-graded modules. In the special case that the $\mathbb{Z}^n$-graded module is S/I, where I is a monomial ideal, the conjecture says that $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$. A monomial ideal I is called *Stanley ideal* if it satisfies the Stanley's conjecture.

A basic fact in commutative algebra says $\operatorname{depth} S/(I, f) = \operatorname{depth} S/I - 1$ for any homogeneous element of positive degree $f \in S$ which is regular on S/I. In this paper we show that a corresponding statement holds for the Stanley depth. In fact, we show in Theorem 1.1 that $\operatorname{sdepth}(S/(I, u)) = \operatorname{sdepth} S/I - 1$ for any monomial $u \in S$ which is regular on S/I.

Special Stanley decompositions arise from prime filtrations. Let

$$\mathcal{F} : I = I_0 \subset I_1 \subset \ldots \subset I_r = S$$

be a prime filtration of S/I, i.e. $I_j/I_{j-1} \cong S/P_j$ for any $j = 1, \ldots, r$, where $P_j \subset S$ are prime ideals. The support of $\mathcal{F}$, is the set $\operatorname{Supp}(\mathcal{F}) = \{P_1, \ldots, P_r\}$. It is well known that $\operatorname{Ass}(S/I) \subset \operatorname{Supp}(\mathcal{F})$. Recall that the prime filtration $\mathcal{F}$ is called *pretty clean*, if for all $i < j$ with $P_i \subset P_j$ it follows that $P_i = P_j$. If S/I has a pretty clean filtration then S/I is called pretty clean, see [2, Definition 3.3]. For the pretty clean filtration, $\operatorname{Supp}(\mathcal{F}) = \operatorname{Ass}(S/I)$, see [2, Corollary 3.6]. This condition implies, by [3, Proposition 2.2], that I is a Stanley ideal. The prime filtration $\mathcal{F}$ is *clean* if $\operatorname{Supp}(\mathcal{F}) = \operatorname{Min}(S/I)$, where $\operatorname{Min}(S/I)$ is the set of minimal prime ideals of S/I. Note that any clean filtration is pretty clean. If S/I has a clean filtration, that S/I is called clean.

The main result (Theorem 2.1) of the second section is that if $I \subset S$ is a monomial ideal and $u \in S$ is a monomial which is regular on S/I, then S/I has a pretty clean filtration if and only if $S/(I, u)$ has a pretty clean filtration. This result implies that an ideal generated by a regular sequence of monomials is pretty clean. This fact was first proved in [1, Proposition 1.2] by a different method.

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1 Stanley decompositions and regular elements

The aim of this section is to show that the Stanley depth behaves like the ordinary depth with respect to reduction modulo regular elements. Indeed we have the following result:

Theorem 1.1. *Let $I \subset S$ be a monomial ideal of $S = K[x_1, \ldots, x_n]$ and $u \in S$ be a monomial regular on S/I. Then* $\operatorname{sdepth}(S/(I, u)) = \operatorname{sdepth}(S/I) - 1$. *In particular, I is a Stanley ideal if and only if (I, u) is a Stanley ideal.*

We first prove a special case of the theorem:

Lemma 1.2. *Let $m < n$ and $J \subset S' = K[x_1, \ldots, x_m]$ be a monomial ideal. Then for the monomial ideal $I = JS$ and for any x_k with $m < k \leq n$ we have*

$$\operatorname{sdepth}(S/(I, x_k)) = \operatorname{sdepth}(S/I) - 1.$$

Proof: Let $T = S'[x_{m+1}, \ldots, x_{k-1}, x_{k+1} \ldots, x_n]$ and $L \subset T$ be the monomial ideal such that $L = JT$. Then we have $S/(I, x_k) = T/L$. Let

$$\mathcal{D}:\ T/L = \bigoplus_{i=1}^{r} u_i K[Z_i]$$

be a Stanley decomposition of T/L such that $\operatorname{sdepth} \mathcal{D} = \operatorname{sdepth} T/L$. Then

$$\mathcal{D}_1 : S/I = (T/L)[x_k] = \bigoplus_{i=1}^{r} u_i K[Z_i][x_k] = \bigoplus_{i=1}^{r} u_i K[Z_i, x_k]$$

is a Stanley decomposition of S/I. It follows that

$$\operatorname{sdepth} \mathcal{D}_1 = \operatorname{sdepth} \mathcal{D} + 1 = \operatorname{sdepth} T/L + 1$$

and

$$\operatorname{sdepth} \mathcal{D}_1 \leq \operatorname{sdepth} S/I.$$

Hence

$$\operatorname{sdepth} T/L + 1 \leq \operatorname{sdepth} S/I.$$

In order to prove the opposite inequality we consider a Stanley decomposition

$$\mathcal{D}_2:\ S/I = \bigoplus_{i=1}^{s} v_i K[W_i]$$

of S/I with $\operatorname{sdepth} \mathcal{D}_2 = \operatorname{sdepth} S/I$.

Let $\mathcal{I} = \{i \in [s] :\ v_i K[W_i] \cap T \neq \{0\}\}$. We claim that

$$\mathcal{D}_3:\ T/L = L^c = \bigoplus_{i \in \mathcal{I}} v_i K[W_i] \cap T. \tag{1}$$

and $\bigoplus_{i \in \mathcal{I}} v_i K[W_i] \cap T$ is a direct sum decomposition of T/L.

In order to prove (1), choose a monomial $v \in L^c$. We want to show that there exists $i \in \mathcal{I}$ such that $v \in v_i K[W_i] \cap T$. Suppose on the contrary that $v \notin v_i K[W_i] \cap T$ for all $i \in \mathcal{I}$. Since $v \in T$, it implies that $v \notin v_i K[W_i]$, for all i. Hence we have $v \in I = JS$. Since $v \in T$ and $L = JT$, it follows that $v \in L$, a contradiction. Conversely, choose a monomial $w \in v_i K[W_i] \cap T$. This implies that $w \notin I = JS$ and since $L = JT \subset JS = I$, we see that $w \in L^c$.

Now we will show that $\mathcal{D}_3$ is a Stanley decomposition. Indeed, we have

$$v_i K[W_i] \cap T = \begin{cases} v_i K[W_i \setminus \{x_k\}], & \text{if } x_k \text{ does not divide } v_i \\ 0, & \text{otherwise.} \end{cases}$$

Comparing the Stanley decomposition $\mathcal{D}_2$ of S/I with the Stanley decomposition $\mathcal{D}_3$ of T/L we see that $\operatorname{sdepth}(\mathcal{D}_2) \leq \operatorname{sdepth}(\mathcal{D}_3) + 1$. Hence

$$\operatorname{sdepth} S/I = \operatorname{sdepth} \mathcal{D}_2 \leq \operatorname{sdepth}(\mathcal{D}_3) + 1 \leq \operatorname{sdepth} T/L + 1.$$

$\square$

For the proof of Theorem 1.1 we also need the following simple fact:

Lemma 1.3. *Let*
$$I = I_0 \subset I_1 \subset \ldots \subset I_r = S$$
be an ascending chain of monomial ideals of S such that each I_j/I_{j-1} is a cyclic module, and hence $I_j/I_{j-1} \cong S/L_j(-a_j)$ for some monomial ideal L_j and some $a_j \in \mathbb{Z}^n$. Then
$$\operatorname{sdepth}(S/I) \geq \min\{\operatorname{sdepth}(S/L_j) : \ j \in \{1, \ldots, r\}\}.$$

Proof: We have the following decomposition of S/I as a K-vector space:
$$S/I = I_1/I_0 \oplus I_2/I_1 \oplus \cdots \oplus S/I_{r-1}.$$
Since each $I_j/I_{j-1} \cong S/L_j(-a_j)$ we get the isomorphism
$$S/I \cong S/L_1(-a_1) \oplus S/L_2(-a_2) \oplus \cdots \oplus S/L_r(-a_r). \tag{2}$$
For each j let $\mathcal{D}_j : S/L_j = \bigoplus_{k=1}^{r_j} u_{jk} K[Z_{jk}]$ be a Stanley decomposition of S/L_j such that $\operatorname{sdepth} \mathcal{D}_j = \operatorname{sdepth} S/L_j$. Then by the isomorphism (2) we obtain the following Stanley decomposition
$$S/I = \bigoplus_{j=1}^{r} \bigoplus_{k=1}^{r_j} u_j u_{jk} K[Z_{jk}],$$
of S/I, where $u_j = x^{a_j}$ for $j = 1, \ldots, r$. From this Stanley decomposition of S/I the desired inequality follows. $\square$

Proof: [Proof of Theorem 1.1] Without loss of generality we may assume that $I = JS$ where $J \subset S' = K[x_1, \ldots, x_m]$ and that $u = x_{m+1}^{a_1} \ldots x_n^{a_{n-m}}$. We consider an ascending chain of ideals of S between (I, u) and S where two successive members of the chain are of the form
$$(I, x_{m+1}^{b_1} \cdots x_k^{b_k} \cdots x_n^{b_{n-m}}) \subset (I, x_{m+1}^{b_1} \cdots x_k^{b_k - 1} \cdots x_n^{b_{n-m}})$$
and where $b_i \leq a_i$ for all $i = 1, \ldots, n - m$.

Observe that
$$(I, x_{m+1}^{b_1} \cdots x_k^{b_{k-1}} \cdots x_n^{b_{n-m}})/(I, x_{m+1}^{b_1} \cdots x_k^{b_k} \cdots x_n^{b_{n-m}}) \simeq S/(I, x_k).$$
Therefore Lemma 1.2 and Lemma 1.3 imply that
$$\operatorname{sdepth}(S/(I, u)) \geq \operatorname{sdepth}(S/(I, x_k)) = \operatorname{sdepth}(S/I) - 1.$$
In order to prove other inequality, we choose a Stanley decomposition
$$\mathcal{D}' : (I, u)^c = \bigoplus_{i=1}^{r} u_i K[Z'_i]$$

of $S/(I,u)$ with $\operatorname{sdepth}(\mathcal{D}') = \operatorname{sdepth}(S/(I,u))$. We obtain a direct sum of K-vector subspaces $\bigoplus_{i=1}^{r} u_iK[Z_i'] \cap S'$ of S'. We observe that

$$J^c = \bigoplus_{i=1}^{r} u_iK[Z_i'] \cap S'$$

and that $\bigoplus_i u_iK[Z_i'] \cap S'$ is a Stanley decomposition of S'/J, where the sum is taken over those $i \in \{1,\dots,r\}$ for which $u_iK[Z_i'] \cap S' \neq \{0\}$, cf. proof of Lemma 1.2.

We have

$$u_iK[Z_i'] \cap S' = \begin{cases} u_iK[Z_i' \cap \{x_1,\dots,x_m\}], & \text{if } \operatorname{supp}(u_i) \subset \{x_1,\dots,x_m\} \\ 0, & \text{otherwise.} \end{cases}$$

Hence if we set $\Lambda = \{i : \operatorname{supp}(u_i) \subset \{x_1,\dots,x_m\}\}$, then

$$\mathcal{D} : S/I = \bigoplus_{i \in \Lambda} u_iK[Z_i]$$

is a Stanley decomposition of S/I, where $Z_i := \{Z_i' \cap \{x_1,\dots,x_m\}\} \cup \{x_{m+1},\dots,x_n\}$.

We claim that $|Z_i| > |Z_i'|$. Indeed, otherwise $\{x_{m+1},\dots,x_n\} \subset Z_i'$, contradicting the fact that $(u) \cap u_iK[Z_i'] = \{0\}$. Therefore, $\operatorname{sdepth}(\mathcal{D}) \geq \operatorname{sdepth}(\mathcal{D}')+1$.

Hence or final conclusion is that

$$\operatorname{sdepth}(S/(I,u)) = \operatorname{sdepth}(S/I) - 1.$$

$\square$

As an immediate consequence of our theorem we obtain the following result first proved in [1, Proposition 1.2].

Corollary 1.4. *Let I be a monomial ideal generated by regular sequence of monomials. Then I is a Stanley ideal.*

2 Pretty clean filtrations and regular elements

Theorem 2.1. *Let $S = K[x_1, x_2, ..., x_n]$ be a polynomial ring and $I \subset S$ be a monomial ideal and u a monomial in S such that u is regular on S/I. Then S/I is pretty clean if and only if $S/(I,u)$ is pretty clean.*

Proof: Suppose S/I is pretty clean and let

$$\mathcal{F} : \ I = I_0 \subset I_1 \subset \ldots \subset I_r = S$$

be a pretty clean filtration of S/I with $I_j/I_{j-1} \cong S/P_j$ for $j = 1, 2, ..., r$. It is known from [2, Corollary 3.6] that $\operatorname{Ass}(S/I) = \{P_1,\dots,P_r\}$.

We have $I_j = (I_{j-1}, z_j)$ where z_j is a monomial in S. The prime filtration $\mathcal{F}$ induces the following filtration

$$\mathcal{G}: \ (I, u) \subset (I_1, u) \subset \ldots \subset (I_r, u) = S,$$

where

$$(I_j, u)/(I_{j-1}, u) = ((I_{j-1}, u), z_j)/(I_{j-1}, u) \cong S/(I_{j-1}, u) : z_j.$$

Since u is regular on S/I, it follows that u is regular on S/I_j for all j. Indeed, since S/I is pretty clean it follows that S/I_j is pretty clean. Hence $\mathrm{Ass}(S/I_j) = \{P_{j+1}, \ldots, P_r\}$ which is contained in $\mathrm{Ass}(S/I)$. Since $\gcd(u, z_j) = 1$ it follows that

$$(I_{j-1}, u) : z_j = ((I_{j-1} : z_j), u) = (P_j, u).$$

Hence

$$(I_j, u)/(I_{j-1}, u) \cong S/(P_j, u).$$

Suppose, without loss of generality, that

$$P_j = (x_1, ..., x_t) \quad \text{and} \quad u = \prod_{i=t+1}^{n} {x_i}^{a_i}.$$

Then $S/(P_j, u) \cong K[x_{t+1}, ..., x_n]/(u)K[x_{t+1}, ..., x_n]$, which is clean by [3]. Hence we see that $(I_j, u)/(I_{j-1}, u)$ is clean and

$$\mathrm{Ass}((I_j, u)/(I_{j-1}, u)) = \{(P_j, x_i) : \ x_i \mid u\}.$$

Therefore our filtration $\mathcal{G}$ can be refined as follows

$$(I_{j-1}, u) = I_{j-1,0} \subset I_{j-1,1} \subset \ldots \subset I_{j-1,s_j} = (I_j, u)$$

where

$$I_{j-1,k}/I_{j-1,k-1} \cong S/P_{j-1,k}$$

with $P_{j-1,k} \in \{(P_j, x_i) : x_i \mid u\}$.

In the refined filtration of $\mathcal{G}$ if we have $I_{j,k} \subset I_{i,l}$, then either $j < i$ or $j = i$ and $k < l$. Suppose $j < i$ and $P_{j,k} \subset P_{i,l}$. We have $P_{j,k} = (P_{j+1}, x_r)$ for some r and $P_{i,l} = (P_{i+1}, x_s)$ for some s. Since $u \notin \bigcup_{P \in \mathrm{Ass}(S/I)} P$ it follows that $x_s \notin P_{j+1}$. Therefore, $P_{j+1} \subseteq P_{i+1}$. However, since $\mathcal{F}$ is a pretty clean filtration it follows that $P_{j+1} = P_{i+1}$, and hence $P_{j,k} = P_{i,l}$.

Next suppose that $i = j$ and $k < l$ and suppose that $P_{i,k} \subseteq P_{i,l}$. Since $\operatorname{height} P_{i,k} = \operatorname{height} P_{i,l}$ we conclude that $P_{j,k} = P_{i,l}$, also in this case. Thus we have shown that the refinement of $\mathcal{G}$ is a pretty clean filtration of $S/(I, u)$, and hence $S/(I, u)$ is pretty clean.

Conversely, suppose that $S/(I, u)$ is pretty clean. Since u is regular on S/I, we may suppose that $I = JS$ where $J \subset S' = K[x_1, \ldots, x_m]$ for $m < n$ and

$\operatorname{supp}(u) \subset \{x_{m+1}, \ldots, x_n\}$. Since $S/(I,u)$ is pretty clean there exist a pretty clean filtration

$$\mathcal{M}: \ (I,u) = I'_0 \subset I'_1 \subset \ldots \subset I'_r = S$$

such that $I'_j/I'_{j-1} \cong S/P_j$ where $P_j \in \operatorname{Ass}(S/(I,u))$. Recall that

$$\operatorname{Ass}(S/(I,u)) = \{(P', x_k) \colon P' \in \operatorname{Ass}(S'/J) \text{ and } x_k \mid u\}.$$

By taking the intersection of above filtration $\mathcal{M}$ with S', we get the filtration

$$\mathcal{N}: \ J_0 \subseteq J_1 \subseteq \ldots \subseteq J_r = S'$$

of S'/J_0 where $J_j = I'_j \cap S'$ for $j = 0, \ldots, r$. We claim that $J_0 = J$. Let I be generated by the monomials $u_1, \ldots, u_l$. Since $I = JS$ with $J \subset S'$ it follows that $u_i \in S'$ for all i. Choose a monomial $v \in J_0 = (I,u) \cap S'$. Then either $v = eu_i$ where $e \in S'$, or $v = fu$ where $f \in S'$. The second case cannot happen since $v \in S'$. This shows that $J_0 \subset J$. The other inclusion is obvious.

Take an ideal $I'_j \in \mathcal{M}$. Then $I'_j = (I'_{j-1}, w_j)$ where $w_j \in S$ and $(I'_{j-1} : w_j) = (P', x_k)$ for some $P' \in \operatorname{Ass}(S'/J)$ and some x_k such that $x_k \mid u$. Then we have $I'_{j-1} \cap S' = I'_j \cap S'$ if and only if $w_j \notin S'$.

Let $\{r_0, \ldots, r_k\}$ be the subset of $[r]$ for which we have J_{r_i} is properly contained in J_{r_i+1} in the filtration $\mathcal{N}$. Set $L_i = J_{r_i}$ for $i = 0, \ldots, k$ and $L_{k+1} = S'$. Then we obtain the filtration

$$\mathcal{L}: \ J = L_0 \subset L_1 \subset \ldots \subset L_{k+1} = S'.$$

We note that $L_i = (J, w_{r_0+1}, w_{r_1+1}, \ldots, w_{r_{i-1}+1})$ for $i = 0, \ldots, k+1$ with $w_{r_i+1} \in S'$ for all i.

Since $L_i = (L_{i-1}, w_{r_{i-1}+1})$, we have that $L_i/L_{i-1} \cong S'/(L_{i-1} :_{S'} w_{r_{i-1}+1})$ and also we have that $L_i = I'_{r_i} \cap S'$. So $(L_{i-1} :_{S'} w_{r_{i-1}+1}) = (I'_{r_{i-1}} \cap S' :_{S'} w_{r_{i-1}+1})$.

We claim that $(I'_{r_{i-1}} \cap S' :_{S'} w_{r_{i-1}+1}) = (I'_{r_{i-1}} :_S w_{r_{i-1}+1}) \cap S'$. In fact, the inclusion $(I'_{r_{i-1}} \cap S' :_{S'} w_{r_{i-1}+1}) \subset (I'_{r_{i-1}} :_S w_{r_{i-1}+1}) \cap S'$ is obvious. In order to prove the other inclusion we choose a monomial $v \in (I'_{r_{i-1}} :_S w_{r_{i-1}+1}) \cap S'$. Then we have that $v \in (I'_{r_{i-1}} :_S w_{r_{i-1}+1})$ and $v \in S'$. Hence $vw_{r_{i-1}+1} \in I'_{r_{i-1}}$ and $vw_{r_{i-1}+1} \in S'$, since $w_{r_{i-1}+1} \in S'$. Therefore $vw_{r_{i-1}+1} \in I'_{r_{i-1}} \cap S'$ which implies that $v \in (I'_{r_{i-1}} \cap S' :_{S'} w_{r_{i-1}+1})$, as desired.

Now we see that

$$\begin{aligned}(L_{i-1} :_{S'} w_{r_{i-1}+1}) &= (I'_{r_{i-1}} \cap S' :_{S'} w_{r_{i-1}+1}) \\ &= (I'_{r_{i-1}} :_S w_{r_{i-1}+1}) \cap S' = (P', x_k) \cap S' = P',\end{aligned}$$

where $(P', x_k) \in \operatorname{Ass}(S/(I,u))$.

This shows that $\mathcal{L}$ is a prime filtration with the property that the prime ideals in $\operatorname{Supp}(\mathcal{L})$ form a subsequence of $P_1, \ldots, P_r$. Therefore, since $\mathcal{M}$ is a pretty clean filtration, the filtration $\mathcal{L}$ is pretty clean as well. From this fact we will deduce that S/I is pretty clean. This then will complete the proof of the theorem.

Indeed, our filtration $\mathcal{L}$ induces the filtration

$$\mathcal{K} : I = JS = L_0S \subset L_1S \subset \cdots \subset L_{k+1}S = S$$

with $L_iS/L_{i-1}S \cong S/P'S$ where $L_i/L_{i-1} \cong S'/P'$ for $i = 1, \ldots, k+1$. This holds because the extension $S' \to S$ is flat. Now, since $\mathcal{L}$ is a pretty clean filtration of S'/J, it is obvious that $\mathcal{K}$ is a pretty clean filtration of S/I. $\square$

As an immediate consequence we obtain the following result from [1, Proposition 1.2].

Corollary 2.2. *Let $u_1, \ldots, u_k$ be a regular sequence in the polynomial ring S. Then $S/(u_1, \ldots, u_k)$ is pretty clean.*

Proof: We use induction on k. For $k = 1$ the assertion follows from Theorem 2.1 applied to $I = (0)$, or from [3]. By induction hypothesis we may now assume that $S/(u_1, \ldots, u_{k-1})$ is pretty clean. Since u_k is regular on $S/(u_1, \ldots, u_{k-1})$ it follows again from Theorem 2.1 that $S/(u_1, \ldots, u_k)$ is pretty clean. $\square$

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