Boundedness analysis for certain two-dimensional differential systems via a Lyapunov approach
by
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Abstract
In this paper, the problem of boundedness of solutions of a two-dimensional differential system is considered. Based on the Lyapunov function approach, a new boundedness criterion is derived in terms of this system. An example is given to show the effectiveness of our result.

Key Words: Boundedness, two-dimensional differential system.
2010 Mathematics Subject Classification: Primary 34D20, Secondary 34C11.

1 Introduction
In 1980, Sinha [1] discussed asymptotic stability of null solution of the following two-dimensional differential system:
\begin{align*}
x' &= f(x, y) + p_1(t)x + r(t)y, \\
y' &= g(x, y) + s(t)x + p_2(t)y.
\end{align*}

In this paper, instead of the preceding system, we consider the following two-dimensional differential system:
\begin{align}
x' &= f(x, y) + p_1(t)x + r(t)y + p_3(t, x, y), \\
y' &= g(x, y) + s(t)x + p_2(t)y + p_4(t, x, y), \tag{1}
\end{align}
where the prime denotes differentiation with respect to $t$, $t \in \mathbb{R}^+ = [0, \infty)$; $f, g, p_1, p_2, p_3, p_4, r$ and $s$ are continuous functions in their respective arguments on $\mathbb{R}^2$, $\mathbb{R}^2$, $\mathbb{R}^+$, $\mathbb{R}^+ \times \mathbb{R}^2$, $\mathbb{R}^+ \times \mathbb{R}^2$, $\mathbb{R}^+$ and $\mathbb{R}^+$, respectively; $r(t)$ and $s(t)$ are bounded functions, $f(0, y) = g(x, 0) = 0$, and it is also assumed that the derivatives $f_x(x, y) \equiv \frac{\partial f}{\partial x}(x, y)$ and $g_y(x, y) \equiv \frac{\partial g}{\partial y}(x, y)$ exist and are continuous. We assume further that
\[ p_1(t) \neq 0, p_2(t) \neq 0, \]
\[
2 \int_0^t p_1(s) \, ds = R_1(t) + Q_1(t),
\]
\[
2 \int_0^t p_2(s) \, ds = R_2(t) + Q_2(t),
\]
where the functions \( R_i(.) \) and \( Q_i(.) \), \((i = 1, 2)\), are defined on \( \mathbb{R}^+ = [0, \infty) \), and \(|Q_1(t)| < c_1, |Q_2(t)| < c_2, R'_1(t) < 0, R'_2(t) < 0\), in which \( c_1 \) and \( c_2 \) are some positive constants.

The motivation of this paper comes from the paper of Sinha \cite{1}. Our aim is to improve the result established in \cite{1} to the system (1) for boundedness of the solutions. We also give an example to illustrate the effectiveness of our result. In particular, one can refer to the papers of Tunç \cite{2, 3, 4}, C. Tunç and E. Tunç \cite{5}, Tunç and Şevli \cite{6} and the references cited in these papers for some works performed on boundedness of the solutions. It is worth mentioning that our result is new and original.

2 Problem Description

We establish the following theorem.

Theorem. In addition to the basic assumptions imposed on the functions \( f, g, p_1, p_2, p_3, p_4, r \) and \( s \) that appearing in the system (1), we assume that there exist two positive constants \( b_1 \) and \( b_2 \) such that the following conditions hold:

(i) \(-b_1 e^{-Q_1(t)} R'_1(t) > 0, b_1 b_2 e^{-Q_2(t)} R'_1(t) R'_2(t) - K^2(t) > 0\) for all \( t \in \mathbb{R}^+ \),

where \( K(t) = r(t) b_1 e^{-Q_1(t)} + s(t) b_2 e^{-Q_2(t)} \),

\[ f(x, y) \leq 0 \] and \( g(y(x, y) \leq 0 \) for all \( t \in \mathbb{R}^+ \) and \( x, y \in \mathbb{R} \),

(ii) \(|p_3(t, x, y)| \leq q_1(t), |p_4(t, x, y)| \leq q_2(t), q_1(t) \leq q(t) \) and \( q_2(t) \leq q(t)\) for all \( t \in \mathbb{R}^+ \) and \( x, y \in \mathbb{R} \),

where \( q_1, q_2, q \in L^1(0, \infty) \), in which \( L^1(0, \infty) \) is the space of Lebesgue integrable functions. Then, there exists a positive constant \( M \) such that the solution \((x(.), y(.))\) of the system (1) satisfies the inequalities

\[ |x(t)| \leq M, |y(t)| \leq M \]

for all \( t \geq t_0 \geq 0 \).
Proof: We employ a Lyapunov function \( V = V(t, x, y) \) defined by:

\[
V(t, x, y) = b_1 e^{-Q_1(t)} x^2 + b_2 e^{-Q_2(t)} y^2,
\]

in which \( b_1 \) and \( b_2 \) are some positive constants.

It is clear that \( V(t, 0, 0) = 0 \), and \( b_1 e^{-Q_1(t)} \) and \( b_2 e^{-Q_2(t)} \) are bounded since \( |Q_1(t)| < c_1 \) and \( |Q_2(t)| < c_2 \). Hence, it is seen that the Lyapunov function \( V \) is positive definite.

Let \( (x(t), y(t)) \) be an arbitrary solution of the system (1). Differentiating the function \( V \) along the system (1), we have

\[
\frac{d}{dt} V(t, x(t), y(t)) =
\]

\[
= -b_1 Q_1'(t) e^{-Q_1(t)} x^2(t) - b_2 Q_2'(t) e^{-Q_2(t)} y^2(t)
\]

\[
+ 2b_1 e^{-Q_1(t)} x(t) \frac{dx(t)}{dt} + 2b_2 e^{-Q_2(t)} y(t) \frac{dy(t)}{dt}
\]

\[
= -b_1 Q_1'(t) e^{-Q_1(t)} x^2(t) - b_2 Q_2'(t) e^{-Q_2(t)} y^2(t)
\]

\[
+ 2b_1 e^{-Q_1(t)} x(t) \{ f(x(t), y(t)) + p_1(t) x(t) + r(t) y(t) + p_3(t, x(t), y(t)) \}
\]

\[
+ 2b_2 e^{-Q_2(t)} y(t) \{ g(x(t), y(t)) + s(t) x(t) + p_2(t) y(t) + p_4(t, x(t), y(t)) \}.
\]

In view of the assumptions

\[
2 \int_0^t p_1(s) ds = R_1(t) + Q_1(t)
\]

and

\[
2 \int_0^t p_2(s) ds = R_2(t) + Q_2(t),
\]

the preceding equality leads that

\[
\frac{d}{dt} V(t, x(t), y(t)) =
\]

\[
= 2b_1 e^{-Q_1(t)} f(x(t), y(t)) x(t) + 2b_2 e^{-Q_2(t)} g(x(t), y(t)) y(t)
\]

\[
+ 2b_1 e^{-Q_1(t)} x(t) p_3(t, x(t), y(t))
\]

\[
+ 2b_2 e^{-Q_2(t)} y(t) p_4(t, x(t), y(t)) - W_1(t),
\]

where

\[
W_1 = -b_1 e^{-Q_1(t)} R_1'(t) x^2(t) -
\]

\[
- b_2 e^{-Q_2(t)} R_2'(t) y^2(t).
\]

Hence, it is seen that the Lyapunov function \( V \) is positive definite.
It follows that the expression $W_1(t)$ represents a quadratic form, and one can arrange $W_1(t)$ as the following:

$$W_1(t) = [x(t), y(t)] A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

where

$$A = \begin{bmatrix} -b_1 e^{-Q_1(t)} R_1'(t) & -r(t)b_1 e^{-Q_1(t)} - s(t)b_2 e^{-Q_2(t)} \\ -r(t)b_1 e^{-Q_1(t)} - s(t)b_2 e^{-Q_2(t)} & -b_2 e^{-Q_2(t)} R_2'(t) \end{bmatrix}$$

Now, by noting the basic information related to the positive definiteness of a quadratic form, we can conclude that $W_1(t) \geq 0$ provided that

$$-b_1 e^{-Q_1(t)} R_1'(t) > 0$$

and

$$b_1 b_2 e^{-Q_1(t)-Q_2(t)} R_1'(t) R_2'(t) - [r(t)b_1 e^{-Q_1(t)} + s(t)b_2 e^{-Q_2(t)}]^2$$

$$= b_1 b_2 e^{-Q_1(t)-Q_2(t)} R_1'(t) R_2'(t) - K^2(t) > 0.$$

Hence, we have

$$\frac{d}{dt} V(t, x(t), y(t)) \leq$$

$$2b_1 e^{-Q_1(t)} f(x(t), y(t)) x(t) + 2b_2 e^{-Q_2(t)} g(x(t), y(t)) y(t)$$

$$+ 2b_1 e^{-Q_1(t)} x(t) p_3(t, x(t), y(t)) + 2b_2 e^{-Q_2(t)} y(t) p_4(t, x(t), y(t)).$$

Let $W_2(t)$ represent the first two terms included in (4):

$$W_2(t) = 2b_1 e^{-Q_1(t)} f(x(t), y(t)) x(t) + 2b_2 e^{-Q_2(t)} g(x(t), y(t)) y(t).$$

By the fact $f(0, y) = 0$, $g(x, 0) = 0$ and the generalized mean value theorem for the derivative, we obtain that there exist $\theta_1(t)$ and $\theta_2(t) \in [0, 1]$ such that

$$W_2(t) = 2b_1 e^{-Q_1(t)} f(x(t), y(t)) x(t) + 2b_2 e^{-Q_2(t)} g(x(t), y(t)) y(t)$$

$$= 2b_1 e^{-Q_1(t)} f_x(\theta_1 x(t), y(t)) x^2(t) + 2b_2 e^{-Q_2(t)} g_y(\theta_2 y(t)) y^2(t).$$

Making use of the assumptions $f_x(x, y) \leq 0$ and $g_y(x, y) \leq 0$, it follows that $W_2(t) \leq 0$. This fact now yields to the following inequality:

$$\frac{d}{dt} V(t, x(t), y(t)) \leq 2b_1 e^{-Q_1(t)} x(t) p_3(t, x(t), y(t)) + 2b_2 e^{-Q_2(t)} y(t) p_4(t, x(t), y(t)).$$
By noting the assumption (ii), the inequalities \(|y| < 1 + y^2, |z| < 1 + z^2\) and the boundedness of \(b_1 e^{-Q_1(t)}\) and \(b_2 e^{-Q_2(t)}\), the preceding inequality implies that

\[
\frac{d}{dt} V(t, x(t), y(t)) \leq \\
\leq 2b_1 e^{-Q_1(t)} |x(t)| |p_3(t, x(t), y(t))| + 2b_2 e^{-Q_2(t)} |y(t)| |p_4(t, x(t), y(t))| \\
\leq 2b_1 e^{-Q_1(t)} |p_3(t, x(t), y(t))| x^2 + 2b_2 e^{-Q_2(t)} |p_4(t, x(t), y(t))| y^2(t) \\
+2b_1 e^{-Q_1(t)} |p_3(t, x(t), y(t))| + 2b_2 e^{-Q_2(t)} |p_4(t, x(t), y(t))| \\
\leq 2b_1 e^{-Q_1(t)} q_1(t) x^2(t) + 2b_2 e^{-Q_2(t)} q_2(t) y^2(t) \\
+2b_1 e^{-Q_1(t)} q_1(t) + 2b_2 e^{-Q_2(t)} q_2(t) \\
\leq 2V(t, x(t), y(t)) q(t) + k_1 q_1(t) + k_2 q_2(t),
\]

where \(2b_1 e^{-Q_1(t)} \leq k_1, 2b_2 e^{-Q_2(t)} \leq k_2\), \(k_1\) and \(k_2\) are some positive constants, which we now assume.

Integrating (5) from 0 to \(t\), using the assumptions \(q_1 \in L^1(0, \infty), q_2 \in L^1(0, \infty), q \in L^1(0, \infty)\), and Gronwall-Reid-Bellman inequality, we obtain

\[
V(t, x(t), y(t)) \leq V(0, x(0), y(0)) + k_1 A + k_2 B + 2 \int_0^t V(s, x(s), y(s)) q(s) ds \\
\leq \{V(0, x(0), y(0)) + k_1 A + k_2 B\} \exp[2 \int_0^t q(s) ds] \\
= \{V(0, x(0), y(0)) + k_1 A + k_2 B\} \exp(2C) = M_1 < \infty,
\]

where \(M_1 > 0\) is a constant, \(A = \int_0^\infty q_1(s) ds, B = \int_0^\infty q_2(s) ds\) and \(C = \int_0^\infty q(s) ds\).

Now, subject to the above discussion, we arrive at the following:

\[
b_1 e^{-Q_1(t)} x^2 + b_2 e^{-Q_2(t)} y^2 = V(t, x, y) \leq M_1.
\]

Therefore, one can easily conclude, for some positive constant \(M\), that

\[
|x(t)| \leq M, |y(t)| \leq M,
\]

for all \(t \geq t_0 \geq 0\).

The proof is complete. \(\square\)
Example. Consider the following two-dimensional differential system:

\[
x' = -x^3 y^2 + \frac{a}{2} (1 - \varepsilon \sin \lambda t) + \frac{1}{1 + t^2 + \sin^2 x + y^2},
\]
\[
y' = -x^2 y^5 + \frac{b}{2} (1 + \varepsilon \cos \lambda t) + \frac{1}{1 + t^2 + \cos^2 x + y^2},
\]

(6)

where \( \varepsilon \) and \( \lambda \) are some arbitrary constants, and \( a \) and \( b \) are some negative constants.

It is clear that the two-dimensional differential system (6) is a special case of the two-dimensional differential system (1). By comparing (6) with (1) and taking into account the assumptions of the theorem, it follows the following:

\[
f(x, y) = -x^3 y^2,
\]
\[
f(0, y) = 0,
\]
\[
f_x(x, y) = -3x^2 y^2 \leq 0,
\]
\[
p_1(t)x = \frac{a}{2} (1 - \varepsilon \sin \lambda t),
\]
\[
2 \int_0^t p_1(s) ds = a \int_0^t (1 - \varepsilon \sin \lambda s) ds = (at - \frac{a \varepsilon}{\lambda}) + \frac{a \varepsilon}{\lambda} \cos \lambda t = R_1(t) + Q_1(t),
\]
\[
R_1(t) = at - \frac{a \varepsilon}{\lambda},
\]
\[
R_1'(t) = a < 0,
\]
\[
Q_1(t) = \frac{a \varepsilon}{\lambda} \cos \lambda t,
\]
\[
|Q_1(t)| = \left| \frac{a \varepsilon}{\lambda} \cos \lambda t \right| \leq \frac{|a \varepsilon|}{|\lambda|},
\]
\[
r(t) = 0,
\]
\[
p_3(t, x, y) = \frac{2}{1 + t^2 + \sin^2 x + y^2},
\]
\[
|p_3(t, x, y)| \leq \frac{3}{1 + t^2} = q_1(t),
\]
\[
\int_0^\infty q_1(s) ds = \int_0^\infty \frac{3}{1 + s^2} ds = \frac{3\pi}{2} < \infty,
\]
\[
g(x, y) = -x^2 y^5,
\]
\[
g(x, 0) = 0,
\]
\[
g_y(x, y) = -5x^2 y^4 \leq 0,
\]
\[
s(t)x = 0,
\]
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\[ p_2(t)y = \frac{b}{2}(1 + \varepsilon \cos \lambda t)y, \]

\[ 2 \int_0^t p_2(s)ds = b \int_0^t (1 + \varepsilon \cos \lambda s)ds = bt + \frac{b\varepsilon}{\lambda} \sin \lambda t = R_2(t) + Q_2(t), \]

\[ R_2(t) = bt, \]

\[ R'_2(t) = b < 0 \]

\[ Q_2(t) = \frac{b\varepsilon}{\lambda} \sin \lambda t, \]

\[ |Q_2(t)| = \left| \frac{b\varepsilon}{\lambda} \sin \lambda t \right| \leq \frac{|b\varepsilon|}{|\lambda|}, \]

\[ p_4(t, x, y) = \frac{1}{1 + t^2 + \cos^2 x + y^2}, \]

\[ |p_4(t, x, y)| \leq \frac{2}{1 + t^2} = q_2(t) \]

and

\[ \int_0^\infty q_2(s)ds = \int_0^\infty \frac{2}{1 + s^2}ds = \pi < \infty, \]

that is, \( q_2 \in L^1(0, \infty) \).

Let \( q(t) = \frac{1}{1 + t^2} \). Clearly, \( q_1(t) \leq q(t), q_2(t) \leq q(t) \) and \( q \in L^1(0, \infty) \).

Thus, all the assumptions of Theorem hold. That is, all solutions of the system (6) are bounded.

**Conclusion**

By means of Lyapunov function approach this paper obtained a new boundedness criterion for a certain two-dimensional differential system. An example is showed to the importance and applicability of this criterion. Our criterion improves an important result obtained on the stability of the null solution of a two-dimensional differential system in the literature to boundedness of the solutions of an extended two-dimensional differential system.

**References**


Received: 28.05.2009.
Revised: 15.10.2009.

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