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Finding Eulerian Cycle Decompositions and the Rotation distance between binary trees

by
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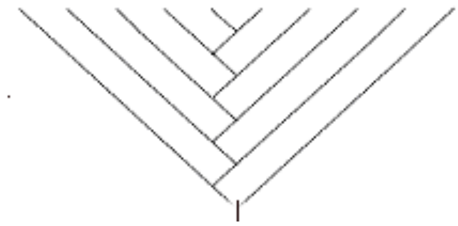
Abstract

To any rooted binary tree we assign a permutation. We define the class "T" of permutations as the set of permutations which come from this assignement.
In 1999, Caprara [4] proved it is NP-hard to find a maximal cycle decomposition of the breakpoint graph of a permutation. His result implies that it is NP-hard to sort permutations by reversals, answering a question relevant to Pevzner-Hannenhalli Theory on sorting genomes by reversals. Using Hyperbolic Geometry, Thurston, Sleator and Tarjan(1988) [9] proved that the diameter of the Rotation Graph is $2n - 6$. In a combinatorial approach to understand how trees evolve under rotations, we found a class "T" of permutations which satisfy the following condition: given any permutation, it is possible to decide in linear time if it belongs to "T". If the answer is positive, it is possible to find a maximal cycle decomposition of its breakpoint graph. There is also numerical and topological information shared by the trees and their associated permutation. We prove a result about a number $w(T)$, associated with any tree.

Key Words: Rooted binary trees, rotation distance, sorting by reversals, breakpoint graphs.
2000 Mathematics Subject Classification: Primary 03D15, Secondary 68R05.

1 Introduction: binary rooted trees and permutations

Planar rooted binary trees are connected graphs without cycles, with n trivalent vertices and $n+2$ univalent vertices, one of them being marked as the root. There are $\frac{1}{n+1}\binom{2n}{n}$ binary trees with n internal vertices.



Two trees are connected by an *elementary move* called rotation if they are identical, except the zones from the picture below, where one edge is moved into another position; we erase an internal edge and one internal vertex and we glue it again in a different location, to get a rooted binary tree.

The Rotation Graph G_n [9] has vertices all rooted binary trees with n internal vertices, two trees being adjacent if they are united by a rotation.

Let T be a rooted binary tree, with n internal vertices. The univalent vertices are labelled $1, 2, ...n + 1$.
The internal zone between the univalent vertices k and $k + 1$ is called zone k. Every internal zone has a unique internal vertex which is the head of that zone.

To every binary rooted tree with n internal vertices T, we associate a permutation $p(T) \in S_n$, the symmetric group of n letters. If the trees evolve by rotations, then $p(t)$ evolve by an insertion.
An insertion in a permutation a:= a(1)a(2)....a(n) is the following transformation applied to a: insert the element $a(y)$ between two consecutive elements in a: $a(x - 1)$ and $a(x)$. Under this transformation, we get the permutation $ar(x, y)$,

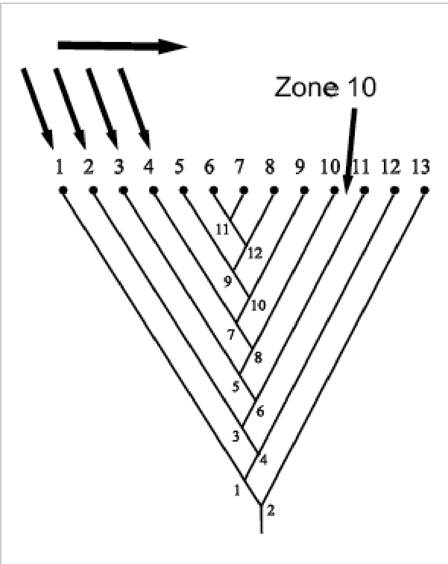
where $r(x, y)$ is the insertion:

$$\text{r(x,y)=} \begin{cases} \text{1,2,...x,x+1,.......y....n} \\ \text{1,2,...y,x,..........y-1...n} \end{cases}$$

There is a unique directed path $P(k)$ which joins every univalent vertex k to the root. We label internal vertices with numbers $1, 2...n$ in the order they appear on the paths $P(1)$, $P(2)...P(n+1)$, from the univalent vertices to the root. So we define a total order on the set of internal vertices: $x \leq y$ if $x \in P(k)$ and $y \in P(m)$ and $k \leq m$, or $k{=}m$ and y is on the path from x to the root of the tree.

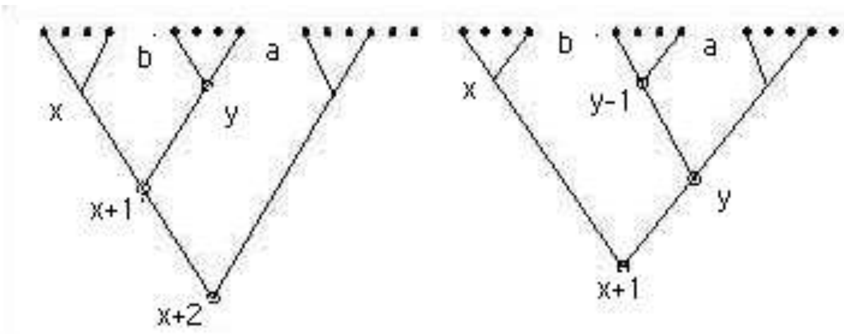
In the permutation associated with the tree, $p(T)(x) = y$, if the internal vertex labelled with x is the vertex associated with the zone y, between the univalent vertices y and $y + 1$.

Example: labelled tree and its permutation:1 12 2 11 3 10 4 9 5 8 6 7



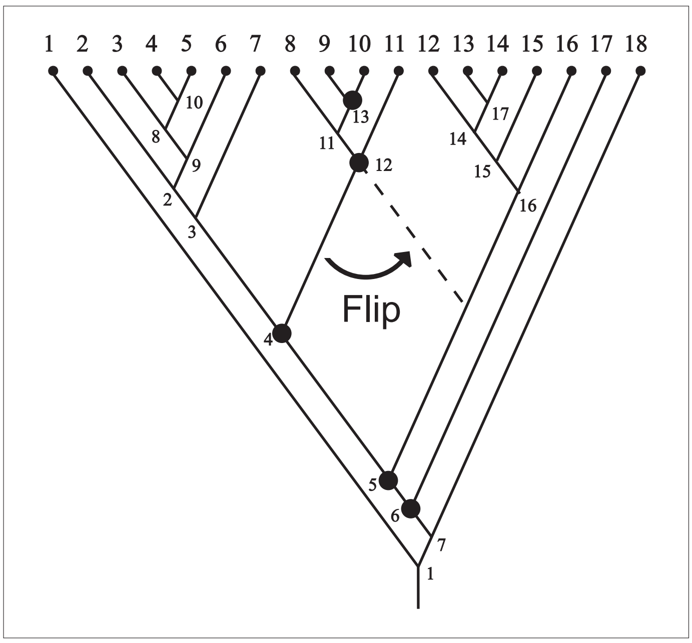
Lemma 1. *At a rotation , the permutation $p(T)$ evolves by an insertion.*

Local pictures of two trees united by a rotation; they are identical except the zones below:



Proof: Zone a, adjacent to zone b in $p(T)$ is moving under rotation , from the position $x+2$ to the position y (insertion). The insertion is from left to the right if the rotation is from left to the right. □

Remark 1. *An insertion involves maximum 5 adjacencies of a permutation; the zones are the 3 zones which characterize the flip:a, b and the zone whose internal vertex is y; and the first two zones, if they exist, which appear in \ direction, immediately after the vertices labelled $x+2$ and y (let us call α and β zones) in the picture above.*



p(T): 1 2 6 7 11 16 17 3 5 4 8 10 9 12 14 15 13
p(flip(T)) : 1 2 6 7 16 17 3 5 4 8 10 11 9 12 14 15 13
The zones involved : 7,11,10,16,9

1.1 The Breakpoint Graph of a permutation

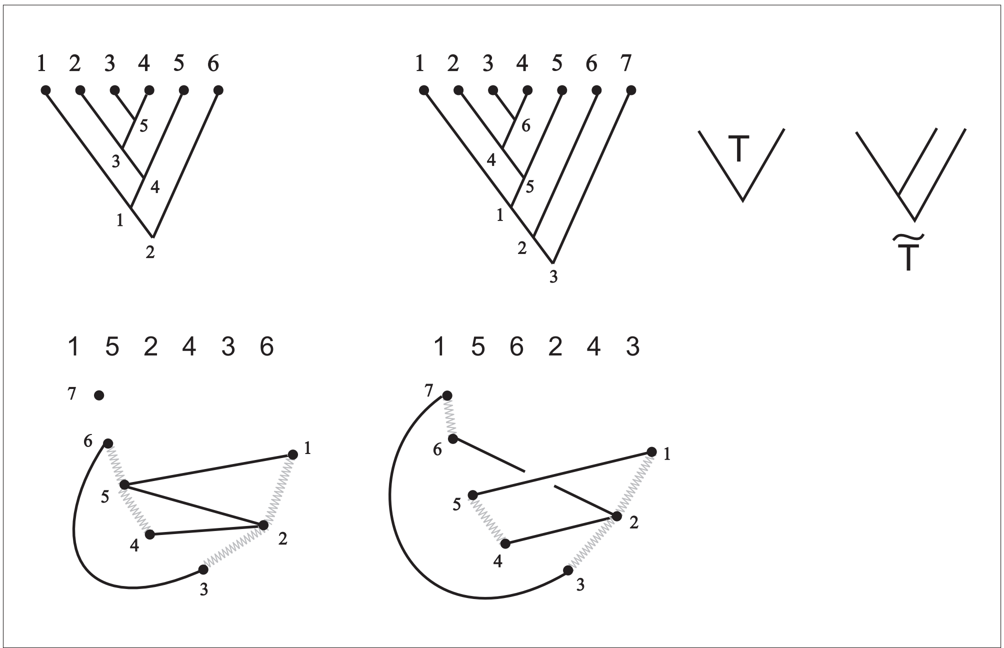
We associate to $p(T)$ its breakpoint graph, defined by Pevzner [2] [4]. Recall the breakpoint graph of a permutation p from the Symmetric Group $S(n)$:

We add 0 and $n+1$: p(0)=0 and $p(n+1) = n+1$; Let G be the graph with vertices $0, 1, 2...n+1$. **The vertices will be drawn equidistantly on a circle.** We join i and $i+1$ by a grey edge if they are not adjacent (on consecutive positions) in p. We join $p(i)$ and $p(i+1)$ by a black edge if they are not adjancent in the identity permutation. We get a bicolored graph, where every vertex has degree 0,2 or 4, with an equal number of grey and black edges. It means there is an Eulerian cycle decomposition in alternating color cycles. In our case, where $p = p(T)$, the vertices of $G(p(T))$ are the zones from the basis of the tree. The zone i is the zone between the univalent vertices i and $i+1$. $p(T)(1)$ is always 1, so 0 is not added. The notion of adjacency in the identity permutation and in $p(T)$ can be defined on the geometric planar representation of the tree. We write $G(T)$ or $G(p)$ or $G(p(T))$ for the breakpoint graph of a permutation associated with the tree T as above.

Definition 1. *The dimension of a tree is the number of its internal vertices.* $c(p)=|C(p)|$ *= the maximal number of alternating cycles in an Eulerian decomposition* $C(p)$ *of* $G(p)$*, among all possible Eulerian decompositions of* $G(p)$*.*

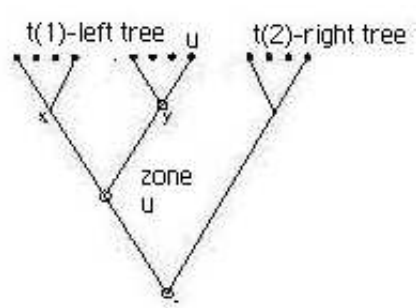
1.2 The "hook" transformation

Let T be a tree. Let *hook*(T) be the tree obtained from T by attaching a hook in the right hand side: so the left tree of *hook*(T) is T, and the right tree is one leaf with an univalent vertex. We add $n+1$ to $p(T)$. Then $p(hook(T))$ is built by insertion of $n+1$ in another place of $p(T)$.



Lemma 2. *Let T be an n-vertex tree, T=$t(1) \vee t(2)$ its decomposition in its left and right trees. Then $G(T)$ is the join of the graphs $G(hook(t(1)))$ and $G(t(2))$. The join is obtained by the identification of the vertex |t(1)| +2 from the first graph with the vertex labelled 1 from the second graph.*
We write $G(T)$=$G(hook(t(1)))\vee$ G(t(2))

Proof: $p(T)$=$p(hook(t(1))p(+t(2))$, where $p(+t(2))$ is obtained from $p(t(2))$ by adding dim($t(1)$)+1 to every number. U is the middle zone; p(u+1)=u+1



□

As a corollary, we have: There is a maximal cycle decomposition of $G(p(T))$ given by the decomposition of T in its left and right trees: if T=t(1)$\vee$ t(2), then $C(G(T))$=C(G(hook(t(1))))$\vee$C(G(t(2))).
We denoted by $C(p)$ an Eulerian decomposition with maximal number of alternating cycles.

Proof: In an Eulerian decomposition, there is no cycle which pass the zone u between $t(1)$ and $t(2)$, because there is no edge which connect two vertices $x < u$ and $y > u$ from the breakpoint graph.
Note. u+1 is a separating fixed point of the permutation $p(T)$: u+1 is a fixed point and $x < u$+1 if and only if $p(T)$(x)< u+1.

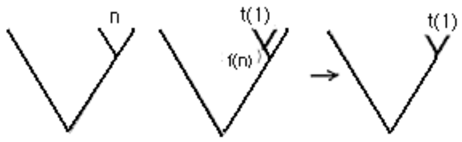
□

1.3 The Class "T" of permutations associated with trees

Let p be a permutation from the symmetric group $S(n)$.
We describe below a linear time algorithm to decide if p is a permutation associated with a tree (p is named a "tree permutation"), and if the answer is positive, we get a tree T such that $p(T) = p$. During this section we explain why this procedure return a correct answer for our question.

For every k from 1 to n, let q_k be the permutation obtained from p after we delete the numbers $k+1, k+2...n$. Define $f(k)$ to be q_k(x-1), if q_k(x)=k.

Lemma 3. *If the permutation p is a "tree permutation", then q_{n-1} is a "tree permutation".*



Proof: If zone n has a triangular shape as in the picture above, when one leaf bounds it, then q_{n-1} is the permutation of the tree obtained by erasing that leaf, connected with the n^{th} univalent vertex. Otherwise, we delete entire zone n and we glue the tree $t(1)$ (which form a part of the boundary of zone n) to the right wall of the tree T. We will get a tree with $n-1$ univalent vertices, whose permutation is q_{n-1}.
From a tree permutation q_{n-1}, we can build the permutation p if and only if zone $f(n)$ has its head (the trivalent vertex which is the head of the zone) on the right wall of the big tree T. The entire algorithm described in this section is based on the fact we can built p out of q_{n-1}.

From p, we delete the numbers 6,7,8...n. We will get a permutation z= q_5 from $S(5)$.
Case I- If there is no tree $T(5)$ with 5 internal vertices such that $p(T(5)) = z$, then p is not a tree permutation.
By applying the lemma above, we get: p is a tree permutation if and only if, for all k, q_k is a tree permutation. In particular, for k=5, we prove Case I.

Case II- Suppose there are trees $U(1)....U(q)$, q smaller than $\frac{1}{6}\binom{10}{5}$ such that $p(U(i))))$=z.

For every such tree $U(i)$ as above, we will try to add leaves (edges connected by an univalent vertex). Eventually by modifying the tree, after $n-5$ steps, we get a tree such that $p(T)$=p. For every step k, the tree T_k satisfies $p(T_k)$=q_{k+5}.

Let Q be a fixed tree with 5 internal vertices from the list above. At every step, we can decide if we have an obstruction which says that p is not a tree permutation, or we can move on to a tree with more internal vertices.

Suppose we have a tree T_{k-1} with $k-1$ internal vertices, such that p(T_{k-1})is the permutation obtained from p by deleting $k, k+1, k+2...n$. We want to add zone k. If the head of the zone $f(k)$ is not on the right wall of the tree, then we have an obstruction: p is not a tree permutation.

If $f(k)$ is on the right wall, we have two cases:
Case I: $f(k)$ is not k-1; there is a unique way to evolve from T_{k-1} to another tree which bear the permutation q_k obtained from p by deleting $k+1, k+2...n$. We reverse the procedure from Lemma 3, by adding zone k.
Case II: $f(k)$=$k-1$; we add a left to the right leaf to create zone k, by dividing zone $k-1$ in two zones, as in the first case of Lemma 3.
This choice does not affect the property to find or not to find an obstruction in the future. An obstruction appears at step M when $f(M)$ is not on the right wall of the tree. For any M, $f(M)$ is unique. The property of the internal vertex $f(M)$ of being on the right wall is not affected in case II by that choice of the leaf to create zone k.

□

Corollary 1. *There are at most* $\frac{1}{6}\binom{10}{5}\,2^{n-5}$ *trees which have the same permutation.*

Indeed, only in case II, which can be repeated at most $n-5$ times, there are two choices to add the leaf which creates zone k.

Definition 1.1. *Given a tree* T*, the set of its subtrees is defined recursively: the left and the right trees of* T *are subtrees. Any subtree is a left or a right tree for a subtree with more internal vertices.*

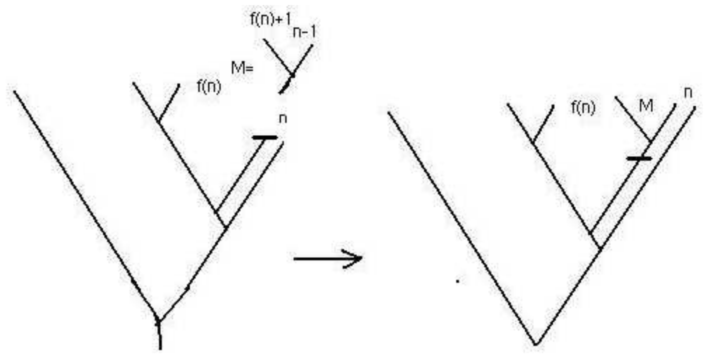
Note: The procedure above has a geometric character. Given p, we have to draw a tree at every stage, and to check if $f(k)$ is on the right wall of the tree.

Permutations from class T have the following property: they have separating fixed points, as described by the Note after Lemma 2. Also, the intervals $[f(k), k]$ are disjoint or are included one into the other. They can have one number in common (an end of an interval). They form a nested set of (non-crossing) intervals. Let us call a permutation which satisfy this condition a permutation of type (A).

Theorem 1. *A permutation* $p \in S_n$ *is a tree permutation if and only if* p *is of type* (A) .

Proof: We apply induction over n. If n is a fixed point of p, we apply induction for q_{n-1}. There is a tree whose permutation is q_{n-1}; we add one leaf to get p. If n is not a fixed point and $f(n) \leq n$-2, we apply induction for the permutation q, obtained from p by deleting the numbers: $f(n)+1....n-1$. There is a tree

$T(q)$ such that $p(T(q)) = q$. There is a tree M such that $p(M)$ is the permutation obtained from p by deleting the permutation q. We graft the trees $T(q)$ and M and we will get a tree whose permutation is p.



If $f(n) = n-1$, we consider $f(n-1)$=$f(f(n))$. If p is not the identity, after k iterations we get f^k(n)$\leq f^{k-1}$(n)-2; we apply induction for the permutation q, obtained from p by deleting the numbers: f^k(n)+1,... f^{k-1}(n)-1. There is also a tree M such that $p(M)$ is the permutation obtained from p by deleting the permutation q. We apply the grafting procedure above. □

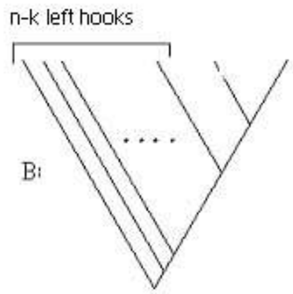
2 The breakpoint graphs of permutations associated with trees

2.1 The recursive approach

There is a linear procedure to decide if a permutation p from S_n is in class "T". We will see in this section that we can find a maximal cycle decomposition of $G(p(T))$ based on the subtrees of T (using Lemma 2), which suggest a polynomial time to find this decomposition. We will study families of trees, graded by the number of their internal vertices.

We analyze $G(p(T))$, for any tree T= $t(1) \vee t(2)$.
Let G_n be the set of breakpoint graphs which come from $p(T)$, for any tree T with n internal vertices. We describe the relation between G_n and G_k for all k $\leq n$.

G_k is included in G_n : we attach $n-k$ left hooks to any tree from G_k; its breakpoint graph will receive $n-k$ isolated points: $1, 2, 3...n-k$, which are fixed points for the tree permutations built in this way.



Case I) If $t(2)$ is not a leaf (a dimension 1 tree), Lemma 2 implies that $G(p(T))$ from G_n (having $n+1$ vertices) are formed by the identification of two endpoints (the join) of two graphs, one from G_k and the second one from G_{n-k} .

Case II) If $t(2)$ is a leaf, then T is the hook transformation of $t(1)$;$t(1)=t(3) \vee t(4)$.
We have two subcases:

Case II a) If $t(4)$ is not a leaf and $G(t(4))$ is non-trivial, we will prove that any maximal cycle decompositions of $G(T)=G(hook(t(1))$ can be build using maximal cycle decompositions the graphs of $hook(t(3))$ and $hook(t(4))$, which have dimensions lower than $G(T)$.

Notation: for any tree T, $v(T)=$ the zone which appears after the middle zone of T in $p(T)$
$u(T)=$ the middle zone of T

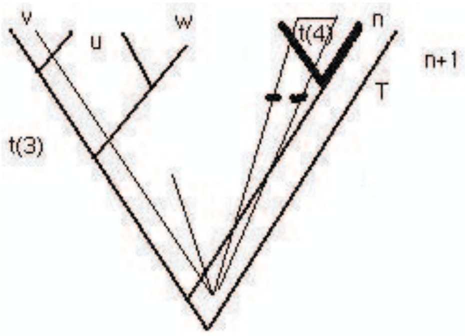
-The edges $nu(t(4))$ and $nv(t(4))$ appear in $G(hook(t(4))$, but they do not appear in $G(T)$. They are replaced by the edges $nv(t(3))$ and $nu(t(1))$. From a maximal cycle decomposition of $G(hook(t(4))$, we take all cycles, except the cycles (one or two) which contain the edges $nu(t(4)$ and $nv(t(4))$. Let us call the ignored cycles from $G(hook(t(4))$ C and D (they can be identical). C, D, the edge $u(t(4))v(t(4))$, $nv(t(3))$ and $nu(t(1))$ will be part of the same new cycle. The remaining edges of this new cycle are the edges of a cycle Y from $G(hook(t(3))$ which contain $u(T)v(t(3))$. This edge does not appear in $G(T)$. We replace $u(T)v(t(3))$ from Y by the sequence of edges: $u(T)n\ C\ u(t(4))v(t(4))\ D\ vt((3))$.
The edge $u(t(4))v(t(4))$ appears in $G(T)$. It does not appear in $G(hook(t(4))$. It is used to form a bridge between C and D.

Any maximal cycle decomposition of $G(T)$ is obtained in this way: $nu(t(1))$ and

$nv(t(3))$ have to belong to the same cycle X. The intersection between X and the vertices of the graphs of $hook(t(3))$ and $hook(t(4))$ has to generate maximal cycle decompositions of these graphs.

We have: $c(p(hook(t(3) \vee t(4)))) = c(p(hook(t(3)))) + c(p(hook(t(4))))$-$x$, where x is 1 or 2; x=1 if and only if C=D.

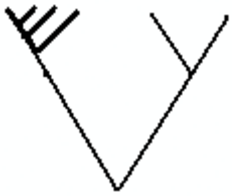
$c(p(hook(T \vee Q)))$ - $c(TvQ) = c(p(hook(Q)))$ - $c(p(Q))$ -x
c is the maximal number of alternating cycles in a decomposition of $G(x)$.



The breakpoint graph of a tree permutation can be drawn on the tree: the vertices are the zones from the base of the tree. In the picture above, all important edges mentioned above appear.
Zone n takes the role of the last vertex for $G(hook(t(4)))$. It is also a last vertex for $G(hook(t(3)))$, instead of the middle zone of $t(1)$ (which is connected with $v(t(3))$ in $G(hook(t(3))$, but not in $G(T)$). Two edges from the triangle above which contains n do not appear in $G(T)$. They appear in $G(hook(t(4))$.

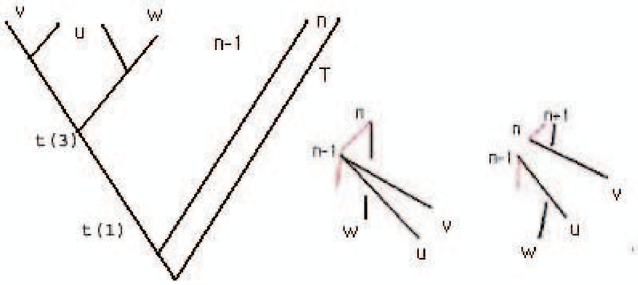
We can combine Theorem 1 with the recursive procedure above to find a maximal cycle decomposition of a tree permutation p. Instead of the geometric picture of a tree T which satisfy $p(T) = p$, we can use paranthetizations of $n + 1$ variables: for example, the inductive step of the proof of Theorem 1 says that we can put parantheses between $f(n) + 1....n - 1$. The middle zone of the tree is a point where there are $n - k$ left parantheses and k right parantheses, so we can find the parameter $u(T)$. Given p, the tree T is not unique.

Case II b) This is a case when we cannot really apply Case II a), and this case is possible when at least one of the edges $nu(t(4)), nv(t(4)), nu(t(3))$ and $nv(t(3))$ does not exist in $G(hook(t(4))$ or $G(hook(t(3))$. This is possible in the following situations:
a)$t(4)$ or $t(3)$ is a leaf.
b)$t(3)$ and $t(4)$ are not a leaves, and at least one of $t(3)$ or $t(4)$ has the following structure:



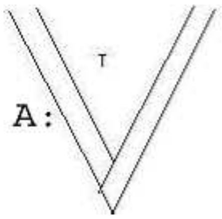
In the figure above, the right tree has dimension 0 or 1.
If the left tree of $t(4)$ (or $t(3)$) has dimension bigger than 1, $G(hook(t(4)))$ has isolated vertices and $G(T)$ will have the structure of the breakpoint graph of a lower dimension tree; so we can suppose that its dimension is 0 or 1.

If $t(4)$ is a leaf, then $G(T)$ is built from $G(t(1)) = G(hook(t(3)))$ (a lower dimension tree) using the folowing procedure:
Let us say that $t(3)$ has $n-1$ univalent vertices. The vertex n from $G(hook(t(3)))$ is split into 2 vertices, labelled n and $n+1$, which will receive half of the edges of n.



From Case II a), any maximal cycle decomposition of $G(hook(t(3))$ will have the vertices: u, v, $n-1$, n, w on the same cycle. This means that there is a bijection between the eulerian decompositions of $G(hook(t(3)))$ and $G(hook(hook(t(3)))) = G(T)$; in particular this is true for Eulerian decompositions with maximal number of cycles.

If $t(3)$ is a leaf, we are in the situation described in the figure below:



$p(A)$=0 n $p(T)$, where we labelled the first zone of the tree A with 0.
The vertices of $G(A)$ are 0,1,...$n+1$.
$p(T)$ begins with 1, so there is a black edge between 1 and n and between 0 and n and a grey edge between n and $n+1$ and between 0 and 1.

In $G(T)$, n is joined by a black edge with $p(T)(n-1) = v$. In $G(A)$, v is joined with $n+1$

Proposition 1. *A and T have the same structure of their breakpoint graph.*

The sequence of edges $[n-1, n][n, v]$ from a cycle decomposition of $G(T)$ is replaced by $n-1, n, 0, 1, n, n+1, v$ from $G(A)$.

The remaining cases, when $t(3)$ or $t(4)$ are not leaves, but have dimensions at most 3 can be treated similarly. At most one cycle of length 4 can differentiate the structure of the maximal Eulerian decomposition of
$G(hook(t(3)\vee t(4)))$ from $G(t(3) \vee t(4))$.

3 The structure of a maximal cycle decomposition of the breakpoint graph of a tree

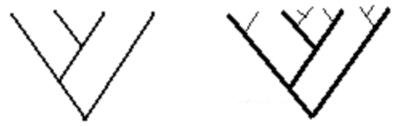
We define and identify several combinatorial pieces of a maximal cycle decomposition of $G(p(T))$. Their existence is based on the recursive approach above, on the geometry of the tree and on the fact that we draw the Breakpoint graph on the vertices of a regular $n+1$-gon. In $C(G(T))$ we will meet: long cycles, cycles of length 4, curls and twisted curls, triangles and bounded regions in the plane formed by a long cycles. The number of bounded regions formed by the long cycles is denoted $w(T)$, as a parameter of the maximal cycle decomposition of a $G(p)$, which depends on the tree T; $p(T)$=p. We also define a special class of trees called w-trees.

Definition 2. *A long cycle in a maximal decomposition of $G(p)$ is a cycle of length greater than 4, or a cycle of length 4 whose the remaining two edges (out of the 6 which can be formed by its vertices) are not in $G(p)$*

Definition 3. *A connected component (or structure) of a breakpoint graph is the union of a long cycle with the length 4 cycles (c4 for short) attached to it.*

Definition 4. *A w-tree T is a tree whose maximal cycle decomposition in alternating cycles of $G(T)$ has at most one long cycle, which bounds only one region in the plane, and all other cycles of length 4, if they exist, are attached by this long cycle. (they have two vertices in common with this long cycle).*

A w-structure is a minor of tree of the shape as in the picture below. Define $w(G)$ as the maximal number of w-structures which stay linearly and with disjoint interiors (un-nested) on the basis of the tree. A w-tree is a tree with $w(G)$=1.



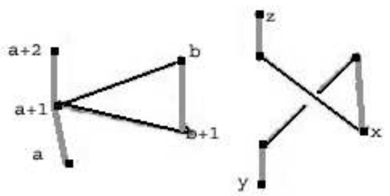
The best way to understand the w-trees is recursively, using left-right subtree decomposition. If t=$t(1) \vee t(2)$ is a w-tree, then exactly one from *hook*$(t(1))$ and $t(2)$ is a w-tree. The other one is a comb-tree.

Long cycles can have the following structures attached to them: triangles, twisted curls, and curls with one or two cycles of length 4 attached.

Definition 5. *A triangle in a maximal cycle decomposition of $G(T)$ is a sequence of 5 consecutive points from a long cycle, as in the figure below. The third point will be called the base of the triangle.*

Definition 6. *A twisted curl in a maximal cycle decomposition of $G(T)$ is a sequence of 6 consecutive points from a long cycle, as in the figure below. The midpoint of the second and the fifth point will be called the base of the twisted curl.*

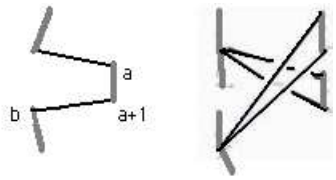
We draw the vertices of the breakpoint graph as the vertices of an n-gon. The grey edges will be some of the edges of the n-gon. The numbers (labels) of the vertices which form the basis of a movable structure are greater than the numbers of the vertices of the grey edge. In the figure below, $b+2 \leq a$ and $x+1 \leq y \leq z-1$.



Definition 7. *A movable structure is a triangle or a twisted curl.*

There are consequences of the inequalities among the numbers envolved in the geometry of the tree:

1)Two curls of the same connected structure do not intersect.
2)A sequence of 6 consecutive points of a long cycle as in the picture below cannot exist.
So, the following structures of a breakpoint graph drawn on the vertices of a regular n-gon do not exist on a long cycle in the case of permutations which came from trees.

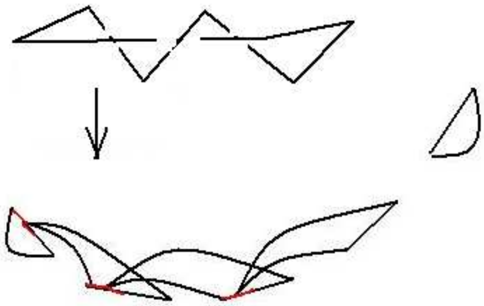


A curl (6 consecutive points of a long cycle) is twisted, or several points among these 6 points are also vertices for one or two cycles of length 4.

Theorem 2. *Let T be a tree. Let $w(T)$ be the maximal number of w-structures which stay linearly and un-nested on the basis of T. Then any maximal cycle decomposition of $G(T)$ will have the following structure:*
a)it will contain several long cycles, which are plane projections of 3-dimensional circles (trivial knots). The number of bounded regions is $w(T)$.

b)the long cycles contain triangles, triangular structures associated with a w-structure, twisted curls and curls whose walls can be diagonals for 1 or 2 cycles of length 4.

So, in a maximal Eulerian decomposition of the breakpoint graph of a tree there are two types of long cycles: cycles which bound only one region and cycles which bounds more than one region in the plane, forming a chain of triangular structures as in the figure below.



A long cycle which bounds several regions in the plane is the projection of a 3-dimensional knot. There is a linear order on the set of long cycles induced by the linear order of the w-structures of a tree.

Proof: We use induction over the dimension of the tree T, using left/right subtrees decomposition. If the right tree has dimension greater then 1, we apply induction using Lemma 2. The topological structure of $C(G(T))$ is given by the topological structures of smaller trees.
If the right tree is a leaf we apply the hook transformation analysis for the left tree Q. T=$hook(Q)$. T and Q have the same number of w-structures.
We saw how we get a maximal cycle decomposition of $hook(Q = A \vee B)$ from $hook(A)$ and $hook(B)$.
The difference between $w(Q)$ and the number of bounded regions formed by long cycles does not change by applying the hook transformation, establishing a). The structures of the long cycles of $hook(A)$ and $hook(B)$ (triangles and curls) will be found in $G(T)$, establishing b).

□

4 The evolution of rooted binary trees under rotations

The way rooted binary trees evolve under rotations is not fully understood. Recently, Kauffman, Eliahou and Kryuchkov [6] [7] reformulated the Four Color Theorem in terms of signed diagonal flips . Also, it is not known if the problem to compute the rotation distance between any two trees is NP. The initial approach of the Rotation Graph, made by Thurston, Sleator and Tarjan [9], used hyperbolic geometry.

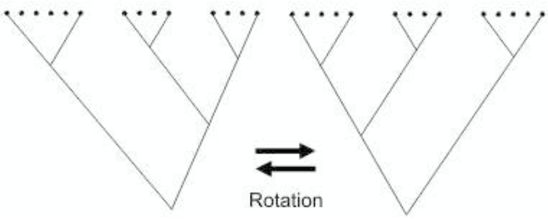
An insertion is a composition of 2 reversals. These transformations were already studied by Bafna, Pevzner and Christie [2] [3]in the case of Genome Rearrangements: genomes are modeled as permutations, which evolve under reversals, transpositions or mutations.

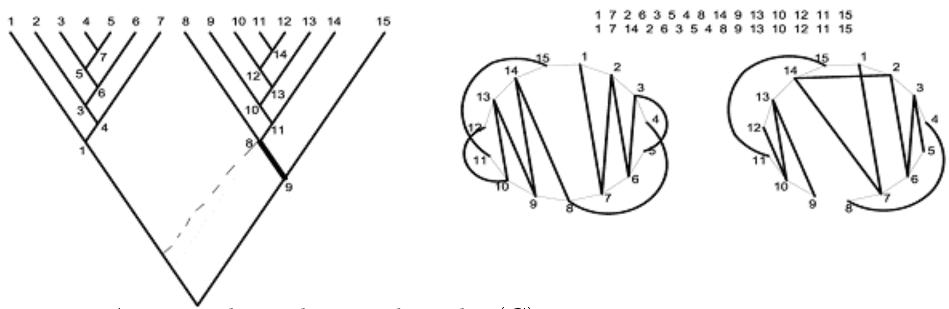
Theorem (Pevzner). Let p_1 and p_2 be permutations connected by a reversal. Let b_i and c_i be the number of black edges and of cycles in a maximal decomposition of G(p_i). Then:
$|\Delta\text{c}| \leq 1$
$|\Delta\text{b}| \leq 2$
$|\Delta\text{b-c}| \leq 1$, where Δx is the difference in the values of x for two permutations connected by a reversal.

Applying this theorem two times, in the case of 2 permutations of trees, we get the following theorem concerning the change of c and b at a rotation:
$|\Delta\text{c}| \leq 2$
$|\Delta\text{b}| \leq 3$(here, we use the fact that the permutations are associated with trees; at a rotation we cannot modify the number of black edges by 4)
$|\Delta\text{b-c}| \leq 2$

Lemma 4. *A rotation changes $w(G)$ by at most 1. It acts on maximum 2 connected structures associated to the first 2 subtrees of the rotation.*

Proof: A rotation modifies the adjacencies from the first 2 subtrees of the flip (the first two small subtrees in the picture below). $w(G)$ is changed if the moving internal edge of the tree will play a role in a w-structure.
We can apply induction over the dimension of the tree. If the figure below represent the tree, then $w(G)$ is changed by 1 if w(G)=1 for the tree of the righthand side. If the figure below represent just a local picture of the tree, then we apply induction for the (left or right) subtree which contain it.





A tree with one long cycle and $w(G)$=2

4.1 Final remarks

Lower bounds for the rotation distance between planar binary trees can be derived using the combinatorial structures above.The insertion distance is not strong enough to characterize the rotation distance. The breakpoint graph seems to be very useful to record structural changes under flips.

The recursive approach on the structure of the trees implies a polynomial time algorithm to find a maximal cycle decomposition of a permutation from class T: for every permutation p, we build a tree T whose permutation is p. This tree has at most n subtrees of dimension lower than 6, with very simple structure of their breakpoint graphs. Based on their relations in the big tree T, we can build a maximal Eulerian decomposition of $G(p)$.
It is also a challenge to enlarge the class T of permutations which are not covered

by Caprara's NP-result on maximal cycle decomposition. There is also the question if, given two permutations from class T, it is possible to find the breakpoint graph maximal decomposition in alternating cycles of $\mathrm{G}(g_1^{-1} \circ g_2)$ based on their associated trees.

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