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The Aleksandrov-Fenchel inequality for p-dual volumes

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Abstract

In this paper, we establish the Aleksandrov-Fenchel inequality for p-dual mixed volumes.

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One might say the history of intersection bodies began with the paper[1] of Busemann. Intersection bodies were first explicily defined and named by Lutwak[2]. It was here that the duality between intersection bodies and projection bodies was first made clear. Despite the considerable ingenuity of earlier attacks on the Busemann-Petty problem, it seems fair to say that the work [2] of Lutwak represents the beginning of its eventual solution. In [2], Lutwak also showed that if a convex body is sufficiently smooth and not an intersection body, then there exists a centred star body such that the conditions of Busemann-Petty problem holds, but the result inequality is reversed. Following Lutwak, the intersection body of order i of a star body is introduced by Zhang[3]. It follows from this definition that every intersection body of order i of a star body is an intersection body of a star body, and vice versa. As Zhang observes, the new definition of intersection body allows a more appealing formulation, namely: The Busemann-Petty problem has a positive answer in n-dimensional Euclidean space if and only if each centered convex body is an intersection body. The intersection body plays an essential role in Busemann's theory[4] of area in Minkowski spaces. The intersection body also is an important matter of the Brunn-Minkowski theory.

In recent years some author including Ball[5-6], Bourgain[7], Gardner[8-10], Schneider[11] and Lutwak[12-18] et al have given considerable attention to the Brunn-Minkowski theory and their various generalizations. The purpose of this

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paper first is to establish the Minkowski inequality for the dual Quermassintegral sum, which is a generalization of the Minkowski inequality for mixed intersection bodies. Then, the Brunn-Minkowski inequality and the Aleksandrov-Fenchel inequality for mixed intersection bodies are proved and some related results also be given. In this work we shall derive, for intersection bodies, all the analogous inequalities for Lutwak's mixed projection body inequalities[15]. Thus, this work may be seen as presenting additional evidence of the natural duality between intersection and projection bodies.

1 Notation and Preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathbb{C}^n$ denote the set of non-empty convex figures(compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathbb{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u. We use $V(K)$ for the n-dimensional volume of convex body K. The support function of $K \in \mathcal{K}^n$, $h(K, \cdot)$, defined on $\mathbb{R}^n$ by $h(K, \cdot) = Max\{x \cdot y : y \in K\}$. Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, its radial function $\rho(K, \cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by $\rho(K, u) = Max\{\lambda \geq 0 : \lambda u \in K\}$. If $\rho(K, \cdot)$ is positive and continuous, K will be called a star body. Let φ^n denote the set of star bodies in $\mathbb{R}^n$.

1.1 Dual mixed volumes

If $K_1, \ldots, K_r \in \varphi^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, then the radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, is defined by $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i\}$.

The following property will be used later. If $K, L \in \varphi^n$ and $\lambda, \mu \geq 0$

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot) \qquad (1.1.1).$$

For $K_1, \ldots, K_r \in \varphi^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski liner combination $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is a homogeneous nth-degree polynomial in the λ_i [11],

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum \tilde{V}_{i_1, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n} \qquad (1.1.2)$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in (1.1.2) to be symmetric in their arguments, then they are uniquely determined. The coefficient $\tilde{V}_{i_1,\ldots,i_n}$ is nonnegative and depends only on the bodies $K_{i_1},\ldots,K_{i_n}$. It is written as $\tilde{V}(K_{i_1},\ldots,K_{i_n})$ and is called the *dual mixed volume* of $K_{i_1},\ldots,K_{i_n}$. If $K_1=\cdots=K_{n-i}=K$, $K_{n-i+1}=\cdots=K_n=L$, the dual mixed volumes is written as $\tilde{V}_i(K,L)$. The dual mixed volumes $\tilde{V}_i(K,B)$ is written as $\tilde{W}_i(K)$.

Dual Quermassintegral is special case of the p-th dual volume:

$$\tilde{V}_p(K)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^p dS(u),\quad -\infty<p<\infty. \tag{1.1.3}$$

Taking for $p=n-i$ in $\tilde{V}_p(K)$, $\tilde{V}_p(K)$ changes to the well know dual Quermassintegral $\tilde{W}_i(K)$.Taking for $p=n$ in $\tilde{V}_p(K)$, $\tilde{V}_p(K)$ changes to the well know general volume $V_i(K)$.

If $K_i\in\varphi^n(i=1,2,\ldots,n-1)$, then the dual mixed volume of $K_i\cap E_u(i=1,2,\ldots,n-1)$ will be denoted by $\tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)$. If $K_1=\ldots=K_{n-1-i}=K$ and $K_{n-i}=\ldots=K_{n-1}=L$, then $\tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)$ is written $\tilde{v}_i(K\cap E_u,L\cap E_u)$. If $L=B$, then $\tilde{v}_i(K\cap E_u,B\cap E_u)$ is written $\tilde{w}_i(K\cap E_u)$.

1.2 Intersection bodies

For $K\in\varphi^n$, there is a unique star body IK whose radial function satisfies for $u\in S^{n-1}$,

$$\rho(IK,u)=v(K\cap E_u), \tag{1.2.1}$$

It is called the *intersection bodies* of K. From a result of Busemann, it follows that IK is a convex if K is convex and centrally symmetric with respect to the origin. Clearly any intersection body is centred.

Volume of the intersection bodies is given by $V(IK)=\frac{1}{n}\int_{S^{n-1}}v(K\cap E_u)^n dS(u)$.

The mixed intersection bodies of $K_1,\ldots,K_{n-1}\in\varphi^n$, $I(K_1,\ldots,K_{n-1})$, whose radial function is defined by

$$\rho(I(K_1,\ldots,K_{n-1}),u)=\tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u), \tag{1.2.2}$$

where $\tilde{v}$ is $(n-1)$-dimensional dual mixed volume.

If $K\in\varphi^n$ with $\rho(K,u)\in C(S^{n-1})$, and $i\in\mathbb{R}$ is positive, the *intersection body of order* i of K is the centered star body I_iK such that[3] $\rho(I_iK)=\frac{1}{n-1}\int_{S^{n-1}}\rho(K,u)^{n-i-1}dS(u)$, for $u\in S^{n-1}$, where $I_iK=I(\underbrace{K,\ldots,K}_{n-i-1},\underbrace{B,\ldots,B}_{i})$.

If $K_1=\cdots=K_{n-i-1}=K, K_{n-i}=\cdots=K_{n-1}=L$, then $I(K_1,\ldots,K_{n-1})$ is written as $I_i(K,L)$. If $L=B$, then $I_i(K,L)$ is written as I_iK is called the ith intersection body of K. For I_0K simply write IK. The term is introduced by Zhang[3].

2 Main results

2.1 Two Lemmas

The following results will be required to prove our main Theorems.

Lemma A *If* $K, L \in \varphi^n$, $-\infty < p < \infty$, *then*

$$\tilde{V}_p(I(K_1, \ldots, K_{n-1})) = \frac{1}{n}\int_{S^{n-1}} v(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)^p dS(u).$$

To prove this we use (1.1.3) in conjunction with the fact (1.2.2).

Lemma B[14] *If* $K_1, \ldots, K_n \in \varphi^n$, *then*

$$\tilde{V}(K_1, \ldots, K_n)^r \leq \prod_{j=1}^{r} \tilde{V}(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_n).$$

with equality if and only if $K_1, \ldots, K_n$ *are all dilations of each other.*

2.2 The Aleksandrov-Fenchel inequality for p-dual mixed volumes

The Aleksandrov-Fenchel inequality for mixed intersection bodies which be proven: If $K_1, \ldots, K_{n-1} \in \varphi^n$, then

$$V(I(K_1, \ldots, K_{n-1})) \leq \prod_{j=1}^{r} V(I(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}))$$

with equality if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

This is just a special case of the following result.

Theorem 2.2.1 *If* $K_1, \ldots, K_{n-1} \in \varphi^n$, $-\infty < p < \infty, 0 < j < n-1$ *and* $0 < r \leq n-1$ *then*

$$\tilde{V}_p(I(K_1, \ldots, K_{n-1}))^r \leq \prod_{j=1}^{r} \tilde{V}_p(I(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})). \qquad (2.2.1)$$

with equality if and only if $K_1, \ldots, K_{n-1}$ *are all dilations of each other.*

Proof: From (1.1.3) and (1.2.2), we have that

$$\tilde{V}_p(I(K_1, \ldots, K_{n-1})) = \frac{1}{n}\int_{S^{n-1}} \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)^p dS(u). \qquad (2.2.2)$$

By using the inequality in Lemma B, we easy get that

$$\tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)^{n-i} \leq$$

$$\left(\prod_{j=1}^{r}\tilde{v}(\underbrace{K_j\cap E_u,\ldots,K_j\cap E_u}_{r},K_{r+1}\cap E_u,\ldots,K_{n-1}\cap E_u)\right)^{\frac{n-i}{r}}, \qquad (2.2.3)$$

with equality if and only if $K_1\cap E_u,\ldots,K_{n-1}\cap E_u$ are all dilations of each other, it follows if and only if $K_1,\ldots,K_{n-1}$ are all dilations of each other.

On the other hand, the Hölder's inequality can be stated as

$$\int_{S^{n-1}}\prod_{i=1}^{m}f_i(u)dS(u)\le\prod_{i=1}^{m}\left(\int_{S^{n-1}}(f_i(u))^m dS(u)\right)^{1/m}, \qquad (2.2.4)$$

with equality if and only if all f_i are proportional.

From (2.2.2),(2.2.3) and (2.2.4), we obtain that

$$\tilde{V}_p(I(K_1,\ldots,K_{n-1}))=\frac{1}{n}\int_{S^(n-1)}\tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)^p dS(u)$$

$$\le\frac{1}{n}\int_{S^u}\left(\prod_{j=1}^{r}\tilde{v}(\underbrace{K_j\cap E_u,\ldots,K_j\cap E_u}_{r},K_{r+1}\cap E_u,\ldots,K_{n-1}\cap E_u)\right)^{\frac{p}{r}}dS(u)$$

$$\le\left(\prod_{j=1}^{r}\frac{1}{n}\int_{S^{n-1}}\tilde{v}(\underbrace{K_j\cap E_u,\ldots,K_j\cap E_u}_{r},K_{r+1}\cap E_u,\ldots,K_{n-1}\cap E_u)^p dS(u)\right)^{r}$$

$$=\left(\prod_{j=1}^{r}\tilde{V}_p(I\,(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1}))\right)^{r}.$$

In view of the equality conditions of (2.2.3) and (2.2.4), it follows that the equality holds if and only if $K_1,\ldots,K_{n-1}$ are all dilations of each other.

The proof is complete. □

Remark 2.2.1 From the case $r=n-1$ of inequality (2.2.1), it follows that

Corollary 2.2.1 *If $K_1,\ldots,K_{n-1}\in\varphi^n$, and $-\infty<p<\infty$ then*

$$\tilde{V}_p(I(K_1,\ldots,K_{n-1}))^{n-1}\le\tilde{V}_p(IK_1)\cdots\tilde{V}_p(IK_{n-1}), \qquad (2.2.5)$$

with equality if and only if $K_1,\ldots,K_{n-1}$ are all dilations of each other.

Taking $K_1=\cdots=K_r=K$, $K_r=L$, and $K_{r+1}=\cdots=K_{n-1}=B$ to (2.2.1), (2.2.1) changes to

Corollary 2.2.2 *If $K, L \in \varphi^n$, and $-\infty < p < \infty$ and $0 \leq j < n-1$, then*

$$\tilde{V}_p(I(\underbrace{K,\ldots,K}_{n-j-2},\underbrace{B,\ldots,B}_{j},L))^{n-j-1} \leq \tilde{V}_p(I_jK)^{n-j-2}V_p(I_jL),$$

with equality if and only if K and L are dilates.

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