

## The Reticulation of a Residuated Lattice

by  
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### Abstract

In this paper we define the reticulation of a residuated lattice, prove that it has “good properties“, present two constructions for it, prove its uniqueness up to an isomorphism, define the reticulation functor and give several examples of finite residuated lattices and their reticulations.

**Key Words:** Residuated lattice, reticulation, prime spectrum.

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### 1 Introduction

The reticulation of an algebra  $A$  is, basically, a pair  $(L(A), \lambda)$  consisting of a bounded distributive lattice  $L(A)$  and a surjection  $\lambda : A \rightarrow L(A)$  so that the function given by the inverse image of  $\lambda$  induces (by restriction) a homeomorphism of topological spaces between the prime spectrum of  $L(A)$  and that of  $A$ . This construction allows many properties to be transferred between  $L(A)$  and  $A$ .

The reticulation was first defined for commutative rings by Simmons ([14]) and it was extended by Belluce to non-commutative rings ([3]). As for the algebras of fuzzy logics, Belluce also constructed the reticulation of an MV-algebra ([2]), G. Georgescu defined the reticulation of a quantale ([6]) and L. Leuştean made this construction for BL-algebras ([11], [12]).

In the following we will define the reticulation of a residuated lattice, prove several properties of it and conclude by giving examples of reticulations of residuated lattices.

The present paper contains the following original results: an axiomatic purely algebraic definition of the reticulation (given by conditions 1)-5) of Section 3), in contrast with previous work, in which the reticulation was defined by its construction; the proof of the existence of the reticulation, accomplished through two constructions of the reticulation: one based on a certain congruence relation and

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\*Dedicated to the memory of my dear grandfather, Gheorghe Mureșan

one involving principal filters; the uniqueness of the reticulation, which is implied by its axiomatic definition; several examples of construction of the reticulation of a residuated lattice.

The main results of this paper are: Theorem 3.1 (which proves that the axiomatic algebraic definition of the reticulation that we give is in accordance with the general notion of reticulation), Theorem 4.1 (the first construction of the reticulation), Theorem 4.2 (the second construction of the reticulation), Theorem 4.3 (uniqueness of the reticulation) and Proposition 4.5 (which proves that the definition of the reticulation functor is correct).

## 2 Preliminaries

In this section we recall several basic notions from the theory of lattices and residuated lattices: filters, morphisms, the Stone topology (see also [7], [15], [13]). We are following the terminology of [15].

**Definition 2.1.** A (commutative) residuated lattice is an algebraic structure  $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ , with the first 4 operations binary and the last two constants, so that  $(A, \vee, \wedge, 0, 1)$  is a bounded lattice,  $(A, \odot, 1)$  is a commutative monoid and the following property, called residuation, is satisfied: for all  $a, b, c \in A$ ,  $a \leq b \rightarrow c \Leftrightarrow a \odot b \leq c$ , where  $\leq$  is the partial order of the lattice  $(A, \vee, \wedge, 0, 1)$  (for each  $x, y \in A$ ,  $x \leq y$  iff  $x = x \wedge y$  iff  $y = x \vee y$ ).

The proof of the following remark can be found in [10], [7], [13], [15].

**Remark 2.1.** Let  $A$  be a residuated lattice and  $a, b, c, d \in A$ . Then:

- (i) if  $a \leq b$ , then  $a \odot c \leq b \odot c$ ;
- (ii) if  $a \leq b$  and  $c \leq d$ , then  $a \odot c \leq b \odot d$ ;
- (iii)  $a \odot b \leq a \wedge b$ ;
- (iv)  $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$ .

**Notation 2.1.** Let  $A$  be a residuated lattice and  $a \in A$ . We shall denote by  $a^n$  the following element of  $A$ :  $\underbrace{a \odot \dots \odot a}_{n \text{ of } a}$ .

**Definition 2.2.** Let  $A$  be a residuated lattice. A nonempty subset  $F$  of  $A$  is called a filter of  $A$  iff it satisfies the following conditions:

- (i)  $(\forall x, y \in F) x \odot y \in F$ ;
- (ii)  $(\forall x \in F)(\forall y \in A)$  if  $x \leq y$  then  $y \in F$ .

The set of all filters of  $A$  is denoted  $\mathcal{F}(A)$ .

A filter  $F$  of  $A$  is said to be proper iff  $F \neq A$ .

**Definition 2.3.** Let  $A$  be a residuated lattice. A filter  $P$  of  $A$  is called a prime filter iff  $P \neq A$  and, for all  $x, y \in A$ , if  $x \vee y \in P$ , then  $x \in P$  or  $y \in P$ . The set of all prime filters of  $A$  is called the (prime) spectrum of  $A$  and is denoted  $\text{Spec}(A)$ .

**Notation 2.2.** Let  $A$  be a residuated lattice. For any  $X \subseteq A$ , we shall denote  $D(X) = \{P \in \text{Spec}(A) \mid X \not\subseteq P\}$ . For any  $a \in A$ ,  $D(\{a\})$  will be denoted  $D(a)$ .

**Definition 2.4.** For any residuated lattice  $A$ , the family  $\{D(X) \mid X \subseteq A\}$  is a topology on  $\text{Spec}(A)$ , called the Stone topology.

**Proposition 2.1.** Let  $A$  be a residuated lattice. Then the family  $\{D(a) \mid a \in A\}$  is a basis for the Stone topology.

The proof of the previous proposition can be found in [11], [12], where it is proven for BL-algebras; the proof is valid also for residuated lattices.

**Definition 2.5.** Let  $A$  be a residuated lattice. A filter  $M$  of  $A$  is called a maximal filter iff it is a maximal element of the set of all proper filters of  $A$ . The set of all maximal filters of  $A$  is called the maximal spectrum of  $A$  and is denoted  $\text{Max}(A)$ .

**Notation 2.3.** Let  $A$  be a residuated lattice. For any  $X \subseteq A$ , we shall denote  $D_{\text{Max}}(X) = \{P \in \text{Max}(A) \mid X \not\subseteq P\}$ . For any  $a \in A$ ,  $D_{\text{Max}}(\{a\})$  will be denoted  $D_{\text{Max}}(a)$ .

It is a known fact that any maximal filter of a residuated lattice is a prime filter. Therefore, if  $A$  is a residuated lattice, then the following family of filters is a topology on  $\text{Max}(A)$ :  $\{D_{\text{Max}}(X) \mid X \subseteq A\}$ ; this is the topology induced on  $\text{Max}(A)$  by the Stone topology. The proposition below follows.

**Proposition 2.2.** Let  $A$  be a residuated lattice. Then the family  $\{D_{\text{Max}}(a) \mid a \in A\}$  is a basis for the topology induced on  $\text{Max}(A)$  by the Stone topology.

The following theorem is well-known (its proof can be found, for example, in [13]).

**Theorem 2.1** (the prime filter theorem). Let  $A$  be a residuated lattice,  $F$  be a filter of  $A$  and  $a \in A \setminus F$ . Then there exists a prime filter of  $A$  that includes  $F$  and does not contain  $a$ .

An easy to prove corollary of the prime filter theorem (Theorem 2.1) is the following.

**Corollary 2.1.** Any filter of a residuated lattice is equal to the intersection of the prime filters that include it.

It is obvious by the definition of a filter that the intersection of an arbitrary family of filters of a residuated lattice is a filter. This ensures us of the correctness of the following definition.

**Definition 2.6.** Let  $A$  be a residuated lattice,  $X \subseteq A$  and  $a \in A$ . The least filter of  $A$  that includes  $X$  (that is: the intersection of all filters of  $A$  that include  $X$ ) is called the filter of  $A$  generated by  $X$  and is denoted by  $\langle X \rangle$ . The filter of  $A$  generated by  $\{a\}$  is denoted by  $\langle a \rangle$  and is called the principal filter of  $A$  generated by  $a$ .

**Lemma 2.1.** *Let  $A$  be a residuated lattice and  $a \in A$ . Then  $\langle a \rangle = \{b \in A \mid (\exists n \in \mathbb{N}^*) a^n \leq b\}$ .*

**Proof:** Let  $F = \{b \in A \mid (\exists n \in \mathbb{N}^*) a^n \leq b\}$ . From Definition 2.6, we deduce that  $\langle a \rangle \subseteq F$ . Therefore it is sufficient to prove that  $F$  is a filter. Let  $b, c \in F$ . So  $(\exists n, m \in \mathbb{N}^*) a^n \leq b$  and  $a^m \leq c$ . By Remark 2.1, (ii), this implies that  $a^{n+m} \leq b \odot c$ . So  $b \odot c \in F$ . Let  $b \in F$  (so  $(\exists n \in \mathbb{N}^*) a^n \leq b$ ) and  $c \in A$  so that  $b \leq c$ . Then  $a^n \leq c$ , so  $c \in F$ . Hence  $F$  is a filter of  $A$ .  $\square$

**Definition 2.7.** *Let  $A$  and  $B$  be two residuated lattices. A function  $h : A \rightarrow B$  is called a morphism of residuated lattices iff it is a morphism of bounded lattices and a morphism of monoids and it satisfies:  $(\forall a, b \in A) h(a \rightarrow b) = h(a) \rightarrow h(b)$ .*

**Definition 2.8.** *Let  $L$  be a lattice. A nonempty subset  $F$  of  $L$  is called a (lattice) filter of  $L$  iff it satisfies the following conditions:*

- (i)  $(\forall x, y \in F) x \wedge y \in F$ ;
- (ii)  $(\forall x \in F)(\forall y \in L)$  if  $x \leq y$  then  $y \in F$ .

*The set of all filters of  $L$  is denoted  $\mathcal{F}(L)$ .*

*A lattice filter  $F$  of  $L$  is said to be proper iff  $F \neq L$ .*

**Definition 2.9.** *Let  $L$  be a lattice. A lattice filter  $P$  of  $L$  is called a prime lattice filter iff  $P \neq L$  and, for all  $x, y \in L$ , if  $x \vee y \in P$ , then  $x \in P$  or  $y \in P$ . The set of all prime lattice filters of  $L$  is called the (prime) spectrum of  $L$  and is denoted  $\text{Spec}(L)$ .*

**Notation 2.4.** *Let  $L$  be a lattice. For any  $X \subseteq L$ , we shall denote  $D(X) = \{P \in \text{Spec}(L) \mid X \not\subseteq P\}$ . For any  $a \in L$ ,  $D(\{a\})$  will be denoted  $D(a)$ .*

**Definition 2.10.** *For any lattice  $L$ , the family  $\{D(X) \mid X \subseteq L\}$  is a topology on  $\text{Spec}(L)$ , called the Stone topology.*

**Proposition 2.3.** *Let  $L$  be a lattice. Then the family  $\{D(l) \mid l \in L\}$  is a basis for the Stone topology.*

**Proof:** Analogous to the corresponding proof for residuated lattices.  $\square$

**Definition 2.11.** *Let  $L$  be a lattice. A lattice filter  $M$  of  $L$  is called a maximal lattice filter iff it is a maximal element of the set of all proper lattice filters of  $L$ . The set of all maximal lattice filters of  $L$  is called the maximal spectrum of  $L$  and is denoted  $\text{Max}(L)$ .*

**Notation 2.5.** *Let  $L$  be a lattice. For any  $X \subseteq L$ , we shall denote  $D_{\text{Max}}(X) = \{P \in \text{Max}(L) \mid X \not\subseteq P\}$ . For any  $l \in L$ ,  $D_{\text{Max}}(\{l\})$  will be denoted  $D_{\text{Max}}(l)$ .*

It is a known fact from lattice theory that any maximal (lattice) filter of a distributive lattice is prime. Therefore, if  $L$  is a distributive lattice, then the following family of lattice filters is a topology on  $\text{Max}(L)$ :  $\{D_{\text{Max}}(X) \mid X \subseteq L\}$ ; this is the topology induced on  $\text{Max}(L)$  by the Stone topology. The proposition below follows.

**Proposition 2.4.** *Let  $L$  be a distributive lattice. Then the family  $\{D_{\text{Max}}(l) \mid l \in L\}$  is a basis for the topology induced on  $\text{Max}(L)$  by the Stone topology.*

### 3 An axiomatic algebraic definition of the reticulation

In this section we give a definition of the reticulation of a residuated lattice and we prove that any object that satisfies this definition has “good properties“ of similarity between its structure and that of the residuated lattice it is associated with.

**Notation 3.1.** *For any function  $f : X \rightarrow Y$ , we shall denote by  $f^* : P(Y) \rightarrow P(X)$  the function given by its inverse image:  $(\forall M \in P(Y)) f^*(M) = f^{-1}(M)$ .*

The proof of the following remark is easy to obtain.

**Remark 3.1.** *For any function  $f : X \rightarrow Y$ ,  $f$  is surjective iff  $f^*$  is injective.*

Until specified otherwise,  $A$  will be a residuated lattice,  $L$  a bounded distributive lattice and  $\lambda : A \rightarrow L$  a function with the following properties:

- 1)  $(\forall a, b \in A) \lambda(a \odot b) = \lambda(a) \wedge \lambda(b)$ ;
- 2)  $(\forall a, b \in A) \lambda(a \vee b) = \lambda(a) \vee \lambda(b)$ ;
- 3)  $\lambda(0) = 0$ ;  $\lambda(1) = 1$ .

**Lemma 3.1.** *A function  $\lambda$  that verifies conditions 1)-3) also satisfies:*

- a)  $\lambda$  is order-preserving;
- b)  $(\forall a, b \in A) \lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$ ;
- c)  $(\forall a \in A)(\forall n \in \mathbb{N}^*) \lambda(a^n) = \lambda(a)$ .

**Proof:** a) Let  $a, b \in A$  so that  $a \leq b \Leftrightarrow b = a \vee b \Rightarrow \lambda(b) = \lambda(a \vee b)$ , so, applying condition 2),  $\lambda(b) = \lambda(a) \vee \lambda(b) \Leftrightarrow \lambda(a) \leq \lambda(b)$ .

b) Let  $a, b \in A$ . From Remark 2.1, (iii), we have  $a \odot b \leq a \wedge b$ . Using a) we get  $\lambda(a \odot b) \leq \lambda(a \wedge b)$ , and using 1) we get  $\lambda(a) \wedge \lambda(b) \leq \lambda(a \wedge b)$ . But  $a \wedge b \leq a$  and  $a \wedge b \leq b$ , so, by a),  $\lambda(a \wedge b) \leq \lambda(a)$  and  $\lambda(a \wedge b) \leq \lambda(b)$ , therefore  $\lambda(a \wedge b) \leq \lambda(a) \wedge \lambda(b)$ . This and the converse inequality above give  $\lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$ .

c) For all  $a \in A$  and  $n \in \mathbb{N}^*$ ,  $\lambda(a^n) = \lambda(a \odot \dots \odot a) = \lambda(a) \wedge \dots \wedge \lambda(a) = \lambda(a)$ . We applied condition 1).  $\square$

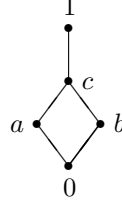
Let us add the following condition on  $\lambda$ :

- 4)  $\lambda$  is surjective.

This condition implies that  $\lambda^*$  is injective, as shown by Remark 3.1.

**Remark 3.2.** *Condition 4) is independent of conditions 1)-3).*

*Proof.* Let  $A = \{0, a, b, c, 1\}$  be the residuated lattice given by:



→	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

and  $\odot = \wedge$  (see [8]).

Let  $L = \{0, x, y, 1\}$  be the linearly ordered bounded distributive lattice given by:  $0 < x < y < 1$  and  $\lambda : A \rightarrow L$ , defined by:  $\lambda(0) = \lambda(a) = 0$ ,  $\lambda(b) = \lambda(c) = x$ ,  $\lambda(1) = 1$ .

A simple verification shows that  $\lambda$  satisfies conditions 1) and 2) and one can see it also satisfies condition 3), but it does not satisfy condition 4).  $\square$

**Proposition 3.1.** *For each  $F \in \mathcal{F}(L)$ , we have  $\lambda^*(F) \in \mathcal{F}(A)$ .*

**Proof:** Let  $F \in \mathcal{F}(L)$ . Then  $F$  is nonempty, so, by condition 4),  $\lambda^*(F)$  is nonempty. Let  $a, b \in \lambda^*(F) = \lambda^{-1}(F)$ .  $\Leftrightarrow \lambda(a), \lambda(b) \in F$ . Since  $F$  is a lattice filter of  $L$ , we get  $\lambda(a) \wedge \lambda(b) \in F$ , so, by 1),  $\lambda(a \odot b) \in F \Leftrightarrow a \odot b \in \lambda^*(F)$ . Let  $a \in \lambda^*(F)$  ( $\Leftrightarrow \lambda(a) \in F$ ) and  $b \in A$  so that  $a \leq b$ . a) gives  $\lambda(a) \leq \lambda(b)$ . But  $\lambda(a) \in F$  and  $F$  is a filter of  $L$ , so  $\lambda(b) \in F \Leftrightarrow b \in \lambda^*(F)$ . So  $\lambda^*(F)$  is a filter of  $A$ .  $\square$

**Proposition 3.2.** *For each  $P \in \text{Spec}(L)$ , we have  $\lambda^*(P) \in \text{Spec}(A)$ .*

**Proof:** By Proposition 3.1,  $\lambda^*(P)$  is a filter of  $A$ . The injectivity of  $\lambda^*$  and the fact that  $P \neq L$  imply  $\lambda^*(P) \neq A$ , because, if  $\lambda^*(P) = A$  then, since  $\lambda^*(L) = A$ , we have  $P = L$ , which is a contradiction. Let  $a, b \in A$  so that  $a \vee b \in \lambda^*(P)$ .  $\Leftrightarrow \lambda(a \vee b) \in P$ . By 2) we get  $\lambda(a) \vee \lambda(b) \in P$ , so, since  $P$  is a prime lattice filter of  $L$ , we have:  $\lambda(a) \in P$  or  $\lambda(b) \in P$ , which is equivalent to  $a \in \lambda^*(P)$  or  $b \in \lambda^*(P)$ . Hence  $\lambda^*(P)$  is a prime filter of  $A$ .  $\square$

According to the previous proposition, we may consider the following restriction of  $\lambda^*: \text{Spec}(L) \rightarrow \text{Spec}(A)$ .

Let us add the following condition on  $\lambda$ :

5) for all  $a, b \in A$ ,  $\lambda(a) \leq \lambda(b)$  iff  $(\exists n \in \mathbb{N}^*)a^n \leq b$ .

**Remark 3.3.** *Condition 5) is independent of conditions 1)-4).*

**Proof:** Let  $A$  be the residuated lattice from the proof of Remark 3.2.

Let  $L = \{0, x, 1\}$  be the linearly ordered bounded distributive lattice given by:  $0 < x < 1$  and  $\lambda : A \rightarrow L$ , defined by:  $\lambda(0) = \lambda(a) = 0$ ,  $\lambda(b) = \lambda(c) = x$ ,  $\lambda(1) = 1$ .

A simple verification shows that  $\lambda$  satisfies conditions 1) and 2) and one can see it also satisfies conditions 3) and 4).

Let us notice that, since  $\odot = \wedge$ , for any  $n \in \mathbb{N}^*$ ,  $a^n = a$ . We have:  $\lambda(a) = 0 \leq x = \lambda(b)$ , but  $a \not\leq b$ . So  $\lambda$  does not satisfy condition 5).  $\square$

**Lemma 3.2.** *Let  $F \in \mathcal{F}(A)$  and  $a, b \in A$  so that  $\lambda(a) = \lambda(b)$ . Then:  $a \in F \Leftrightarrow b \in F$ .*

**Proof:** Assume  $a \in F$ .  $\lambda(a) = \lambda(b) \Rightarrow \lambda(a) \leq \lambda(b) \Leftrightarrow (\exists n \in \mathbb{N}^*)a^n \leq b$ . We used 5). But  $a \in F \Rightarrow a^n \in F$ , so  $b \in F$ . Using  $\lambda(b) \leq \lambda(a)$ , one gets the proof of the converse implication.  $\square$

We could have proven Lemma 3.2 by using the following lemma.

**Lemma 3.3.** *For all  $a, b \in A$ ,  $\lambda(a) = \lambda(b)$  iff  $\langle a \rangle = \langle b \rangle$ .*

**Proof:** Let  $a, b \in A$ .  $\lambda(a) = \lambda(b)$  iff  $\lambda(a) \leq \lambda(b)$  and  $\lambda(b) \leq \lambda(a)$  iff (by condition 5))  $(\exists n \in \mathbb{N}^*)a^n \leq b$  and  $(\exists n \in \mathbb{N}^*)b^n \leq a$  iff (by Lemma 2.1)  $b \in \langle a \rangle$  and  $a \in \langle b \rangle$  iff (by Definition 2.6)  $\langle b \rangle \subseteq \langle a \rangle$  and  $\langle a \rangle \subseteq \langle b \rangle$  iff  $\langle a \rangle = \langle b \rangle$ .  $\square$

**Lemma 3.4.** *For any filter  $F$  of  $A$  and any  $a \in A$ , we have:  $\lambda(a) \in \lambda(F)$  iff  $a \in F$ .*

**Proof:** Let  $a \in A$  so that  $\lambda(a) \in \lambda(F)$ . Then there exists  $b \in F$  so that  $\lambda(a) = \lambda(b)$ . This, by Lemma 3.2, implies that  $a \in F$ . The converse implication is obvious.  $\square$

**Lemma 3.5.** *For any  $F \in \mathcal{F}(A)$ ,  $\lambda^*(\lambda(F)) = F$ .*

**Proof:** Let  $a \in A$ . By Lemma 3.4,  $a \in F$  iff  $\lambda(a) \in \lambda(F)$  iff  $a \in \lambda^{-1}(\lambda(F))$  iff  $a \in \lambda^*(\lambda(F))$ . Hence  $F = \lambda^*(\lambda(F))$ .  $\square$

**Lemma 3.6.** *For any filter  $F$  of  $A$ ,  $\lambda(F)$  is a lattice filter of  $L$ .*

**Proof:** For all  $a, b \in F$ ,  $a \odot b \in F$ , so  $\lambda(a \odot b) = \lambda(a) \wedge \lambda(b) \in \lambda(F)$ . Let  $a \in F$  and  $l \in L$  so that  $\lambda(a) \leq l$ ;  $\lambda$  is surjective (condition 4), so there exists  $b \in A$  so that  $\lambda(b) = l$ ; hence  $\lambda(a) \leq \lambda(b)$ ; then, by condition 5), there exists  $n \in \mathbb{N}^*$  so that  $a^n \leq b$ . But  $a^n \in F$ , so  $b \in F$ , so  $\lambda(b) \in \lambda(F)$ . Therefore  $\lambda(F) \in \mathcal{F}(L)$ .  $\square$

**Lemma 3.7.** *For any  $Q \in \text{Spec}(A)$ ,  $\lambda(Q) \in \text{Spec}(L)$ .*

**Proof:** Let  $Q \in \text{Spec}(A)$ . Lemma 3.6 shows that  $\lambda(Q)$  is a lattice filter of  $L$ . But  $Q$  is a prime filter of  $A$ , so it is a proper filter of  $A$ , hence there exists  $a \in A$  so that  $a \notin Q$ . By Lemma 3.4, this implies that  $\lambda(a) \notin \lambda(Q)$ , so  $\lambda(Q)$  is a proper lattice filter of  $L$ . Let  $l, m \in L$  so that  $l \vee m \in \lambda(Q)$ .  $\lambda$  is surjective, so  $(\exists a, b \in A)\lambda(a) = l, \lambda(b) = m$ . Therefore  $\lambda(a) \vee \lambda(b) \in \lambda(Q)$ .  $\Leftrightarrow \lambda(a \vee b) \in \lambda(Q) \Leftrightarrow (\exists c \in Q)\lambda(a \vee b) = \lambda(c)$ , which is equivalent to  $a \vee b \in Q$ , as Lemma 3.2 shows. But  $Q$  is a prime filter of  $A$ , therefore  $a \in Q$  or  $b \in Q$ , so  $\lambda(a) \in Q$  or  $\lambda(b) \in Q$ , i. e.  $l \in Q$  or  $m \in Q$ . So  $\lambda(Q) \in \text{Spec}(L)$ .  $\square$

**Proposition 3.3.**  *$\lambda^* : \text{Spec}(L) \rightarrow \text{Spec}(A)$  is surjective.*

**Proof:** Let  $Q \in \text{Spec}(A)$ . According to Lemma 3.7, we have  $\lambda(Q) \in \text{Spec}(L)$ , and, according to Lemma 3.5,  $\lambda^*(\lambda(Q)) = Q$ . This proves that  $\lambda^* : \text{Spec}(L) \rightarrow \text{Spec}(A)$  is surjective.  $\square$

**Proposition 3.4.**  *$\lambda^* : \text{Spec}(L) \rightarrow \text{Spec}(A)$  is continuous and open.*

**Proof:** Let  $a \in A$ .

$$\begin{aligned} (\lambda^*)^{-1}(D(a)) &= \{P \in \text{Spec}(L) \mid \lambda^*(P) \in D(a)\} = \\ &= \{P \in \text{Spec}(L) \mid \lambda^{-1}(P) \in D(a)\} = \\ &= \{P \in \text{Spec}(L) \mid a \notin \lambda^{-1}(P)\} = \\ &= \{P \in \text{Spec}(L) \mid \lambda(a) \notin P\} = D(\lambda(a)). \end{aligned}$$

Therefore  $\lambda^*$  is continuous.

Let  $l \in L$ . Since  $\lambda$  is surjective, there exists  $a \in A$  so that  $\lambda(a) = l$ .

$$\begin{aligned} \lambda^*(D(l)) &= \lambda^*(D(\lambda(a))) = \{\lambda^*(P) \mid P \in D(\lambda(a))\} = \\ &= \{\lambda^*(P) \mid P \in \text{Spec}(L), \lambda(a) \notin P\} = \\ &= \{\lambda^*(P) \mid P \in \text{Spec}(L), a \notin \lambda^{-1}(P)\} = \\ &= \{\lambda^*(P) \mid P \in \text{Spec}(L), a \notin \lambda^*(P)\} = \end{aligned}$$

$$= \{Q \in \lambda^*(\text{Spec}(L)) \mid a \notin Q\}.$$

But  $\lambda^* : \text{Spec}(L) \rightarrow \text{Spec}(A)$  is surjective, so this last set is equal to:

$$\{Q \in \text{Spec}(A) \mid a \notin Q\} = D(a).$$

Hence  $\lambda^*$  is open. □

Summing all of the above, we obtain the following theorem.

**Theorem 3.1.** *For any function  $\lambda$  satisfying conditions 1)-5), the function  $\lambda^* : \text{Spec}(L) \rightarrow \text{Spec}(A)$  is a homeomorphism of topological spaces.*

The previous results suggest the following definition of the reticulation of a residuated lattice.

**Definition 3.1.** *Let  $A$  be a residuated lattice. A reticulation of  $A$  is a pair  $(L, \lambda)$ , where  $L$  is a bounded distributive lattice and  $\lambda : A \rightarrow L$  is a function that satisfies conditions 1)-5).*

For the rest of this section, let  $A$  be a residuated lattice and  $(L, \lambda)$  be a reticulation of  $A$ .

For any filter  $F$  of  $A$ , let us denote  $L(F) = \{\lambda(a) \mid a \in F\} = \lambda(F)$ . By Lemma 3.6, we have defined a function  $L : \mathcal{F}(A) \rightarrow \mathcal{F}(L)$ . We consider that there is no confusion between the function  $L$  and the lattice  $L$ .

By Proposition 3.1, we have a mapping  $\lambda^*$  from  $\mathcal{F}(L)$  to  $\mathcal{F}(A)$ .

**Lemma 3.8.** (i) *For any  $F \in \mathcal{F}(A)$ ,  $\lambda^*(L(F)) = F$ .*  
(ii) *For any  $H \in \mathcal{F}(L)$ ,  $L(\lambda^*(H)) = H$ .*

**Proof:** (i) This is exactly Lemma 3.5.

(ii) Let  $l \in L$ . Then, by condition 4), there exists  $a \in A$  so that  $l = \lambda(a)$ . We have:  $l \in H$  iff  $\lambda(a) \in H$  iff  $a \in \lambda^{-1}(H)$  iff  $a \in \lambda^*(H)$  iff  $\lambda(a) \in L(\lambda^*(H))$  (by Lemma 3.4) iff  $l \in L(\lambda^*(H))$ . Therefore  $H = L(\lambda^*(H))$ . □

**Notation 3.2.** *For all filters  $F, G$  of  $A$ , we denote  $\langle F \cup G \rangle$  by  $F \vee G$ .*

**Notation 3.3.** *For all lattice filters  $F, G$  of  $L$ , we denote  $\langle F \cup G \rangle$  by  $F \vee G$ .*

The following proposition is well-known and easy to prove.

**Proposition 3.5.**  *$(\mathcal{F}(A), \vee, \cap, \{1\}, A)$  and  $(\mathcal{F}(L), \vee, \cap, \{1\}, L)$  are bounded distributive lattices, whose order relation is  $\subseteq$ .*

**Proposition 3.6.** *The function  $L : \mathcal{F}(A) \rightarrow \mathcal{F}(L)$  defined above is a bounded lattice isomorphism, whose inverse is  $\lambda^*$ .*

*Proof.* It is obvious from the definition of  $L$  that it is order-preserving, so it is a lattice morphism. Condition 3) shows that  $L(\{1\}) = \{1\}$  and condition 4) shows that  $L(A) = L$ . By Lemma 3.8,  $\lambda^*$  is the inverse of  $L$ . It is obvious by the definition of  $\lambda^*$  that it is order-preserving. So  $L$  is a lattice isomorphism.  $\square$

**Remark 3.4.**  $L$  is the inverse of the homeomorphism  $\lambda^* : \text{Spec}(L) \rightarrow \text{Spec}(A)$ .

**Proof:** Proposition 3.6 proves that  $L$  is the inverse of the homeomorphism  $\lambda^* : \text{Spec}(L) \rightarrow \text{Spec}(A)$ , since, as Lemma 3.7 shows, for any  $Q \in \text{Spec}(A)$ ,  $L(Q) \in \text{Spec}(L)$ .  $\square$

**Proposition 3.7.** For each lattice filter  $M$  of  $L$ , we have:  $M \in \text{Max}(L)$  iff  $\lambda^*(M) \in \text{Max}(A)$ .

**Proof:** Let  $M$  be a lattice filter of  $L$ . We have the following equivalences:  $M \in \text{Max}(L)$  iff (by Definition 2.11)  $M$  is a proper lattice filter of  $L$  and, for any proper lattice filter  $F$  of  $L$  so that  $M \subseteq F$ , we have  $M = F$  iff (by Proposition 3.6)  $\lambda^*(M)$  is a proper filter of  $A$  and, for any proper lattice filter  $F$  of  $L$  so that  $M \subseteq F$ , we have  $M = F$ . But, according to Proposition 3.6, the proper filters of  $A$  are of the form  $\lambda^*(F)$ , with  $F$  a proper lattice filter of  $L$ , and, for any lattice filter  $F$  of  $L$ ,  $M \subseteq F \Leftrightarrow \lambda^*(M) \subseteq \lambda^*(F)$  and  $M = F \Leftrightarrow \lambda^*(M) = \lambda^*(F)$ . Combining this with the above we get the following equivalences:  $M \in \text{Max}(L)$  iff  $\lambda^*(M)$  is a proper filter of  $A$  and, for any proper filter  $G$  of  $A$  so that  $\lambda^*(M) \subseteq G$ , we have  $\lambda^*(M) = G$  iff (by Definition 2.5)  $\lambda^*(M) \in \text{Max}(A)$ .  $\square$

Hence we also have a function  $\lambda^* : \text{Max}(L) \rightarrow \text{Max}(A)$ .

**Proposition 3.8.** The function  $\lambda^* : \text{Max}(L) \rightarrow \text{Max}(A)$  is a homeomorphism of topological spaces, whose inverse is  $L$ .

**Proof:** Proposition 3.6 and Proposition 3.7 show that  $\lambda^* : \text{Max}(L) \rightarrow \text{Max}(A)$  is a bijection whose inverse is  $L$ . It remains to prove that  $\lambda^* : \text{Max}(L) \rightarrow \text{Max}(A)$  is continuous and open. In the following we shall use the proof of Proposition 3.4 and the fact that  $\lambda^* : \text{Max}(L) \rightarrow \text{Max}(A)$  is a bijection. Let  $a \in A$ .  $(\lambda^*)^{-1}(D_{\text{Max}}(a)) = (\lambda^*)^{-1}(D(a) \cap \text{Max}(A)) = (\lambda^*)^{-1}(D(a)) \cap (\lambda^*)^{-1}(\text{Max}(A)) = D(\lambda(a)) \cap \text{Max}(L) = D_{\text{Max}}(\lambda(a))$ . Let  $l \in L$ . Then, by condition 4), there exists  $a \in A$  so that  $\lambda(a) = l$ .  $\lambda^*(D_{\text{Max}}(l)) = \lambda^*(D_{\text{Max}}(\lambda(a))) = \lambda^*(D(\lambda(a)) \cap \text{Max}(L)) = \lambda^*(D(\lambda(a))) \cap \lambda^*(\text{Max}(L)) = D(a) \cap \text{Max}(A) = D_{\text{Max}}(a)$ .  $\square$

We shall use the notations of the conditions 1)-5) and of the properties a)-c) in the following sections also.

#### 4 Existence and uniqueness of the reticulation

In this section we present two alternate constructions of the reticulation of a residuated lattice, we prove the uniqueness of the reticulation up to an isomorphism and we define the reticulation functor.

The first construction of the reticulation that we give is analogous to the one that was made by L. Leuştean for BL-algebras. His proofs are also valid for reticulated lattices and they are the starting point for our proofs, which differ by the approach; ours is based on the five axioms that define the reticulation.

Let  $A$  be a residuated lattice. Let us define the following binary relation on  $A$ : for all  $a, b \in A$ ,  $a \equiv b$  iff  $D(a) = D(b)$  iff  $(\forall P \in \text{Spec}(A)) a \notin P \Leftrightarrow b \notin P$  iff  $(\forall P \in \text{Spec}(A)) a \in P \Leftrightarrow b \in P$ .

**Remark 4.1.** *Let  $F$  be a filter of  $A$ . Then, for all  $a, b \in A$ ,  $a \odot b \in F$  iff  $a \wedge b \in F$  iff  $a, b \in F$ .*

**Proof:** Let  $a, b \in A$ . According to Remark 2.1, (iii), we have  $a \odot b \leq a \wedge b \leq a, b$ . From this we deduce the following implications, which prove our remark:  $a \odot b \in F \Rightarrow a \wedge b \in F \Rightarrow a, b \in F \Rightarrow a \odot b \in F$ .  $\square$

**Remark 4.2.** *Let  $P$  be a prime filter of  $A$ . Then, for all  $a, b \in A$ ,  $a \vee b \in P$  iff  $a \in P$  or  $b \in P$ .*

**Proof:** Let  $a, b \in A$ .  $a, b \leq a \vee b$ , so, since  $P$  is prime, we have that  $a \vee b \in P \Rightarrow a \in P$  or  $b \in P \Rightarrow a \vee b \in P$ .  $\square$

**Proposition 4.1.**  $\equiv$  is a congruence relation on  $A$  with respect to  $\odot$ ,  $\wedge$  and  $\vee$ .

**Proof:** It is obvious from the definition of  $\equiv$  that it is an equivalence relation on  $A$ .

Let  $a_1, b_1, a_2, b_2 \in A$  so that  $a_1 \equiv a_2$  and  $b_1 \equiv b_2$ . Let  $P \in \text{Spec}(A)$ . From Remark 4.1, we have:  $a_1 \odot b_1 \in P \Leftrightarrow a_1, b_1 \in P \Leftrightarrow a_2, b_2 \in P \Leftrightarrow a_2 \odot b_2 \in P$ . So, by the definition of the relation  $\equiv$ , we have:  $a_1 \odot b_1 \equiv a_2 \odot b_2$ . Analogously,  $a_1 \wedge b_1 \equiv a_2 \wedge b_2$ . Similarly, but using Remark 4.2 instead of Remark 4.1, we get  $a_1 \vee b_1 \equiv a_2 \vee b_2$ .  $\square$

**Remark 4.3.**  $(\forall a, b \in A) a \odot b \equiv a \wedge b$ .

**Proof:** Let  $a, b \in A$ . According to Remark 4.1,  $(\forall P \in \text{Spec}(A)) a \odot b \in P \Leftrightarrow a \wedge b \in P$ , therefore  $a \odot b \equiv a \wedge b$ .  $\square$

For all  $a \in A$ , we shall denote by  $[a]$  the equivalence class of  $a$  ( $[a] = \{b \in A \mid a \equiv b\}$ ) and let  $A/\equiv$  be the quotient set of  $A$  with respect to the equivalence relation  $\equiv$ . Let  $\lambda : A \rightarrow A/\equiv$  be the canonical surjection:  $(\forall a \in A)\lambda(a) = [a]$ . Let us define on the quotient set  $A/\equiv$  the following operations:  $\vee$ ,  $\wedge$ ,  $0$  and  $1$ , by:

$$\begin{aligned} (\forall a, b \in A)[a] \vee [b] &= [a \vee b], \\ (\forall a, b \in A)[a] \wedge [b] &= [a \odot b] = [a \wedge b], \\ 0 &= [0], \quad 1 = [1]. \end{aligned}$$

**Proposition 4.2.** *The algebra  $(A/\equiv, \vee, \wedge, 0, 1)$  is a bounded distributive lattice.*

**Proof:** According to Proposition 4.1, the operations  $\vee$  and  $\wedge$  on  $A/\equiv$  are well-defined. The fact that  $(A/\equiv, \vee, \wedge, 0, 1)$  is a bounded lattice can easily be deduced from the fact that  $(A, \vee, \wedge, 0, 1)$  is a bounded lattice. We shall prove, for example, the associativity of  $\wedge$ : for all  $a, b, c \in A$ ,  $([a] \wedge [b]) \wedge [c] = [a \wedge b] \wedge [c] = [(a \wedge b) \wedge c] = [a \wedge (b \wedge c)] = [a] \wedge [b \wedge c] = [a] \wedge ([b] \wedge [c])$ . The distributivity of the lattice  $A/\equiv$  can be deduced in the same manner from Remark 2.1, (iv).  $\square$

**Theorem 4.1.**  *$(A/\equiv, \lambda)$  is a reticulation of  $A$ .*

**Proof:** According to Proposition 4.2,  $A/\equiv$  is a bounded distributive lattice. Conditions 1), 2), 3) from the definition of the reticulation are exactly the definitions of the operations  $\wedge$ ,  $\vee$ ,  $0$  and  $1$ . Condition 4) is obvious.

We shall now prove condition 5). Let  $a, b \in A$ .  $\lambda(a) \leq \lambda(b)$  iff  $[a] \leq [b]$  iff  $[a] = [a] \wedge [b]$  iff  $[a] = [a \odot b]$  iff  $(\forall P \in \text{Spec}(A))a \in P \Leftrightarrow a \odot b \in P$  iff  $(\forall P \in \text{Spec}(A))a \in P \Leftrightarrow a, b \in P$  (as Remark 4.1 shows) iff  $(\forall P \in \text{Spec}(A))a \in P \Rightarrow b \in P$ ; let us denote this last condition by  $(\alpha)$ . On the other hand,  $(\exists n \in \mathbb{N}^*)a^n \leq b$  iff  $b \in \langle a \rangle$  (as Lemma 2.1 shows) iff  $(\forall F \in \mathcal{F}(A))a \in F \Rightarrow b \in F$  (as it is easily seen from Definition 2.6); let us denote this last condition by  $(\beta)$ . It is obvious that  $(\beta)$  implies  $(\alpha)$ . The converse implication is easily proven using corollary 2.1. The proof is now complete.  $\square$

In the following we shall present an alternate construction of the reticulation, suggested by Lemma 3.3.

**Remark 4.4.**  $\langle 0 \rangle = A$  and  $\langle 1 \rangle = \{1\}$ .

**Remark 4.5.** *Let  $F, G$  be filters of  $A$ . Then  $F \vee G = \{a \in A \mid (\exists b \in F)(\exists c \in G)b \odot c \leq a\}$ .*

**Proof:** Let  $H = \{a \in A \mid (\exists b \in F)(\exists c \in G)b \odot c \leq a\}$ . For all  $b \in F, c \in G, a \in A$  so that  $b \odot c \leq a$ , we have:  $b \odot c \in F \vee G$ , so  $a \in F \vee G$ . So  $H \subseteq F \vee G$ .  $1 \in F$  and  $1 \in G$ , so  $F \cup G \subseteq H$ . For proving that  $F \vee G \subseteq H$ , it remains to show that  $H$  is a filter. Let  $a_1, a_2 \in H$ . Then there exist  $b_1, b_2 \in F, c_1, c_2 \in G$ , so that  $b_1 \odot c_1 \leq a_1$

and  $b_2 \odot c_2 \leq a_2$ . Then, by Remark 2.1, (ii),  $b_1 \odot b_2 \odot c_1 \odot c_2 \leq a_1 \odot a_2$ . But  $b_1 \odot b_2 \in F$  and  $c_1 \odot c_2 \in G$ , so  $a_1 \odot a_2 \in H$ . Let  $a \in H$  and  $b \in A$  so that  $a \leq b$ . It is obvious by the definition of  $H$  that  $b \in H$ . So  $H$  is a filter. Hence  $F \vee G = H$ .  $\square$

**Proof:** Obvious, by Lemma 2.1 and the fact that, for all  $n \in \mathbb{N}^*$ ,  $1^n = 1$  and, by Remark 2.1, (iii),  $0^n \leq \underbrace{0 \wedge \dots \wedge 0}_{n \text{ of } 0} = 0$ , so  $0^n = 0$ .  $\square$

**Proposition 4.3.** For all  $a, b \in A$ :

- (i)  $a \leq b$  implies  $\langle b \rangle \subseteq \langle a \rangle$ ;
- (ii)  $\langle a \rangle \vee \langle b \rangle = \langle a \wedge b \rangle = \langle a \odot b \rangle$ ;
- (iii)  $\langle a \rangle \cap \langle b \rangle = \langle a \vee b \rangle$ .

**Proof:** (i) Assume  $a \leq b$ . Let  $c \in \langle b \rangle$ . Then, by Lemma 2.1, there exists  $n \in \mathbb{N}^*$  so that  $b^n \leq c$ . By Remark 2.1, (ii),  $a^n \leq b^n$ . So  $a^n \leq c$ . So  $c \in \langle a \rangle$ . Hence  $\langle b \rangle \subseteq \langle a \rangle$ .

(ii) Remark 2.1, (iii), shows that  $a \odot b \leq a \wedge b \leq a, b$ . Applying (i), we get:  $\langle a \rangle \vee \langle b \rangle \subseteq \langle a \wedge b \rangle \subseteq \langle a \odot b \rangle$ . It remains to prove that  $\langle a \odot b \rangle \subseteq \langle a \rangle \vee \langle b \rangle$ . Let  $x \in \langle a \odot b \rangle$ . By Lemma 2.1, there exists  $n \in \mathbb{N}^*$  so that  $a^n \odot b^n \leq x$ . But  $a^n \in \langle a \rangle$  and  $b^n \in \langle b \rangle$ , so, by Remark 4.5,  $x \in \langle a \rangle \vee \langle b \rangle$ . So  $\langle a \odot b \rangle \subseteq \langle a \rangle \vee \langle b \rangle$ .

(iii)  $a \in \langle a \rangle$  and  $b \in \langle b \rangle$ . Since  $a, b \leq a \vee b$ , we get  $a \vee b \in \langle a \rangle$  and  $a \vee b \in \langle b \rangle$ , so  $a \vee b \in \langle a \rangle \cap \langle b \rangle$ , which is a filter, therefore  $\langle a \vee b \rangle \subseteq \langle a \rangle \cap \langle b \rangle$ . For the converse set inclusion, let  $c \in \langle a \rangle \cap \langle b \rangle$ . Applying Lemma 2.1, we get that there exist  $n, m \in \mathbb{N}^*$  so that  $a^n \leq c$  and  $b^m \leq c$ . So  $a^n \vee b^m \leq c$ . Applying several times Remark 2.1, (iv), we get:  $c \geq a^n \vee b^m = (a^n \vee b^m)^m = ((a \vee b)^n)^m = (a \vee b)^{nm}$ . So  $c \in \langle a \vee b \rangle$ . Therefore  $\langle a \rangle \cap \langle b \rangle \subseteq \langle a \vee b \rangle$ . Hence the desired equality.  $\square$

**Notation 4.1.** Let us denote by  $\mathcal{PF}(A)$  the set of principal filters of  $A$ .

Until specified otherwise, we shall denote by  $\lambda : A \rightarrow \mathcal{PF}(A)$  the function given by: for all  $a \in A$ ,  $\lambda(a) = \langle a \rangle$ .

**Theorem 4.2.**  $((\mathcal{PF}(A), \cap, \vee, A, \{1\}), \lambda)$  is a reticulation of  $A$ .

**Proof:** Remark 4.4, Proposition 4.3 ((ii) and (iii)) and the fact that  $(\mathcal{F}(A), \cap, \vee, A, \{1\})$  is a bounded distributive lattice imply that  $((\mathcal{PF}(A), \cap, \vee, A, \{1\}), \lambda)$  is a bounded distributive lattice. Conditions 1) and 2) from the definition of the reticulation are properties (ii) and (iii) from Proposition 4.3. Condition 3) is Remark 4.4. Condition 4) is obvious. Let us prove now condition 5). First, let us remark that the partial order of the lattice  $(\mathcal{PF}(A), \cap, \vee, A, \{1\})$  is  $\supseteq$ . For all  $a, b \in A$ , we have:  $\langle a \rangle \supseteq \langle b \rangle$  iff  $b \in \langle a \rangle$  iff (by Lemma 2.1)  $(\exists n \in \mathbb{N}^*) a^n \leq b$ . So condition 5) is also verified.  $\square$

**Theorem 4.3.** *Let  $A$  be a residuated lattice and  $(L_1, \lambda_1)$ ,  $(L_2, \lambda_2)$  be two reticulations of  $A$ . Then there exists an isomorphism of bounded lattices  $f : L_1 \rightarrow L_2$  so that  $f \circ \lambda_1 = \lambda_2$ .*

**Proof:** Let  $f : L_1 \rightarrow L_2$ , so that  $(\forall a \in A)f(\lambda_1(a)) = \lambda_2(a)$ .  $\lambda_1$  is surjective, so  $f : L_1 \rightarrow L_2$  is completely defined.

Let us prove that  $(\forall a, b \in A)\lambda_1(a) = \lambda_1(b) \Leftrightarrow \lambda_2(a) = \lambda_2(b)$ . Let  $a, b \in A$ . The following equivalences hold:  $\lambda_1(a) = \lambda_1(b) \Leftrightarrow \lambda_1(a) \leq \lambda_1(b)$  and  $\lambda_1(b) \leq \lambda_1(a) \Leftrightarrow (\exists n \in \mathbb{N}^*)a^n \leq b$  and  $(\exists n \in \mathbb{N}^*)b^n \leq a \Leftrightarrow \lambda_2(a) \leq \lambda_2(b)$  and  $\lambda_2(b) \leq \lambda_2(a) \Leftrightarrow \lambda_2(a) = \lambda_2(b)$ . The direct implication proves that  $f$  is well-defined and the converse implication proves that  $f$  is injective. The surjectivity of  $\lambda_2$  implies that  $f$  is surjective.

For proving that  $f$  is a bounded lattice isomorphism it remains to show that  $f$  commutes with the bounded lattice operations. Using condition 3) we get:  $f(0) = f(\lambda_1(0)) = \lambda_2(0) = 0$  and  $f(1) = f(\lambda_1(1)) = \lambda_2(1) = 1$ . Now let  $l, m \in L_1$ . Since  $f$  is surjective,  $(\exists a, b \in A)l = \lambda_1(a), m = \lambda_1(b)$ . Applying b) we obtain the following equalities:  $f(l \wedge m) = f(\lambda_1(a) \wedge \lambda_1(b)) = f(\lambda_1(a \wedge b)) = \lambda_2(a \wedge b) = \lambda_2(a) \wedge \lambda_2(b) = f(\lambda_1(a)) \wedge f(\lambda_1(b)) = f(l) \wedge f(m)$ . The proof of  $f(l \vee m) = f(l) \vee f(m)$  is analogous, except it uses 2) instead of b).  $\square$

**Remark 4.6.** *In the absence of condition 5), uniqueness is not satisfied.*

**Proof:** Let  $A$  be the residuated lattice from the proof of Remark 3.2 and  $L$  and  $\lambda$  be the lattice and respectively the function from the proof of Remark 3.3. Let us consider another bounded distributive lattice:  $L_1 = \{0, 1\}$ , with  $0 < 1$ , and another function:  $\lambda_1 : A \rightarrow L_1$ , defined by:  $\lambda_1(0) = \lambda_1(a) = \lambda_1(b) = \lambda_1(c) = 0$  and  $\lambda_1(1) = 1$ . Obviously, both  $\lambda$  and  $\lambda_1$  satisfy conditions 1)-4), but neither of them satisfies condition 5) (see the proof of Remark 3.3). Since no bijection exists between the sets  $L$  and  $L_1$ , it follows that the lattices  $L$  and  $L_1$  are not isomorphic.  $\square$

Let  $A, B$  be two residuated lattices and  $(L(A), \lambda_A)$ ,  $(L(B), \lambda_B)$  be their reticulations, defined like in Theorem 4.1. Let  $h : A \rightarrow B$  be a morphism of residuated lattices. Let us define the following function:  $L(h) : L(A) \rightarrow L(B)$ ,  $(\forall a \in A)L(h)([a]) = [h(a)]$ .

**Proposition 4.4.** *Let  $Q \in \text{Spec}(B)$ . Then  $h^{-1}(Q) \in \text{Spec}(A)$ .*

**Proof:** Let  $a, b \in h^{-1}(Q)$ .  $\Leftrightarrow h(a), h(b) \in Q \Rightarrow h(a) \odot h(b) \in Q \Leftrightarrow h(a \odot b) \in Q \Leftrightarrow a \odot b \in h^{-1}(Q)$ . Let  $a \in h^{-1}(Q)$  ( $\Leftrightarrow h(a) \in Q$ ) and  $b \in A$  so that  $a \leq b$ .  $\Rightarrow h(a) \leq h(b)$ . So  $h(b) \in Q$ .  $\Leftrightarrow b \in h^{-1}(Q)$ . So  $h^{-1}(Q)$  is a filter of  $A$ . Let  $a, b \in A$  so that  $a \vee b \in h^{-1}(Q)$ .  $\Leftrightarrow h(a \vee b) \in Q \Leftrightarrow h(a) \vee h(b) \in Q \Rightarrow h(a) \in Q$  or  $h(b) \in Q$ , since  $Q$  is a prime filter of  $B$ . Equivalently,  $a \in h^{-1}(Q)$  or  $b \in h^{-1}(Q)$ . So  $h^{-1}(Q)$  is a prime filter of  $A$ .  $\square$

**Proposition 4.5.**  $L(h)$  is well-defined and it is a bounded lattice morphism.

**Proof:** Let  $a, b \in A$  so that  $[a] = [b]$ . This is equivalent to  $D(a) = D(b)$ . Let  $Q \in \text{Spec}(B)$ . So, by Proposition 4.4,  $h^{-1}(Q) \in \text{Spec}(A)$ . We have the following equivalences:  $Q \in D(h(a)) \Leftrightarrow h(a) \notin Q \Leftrightarrow a \notin h^{-1}(Q) \Leftrightarrow h^{-1}(Q) \in D(a) \Leftrightarrow h^{-1}(Q) \in D(b) \Leftrightarrow b \notin h^{-1}(Q) \Leftrightarrow h(b) \notin Q \Leftrightarrow Q \in D(h(b))$ . So  $D(h(a)) = D(h(b))$ , which is equivalent to  $[h(a)] = [h(b)]$ . Therefore  $L(h)$  is well-defined.

Let  $a, b \in A$ .  $L(h)([a] \wedge [b]) = L(h)([a \wedge b]) = [h(a \wedge b)] = [h(a)] \wedge [h(b)] = L(h)([a]) \wedge L(h)([b])$ . Similarly we get that  $L(h)([a] \vee [b]) = L(h)([a] \vee [b]) = L(h)([a]) \vee L(h)([b])$ .  $L(h)([0]) = [h(0)] = [0]$ .  $L(h)([1]) = [h(1)] = [1]$ . So  $L(h)$  is a bounded lattice morphism. We have used essentially the surjectivity of  $\lambda_A$ .  $\square$

Hence we have defined a functor  $\mathcal{L}$  from the category of residuated lattices to the category of bounded distributive lattices. We shall call  $\mathcal{L}$  *the reticulation functor*.

## 5 Examples

In this section we illustrate the construction of the reticulation of a residuated lattice; we point out a case where the residuated lattice is isomorphic to its reticulation and then we give several examples of finite residuated lattices and their reticulations.

In the following remark and examples we shall use the alternate construction of the reticulation in Section 4. We remind the reader that the partial order relation of the reticulation is  $\supseteq$ .

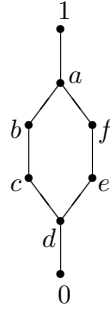
**Remark 5.1.** If  $A$  is a residuated lattice in which  $\odot = \wedge$  and  $(L(A), \lambda)$  is its reticulation, then  $\lambda : A \rightarrow L(A)$  is a bounded lattice isomorphism.

**Proof:** Let us remember that  $L(A) = \{ \langle a \rangle \mid a \in A \}$ ,  $\lambda : A \rightarrow L(A)$ ,  $(\forall a \in A) \lambda(a) = \langle a \rangle$  (see Theorem 4.2) and, for any element  $a \in A$ ,  $\langle a \rangle = \{ b \in A \mid (\exists n \in \mathbb{N}^*) a^n \leq b \}$  (see Lemma 2.1). But, if  $\odot = \wedge$ , then, for all  $n \in \mathbb{N}^*$ ,  $a^n = a$ , so  $\langle a \rangle = \{ b \in A \mid a \leq b \}$ . In this case, for all  $a, b \in A$ , we have:  $\langle a \rangle \supseteq \langle b \rangle$  iff  $b \in \langle a \rangle$  iff  $a \leq b$ . We also have:  $\langle a \rangle = \langle b \rangle$  iff  $\langle a \rangle \supseteq \langle b \rangle$  and  $\langle b \rangle \supseteq \langle a \rangle$  iff  $a \leq b$  and  $b \leq a$  iff  $a = b$ . So  $\lambda$  is injective, and, by condition 4), we get that it is bijective. Properties 2), b) and 3) show us that  $\lambda$  is a bounded lattice morphism. Hence the function  $\lambda$  from  $A$  to  $L(A)$  is a bounded lattice isomorphism.  $\square$

The above remark shows us that we only need to consider examples where  $\odot \neq \wedge$ , because, if these two operations coincide, then the residuated lattice  $A$  and its reticulation are isomorphic lattices, so basically the reticulation is  $A$  without the operations  $\odot$  and  $\rightarrow$ .

We remark the fact that not all finite residuated lattices are isomorphic to their reticulation, as shown by two of the examples below, in which no bijection exists between the two, nor is the condition  $\odot = \wedge$  necessary for such an isomorphism to exist, as shown by example 5.2.

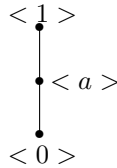
**Example 5.1.** Let  $A = \{0, a, b, c, d, e, f, 1\}$  be the residuated lattice with the partial order relation and operations presented in the Hasse diagram and tables below respectively.



$\rightarrow$	0	a	b	c	d	e	f	1	$\odot$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
a	d	1	a	a	f	f	f	1	a	0	c	c	c	0	d	d	a
b	e	1	1	a	f	f	f	1	b	0	c	c	c	0	0	d	b
c	f	1	1	1	f	f	f	1	c	0	c	c	c	0	0	0	c
d	a	1	1	1	1	1	1	1	d	0	0	0	0	0	0	0	d
e	b	1	a	a	a	1	1	1	e	0	d	0	0	0	d	d	e
f	c	1	a	a	a	a	1	1	f	0	d	d	0	0	d	d	f
1	1	a	b	c	d	e	f	1	1	0	a	b	c	d	e	f	1

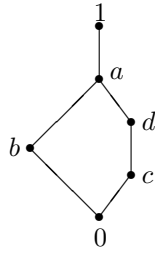
(See [10].)

We have:  $\langle 0 \rangle = \langle d \rangle = \langle e \rangle = \langle f \rangle = A$ ,  $\langle a \rangle = \langle b \rangle = \langle c \rangle = \{c, b, a, 1\}$ ,  $\langle 1 \rangle = \{1\}$ , hence  $L(A) = \{\langle 0 \rangle, \langle a \rangle, \langle 1 \rangle\}$ , with the following lattice structure:



This is a linearly ordered lattice, unlike the reticulations in the examples below.

**Example 5.2.** Let  $A = \{0, a, b, c, d, 1\}$  be the residuated lattice described below.

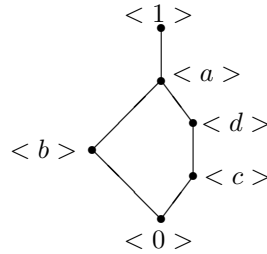


$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	c	c	1
b	c	a	1	c	c	1
c	b	a	b	1	a	1
d	b	a	b	a	1	1
1	0	a	b	c	d	1

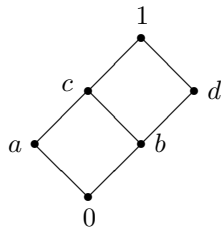
$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	d	d	a
b	c	b	b	0	0	b
c	b	d	0	d	d	c
d	b	d	0	d	d	d
1	0	a	b	c	d	1

(See [10].)

So we have:  $\langle 0 \rangle = A$ ,  $\langle a \rangle = \{a, 1\}$ ,  $\langle b \rangle = \{b, a, 1\}$ ,  $\langle c \rangle = \{c, d, a, 1\}$ ,  $\langle d \rangle = \{d, a, 1\}$ ,  $\langle 1 \rangle = \{1\}$  and hence the reticulation is isomorphic to  $A$ :



**Example 5.3.** Let  $A = \{0, a, b, c, d, 1\}$ , with the following residuated lattice structure:

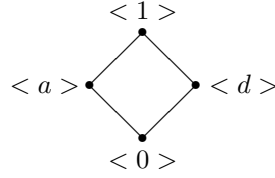


$\rightarrow$	0	$a$	$b$	$c$	$d$	1
0	1	1	1	1	1	1
$a$	$d$	1	$d$	1	$d$	1
$b$	$c$	$c$	1	1	1	1
$c$	$b$	$c$	$d$	1	$d$	1
$d$	$a$	$a$	$c$	$c$	1	1
1	0	$a$	$b$	$c$	$d$	1

$\odot$	0	$a$	$b$	$c$	$d$	1
0	0	0	0	0	0	0
$a$	0	$a$	0	$a$	0	$a$
$b$	0	0	0	0	$b$	$b$
$c$	0	$a$	0	$a$	$b$	$c$
$d$	0	0	$b$	$b$	$d$	$d$
1	0	$a$	$b$	$c$	$d$	1

(see [8]).

The principal filters of this residuated lattice are:  $\langle 0 \rangle = \langle b \rangle = A$ ,  $\langle a \rangle = \langle c \rangle = \{a, c, 1\}$ ,  $\langle d \rangle = \{d, 1\}$ ,  $\langle 1 \rangle = \{1\}$ , so  $L(A) = \{\langle 0 \rangle, \langle a \rangle, \langle d \rangle, \langle 1 \rangle\}$ , with the following lattice structure:



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## References

- [1] R. BALBES, P. DWINGER, *Distributive lattices*, University of Missouri Press, Columbia, Missouri (1974).
- [2] L. P. BELLUCE, Semisimple algebras of infinite valued logic and bold fuzzy set theory, *Can. J. Math.* 38, No. 6 (1986), 1356-1379.
- [3] L. P. BELLUCE, Spectral spaces and non-commutative rings, *Comm. Algebra* 19 (1991), 1855-1865.
- [4] A. DI NOLA, G. GEORGESCU, I. LEUŞTEAN, *MV-algebras*, manuscript.
- [5] A. DI NOLA, G. GEORGESCU, L. LEUŞTEAN, Boolean products of BL-algebras, *J. Math. Anal. Appl.* 251, No. 1 (2000), 106-131.
- [6] G. GEORGESCU, The reticulation of a quantale, *Rev. Roum. Math. Pures Appl.* 40, No. 7-8 (1995), 619-631.
- [7] P. HÁJEK, *Metamathematics of Fuzzy Logic*, Trends in Logic-Studia Logica, Kluwer Academic Publishers, Dordrecht/Boston/London (1998).

- [8] A. IORGULESCU, Classes of BCK algebras-Part III, Preprint Series of the Institute of Mathematics of the Romanian Academy, preprint No. 3 (2004), 1-37.
- [9] P. T. JOHNSTONE, *Stone Spaces*, Cambridge Studies in Advanced Mathematics 3, Cambridge University Press, Cambridge/London/New York/New Rochelle/Melbourne/Sydney (1982).
- [10] T. KOWALSKI, H. ONO, *Residuated Lattices: An algebraic glimpse at logics without contraction*, manuscript.
- [11] L. LEUȘTEAN, The prime and maximal spectra and the reticulation of BL-algebras, *Central European Journal of Mathematics* 1, No. 3 (2003), 382-397.
- [12] L. LEUȘTEAN, *Representations of Many-valued Algebras*, Ph. D. thesis (2003).
- [13] D. PICIU, *Algebras of Fuzzy Logic*, Editura Universitaria Craiova (2007).
- [14] H. SIMMONS, Reticulated rings, *J. Algebra* 66 (1980), 169-192.
- [15] E. TURUNEN, *Mathematics behind fuzzy logic*, Advances in Soft Computing, Physica-Verlag, Heidelberg (1999).

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