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Some new root-finding methods with eighth-order convergence

by
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Abstract

Some new eighth-order iterative methods are obtained by choosing the proper parameters in the general formula with some parameters. Besides, as intermediate results, we get also different families of methods with convergence order five, six and seven. Numerical tests show the superiority of the new eighth-order methods.

Key Words: Non-linear equations, Ostrowski's method, Root-finding, Iterative method, Efficiency
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1. Introduction

Solving non-linear equation is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a simple root of a non-linear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D is a scalar function.

Ostrowski's method [1] with fourth-order convergence is given by

$$\begin{cases} y_n = x_n - f(x_n)/f'(x_n), \\ x_{n+1} = y_n - f(y_n)(x_n - y_n)/(f(x_n) - 2f(y_n)). \end{cases} \tag{1}$$

A variant of the Ostrowski's method with sixth-order convergence is proposed in [2] and given by

$$\begin{cases} y_n = x_n - f(x_n)/f'(x_n), \\ \mu = (x_n - y_n)/(2f(y_n) - f(x_n)), \\ z_n = y_n + \mu f(y_n), \\ x_{n+1} = z_n + \mu f(z_n). \end{cases} \tag{2}$$

This method improves the local order of convergence of Ostrowski's method with an additional evaluation of the function.

Motivated by this method, Sharma and Guha [3] present the following family:

$$\begin{cases} y_n = x_n - f(x_n)/f'(x_n), \\ z_n = y_n - f(y_n)(x_n - y_n)/(f(x_n) - 2f(y_n)), \\ x_{n+1} = z_n - \frac{f(x_n)+af(y_n)}{f(x_n)+(a-2)f(y_n)} \frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{3}$$

where $a \in \mathbb{R}$. For $a = 0$ in (3), the method (2) is obtained.

Furthermore, Chun and Ham [4] present the following family:

$$\begin{cases} y_n = x_n - f(x_n)/f'(x_n), \\ z_n = y_n - f(y_n)(x_n - y_n)/(f(x_n) - 2f(y_n)), \\ x_{n+1} = z_n - H(u_n) \frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{4}$$

where $u_n = f(y_n)/f(x_n)$ and $H(t)$ represents a real-valued function. It is observed that if taking $H(t) = 1/(1-2t)$, then the method defined by (4) reduces to (2). They prove that if any function $H(t)$ satisfies the properties $H(0) = 1, H'(0) = 2, |H''(0)| < \infty$, the method defined by (4) is a sixth-order variant of Ostrowski's method.

Based on (2), a family of modified Ostrowski's methods with seventh-order convergence is presented by Kou et al. in [5]

$$\begin{cases} y_n = x_n - f(x_n)/f'(x_n), \\ z_n = y_n - f(y_n)(x_n - y_n)/(f(x_n) - 2f(y_n)), \\ x_{n+1} = z_n - \left[(1 + H_2(x_n, y_n))^2 + H_\alpha(y_n, z_n)\right] \frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{5}$$

where $H_2(x_n, y_n) = f(y_n)/(f(x_n) - 2f(y_n))$ and $H_\alpha(y_n, z_n) = f(z_n)/(f(y_n) - \alpha f(z_n))$, $\alpha \in \mathbb{R}$.

Anther family of seventh-order methods has been studied in [6]. Recently, some eighth-order methods are also presented in [7, 8, 9, 10, 11]. Kou et al. in [11] propose two families of new methods with eighth-order convergence as the following.

One family of methods is given by

$$\begin{cases} H_\beta(y_n, z_n) = \frac{f(z_n)}{f(y_n) - \beta f(z_n)}, \\ x_{n+1} = z_n - \left[(1 + H_2(x_n, y_n))^2 + (1 + 4H_2(x_n, y_n))H_\beta(y_n, z_n)\right] \frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{6}$$

where $\beta \in \mathbb{R}$.

The other is given by

$$\begin{cases} u_n = z_n - (1 + H_2(x_n, y_n))^2 \frac{f(z_n)}{f'(x_n)}, \\ x_{n+1} = u_n - (1 + 4H_2(x_n, y_n)) \frac{z_n - u_n}{y_n - u_n - \beta(z_n - u_n)} \frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{7}$$

where $\beta \in \mathbb{R}$.

In (6) and (7), y_n, z_n and $H_2(x_n, y_n)$ are defined by

$$\begin{cases} y_n = x_n - f(x_n)/f'(x_n), \\ H_2(x_n, y_n) = f(y_n)/(f(x_n) - 2f(y_n)), \\ z_n = y_n - H_2(x_n, y_n)(x_n - y_n). \end{cases} \tag{8}$$

In this paper, we present a class of new iterative methods with some parameters for the non-linear equation. By choosing the proper parameters, we get different families of methods with convergence order five, six, seven and eight respectively. Numerical tests show the new eighth-order methods work better than the other compared methods. The convergence analysis is also given.

2. Main results

We consider the iteration scheme consisting of two substeps. The first substep is one iterate to get z_n from x_n, namely

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ H_v(x_n,y_n) = \frac{f(y_n)}{f(x_n)-vf(y_n)}, \\ z_n = y_n - \left(H_v(x_n,y_n) + (2-v)H_v(x_n,y_n)^2 + tH_v(x_n,y_n)^3\right)\frac{f(x_n)}{f'(x_n)}, \end{cases} \tag{9}$$

where $v,t \in \mathbb{R}$.

Let

$$K_u(y_n,z_n) = \frac{f(z_n)}{f(y_n) - uf(z_n)}, \tag{10}$$

where $u \in \mathbb{R}$. Denote $H_v = H_v(x_n,y_n)$ and $K_u = K_u(y_n,z_n)$. The second substep is to calculate x_{n+1} from the new point z_n by a family of methods given by

$$x_{n+1} = z_n - (P^T Q)\frac{f(z_n)}{f'(x_n)}, \tag{11}$$

where

$$P = (1,a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8)^T, \tag{12}$$

$$Q = (1,H_v,H_v^2,H_v^3,H_v^4,H_vK_u,H_v^2K_u,K_u,K_u^2)^T, \tag{13}$$

and $a_1 \cdots a_8 \in \mathbb{R}$.

If taking different parameters $a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,u,v,t$, we can obtain different families of methods with convergence order five, six, seven and eight. The convergence analysis is shown in the following theorem.

Theorem 1. *Assume that the function $f: D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D has a simple root $x^* \in D$. Furthermore assume that $f(x)$ is sufficiently smooth in the neighborhood of the root x^*, and z_n is given by* (9)*. If the initial approximation x_0 is sufficiently close to x^*, then*

(I) *for $a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,u,v,t \in \mathbb{R}$, the methods defined by* (11) *can be of fifth-order convergence.*

(II) *for $a_1 = 2,a_2,a_3,a_4,a_5,a_6,a_7,a_8,u,v,t \in \mathbb{R}$, the methods defined by* (11) *can be of sixth-order convergence.*

(III) *for $a_1 = 2,a_2 = 1+2v-v^2+t,a_7 = 1,a_3,a_4,a_5,a_6,a_8,u,v,t \in \mathbb{R}$, the methods defined by* (11) *can be of seventh-order convergence.*

(IV) *for $a_1 = 2,a_2 = 1+2v-v^2+t,a_3 = 6v-2v^2-4+tv+2t,a_5 = 4,a_7 = 1$, a_4,a_6,a_8,u, $v,t \in \mathbb{R}$, the methods defined by* (11) *can be of eighth-order convergence.*

Proof. Using Taylor expansion and taking into account $f(x^*)=0$, we have

$$f(x_n)=f'(x^*)\left[e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+c_5e_n^5+c_6e_n^6+O(e_n^7)\right], \tag{14}$$

where $c_k=(1/k!)f^{(k)}(x^*)/f'(x^*), k=2,3,\cdots$. Furthermore, we have

$$f'(x_n)=f'(x^*)\left[1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+6c_6e_n^5+O(e_n^6)\right]. \tag{15}$$

and hence, we have

$$\begin{aligned} y_n-x^* \;=\;& c_2e_n^2-2\left(c_2^2-c_3\right)e_n^3+\left(4c_2^3-7c_2c_3+3c_4\right)e_n^4-\left(8c_2^4+6c_3^2\right.\\ &\left.+10c_2c_4-20c_2^2c_3-4c_5\right)e_n^5+(5c_6-13c_2c_5-17c_4c_3\\ &+28c_4c_2^2+33c_2c_3^2-52c_2^3c_3+16c_2^5)e_n^6+O(e_n^7). \end{aligned} \tag{16}$$

Again expanding $f(y_n)$ about x^* and from (16), we have

$$\begin{aligned} f(y_n) \;=\;& f'(x^*)[c_2e_n^2-2\left(c_2^2-c_3\right)e_n^3+\left(5c_2^3-7c_2c_3+3c_4\right)e_n^4\\ &-(12c_2^4+6c_3^2+10c_2c_4-24c_2^2c_3-4c_5)e_n^5\\ &+(5c_6-13c_2c_5-17c_4c_3+34c_4c_2^2\\ &+37c_2c_3^2-73c_3c_2^3+28c_2^5)e_n^6+O(e_n^7)]. \end{aligned} \tag{17}$$

Furthermore, we have

$$\begin{aligned} H_v(x_n,y_n) \;=\;& c_2e_n+\left(2c_3+(v-3)c_2^2\right)e_n^2+(3c_4+(4v-10)c_2c_3\\ &+(v^2-6v+8)c_2^3)e_n^3+((6v-14)c_2c_4+(37-32v+6v^2)c_2^2c_3\\ &+(25v-9v^2+v^3-20)c_2^4+(4v-8)c_3^2+4c_5)e_n^4\\ &+[(8v-18)c_2c_5+5c_6+(48-88v+51v^2-12v^3+v^4)c_2^5\\ &+(12v-22)c_3c_4+\left(9v^2-46v+51\right)c_2^2c_4+(55-56v+12v^2)c_2c_3^2\\ &+(8v^3-66v^2+166v-118)c_2^3c_3\right]e_n^5+O(e_n^6). \end{aligned} \tag{18}$$

So it follows that

$$\begin{aligned} z_n-x^* \;=\;& y_n-x^*-\left(H_v(x_n,y_n)+(2-v)H_v(x_n,y_n)^2+tH_v(x_n,y_n)^3\right)\frac{f(x_n)}{f'(x_n)}\\ &=\left((v^2-4v-t+5)c_2^3-c_2c_3\right)e_n^4\\ &+\left[(2v^3-16v^2+40v-3vt+10t-36)c_2^4-2c_3^2-2c_2c_4\right.\\ &\left.+(6v^2-24v-6t+32)c_2^2c_3\right]e_n^5+[(3v^4-34v^3+140v^2-248v\\ &-6v^2t+39vt-62t+170)c_2^5+(9v^2-9t-36v+48)c_2^2c_4\\ &+(296v-122v^2+16v^3-24vt+74t-262)c_2^3c_3-7c_3c_4-3c_2c_5\\ &+(66-48v+12v^2-12t)c_2c_3^2]e_n^6+O(e_n^7). \end{aligned} \tag{19}$$

Taylor expansion of $f(z_n)$ about x^* is

$$f(z_n) = f'(x^*)\left[(z_n - x^*) + c_2(z_n - x^*)^2 + O((z_n - x^*)^3)\right]. \tag{20}$$

Then we have

$$\begin{aligned} K_u(y_n, z_n) \;=\; & \left((v^2 - 4v - t + 5)c_2^2 - c_3\right)e_n^2 \\ & + \left[(2v^3 - 14v^2 + 32v - 3vt + 8t - 26)c_2^3 - 2c_4\right]e_n^3 \\ & + (4v^2 - 16v - 4t + 20)c_2c_3e_n^3 + [(93 + 33vt - 10ut \\ & + 26uv^2 - 6v^2t + v^4u - 40uv - 8v^3u + t^2u - 164v \\ & + 107v^2 - 30v^3 - 41t + 8uvt - 2v^2ut + 3v^4 + 25u)c_2^4 \\ & + (12v^3 - 79v^2 + 172v - 130 - 18vt + 8uv - 2uv^2 + 2ut \\ & + 43t - 10u)c_2^2c_3 + (19 - 4t + u + 4v^2 - 16v)c_3^2 \\ & + (6v^2 - 24v - 6t + 29)c_2c_4 - 3c_5]e_n^4 + O(e_n^5). \end{aligned} \tag{21}$$

Since we get from (11) that

$$\begin{aligned} e_{n+1} \;=\; & z_n - x^* - \left(1 + a_1H_v + a_2H_v^2 + a_3H_v^3 + a_4H_v^4 + a_5H_vK_u \right. \\ & \left. + a_6H_v^2K_u + a_7K_u + a_8K_u^2\right)\frac{f(z_n)}{f'(x_n)}, \end{aligned}$$

then we have

$$e_{n+1} = (a_1 - 2)\left[c_2^2c_3 + (t - v^2 + 4v - 5)c_2^4\right]e_n^5 + \phi_1e_n^6 + \phi_2e_n^7 + \phi_3e_n^8 + O(e_n^9), \tag{22}$$

where

$$\begin{aligned} \phi_1 \;=\; & (96v - 36v^2 - 92 - 15a_1t - 6tv + 25a_1v^2 - 65a_1v - 25a_7 + 24t \\ & + 61a_1 + 4v^3 + 4a_1tv + 2a_7v^2t + 10a_7t + a_2t - 8a_7vt \\ & + 8a_7v^3 - a_2v^2 - a_7t^2 + 4a_2v - 3a_1v^3 - a_7v^4 - 26a_7v^2 \\ & - 5a_2 + 40a_7v)c_2^5 + (a_2 + 83 + 10a_7 - 47a_1 + 8a_1t \\ & + 33a_1v - 8a_1v^2 + 2a_7v^2 - 60v + 15v^2 - 2a_7t - 15t \\ & - 8a_7v)c_2^3c_3 + (2a_1 - 4)c_2^2c_4 + (4a_1 - a_7 - 7)c_2c_3^2, \end{aligned}$$

$$\begin{aligned} \phi_2 \;=\; & (524 + 352v^2 - 688v + 130a_1t + 90tv - 315a_1v^2 + 596a_1v \\ & + a_3t + 8a_5v^3 + 6v^4 + 360a_7 - 12tv^2 - 172t - 440a_1 - 5a_3 \\ & - 76v^3 - 72a_1tv - 76a_7v^2t - 25a_5 - 172a_7t - 18a_2t \\ & + 190a_7vt - 228a_7v^3 + 32a_2v^2 + 20a_7t^2 - 82a_2v + 72a_1v^3 \\ & + 48a_7v^4 - 26a_5v^2 - a_5v^4 + 4a_3v + 40a_5v - a_3v^2 - 6a_1v^4 - a_5t^2 \\ & + 10a_5t - 4a_2v^3 - 4a_7v^5 + 552a_7v^2 + 76a_2 + 2a_5v^2t \\ & + 5a_2tv + 10a_7v^3t + 10a_1tv^2 - 6a_7t^2v - 8a_5vt - 688a_7v)c_2^6 \end{aligned}$$

$$
\begin{aligned}
&+(856v-328v^2-828-141a_1t-57tv+242a_1v^2-624a_1v\\
&-342a_7+214t+597a_1+a_3+38v^3+40a_1tv+20a_7v^2t+10a_5\\
&+124a_7t+10a_2t-86a_7vt+84a_7v^3-10a_2v^2-10a_7t^2+42a_2v\\
&-30a_1v^3-10a_7v^4+2a_5v^2-8a_5v-2a_5t-296a_7v^2-60a_2\\
&+496a_7v)c_2^4c_3+(-4a_7t+124+20a_7+12a_1t+22v^2-12a_1v^2\\
&+50a_1v-16a_7v+2a_2-88v+4a_7v^2-73a_1-22t)c_2^3c_4\\
&+(3a_1-6)c_2^2c_5+(24a_1t+64a_7-24a_1v^2+6a_2-12a_7t\\
&-42t-168v-157a_1+248-a_5+12a_7v^2-48a_7v+42v^2\\
&+102a_1v)c_2^2c_3^2+(14a_1-4a_7-24)c_2c_3c_4+(4a_1-2a_7-6)c_3^3,
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_3 \quad = \quad &\big(218tv^2-24a_8tv^3+3848v-63a_7v^4u-21a_7t^2v^2-3a_8t^2v^2\\
&-90a_5tv^2+3a_7v^4ut-2390a_7tv+78a_8tv^2+3a_8tv^4+300a_7uv\\
&-105a_2tv+15a_2tv^2-7a_5t^2v+12a_5tv^3+224a_5tv-a_7v^6u\\
&-125a_8+962t+2438a_1-6450a_7v^2-25a_6+2a_6v^2t+12a_8t^2v\\
&+30a_7v^4t-330a_7v^3t-120a_8tv+a_7t^3u-210a_1tv^2-8a_6vt\\
&+12a_7v^5u+6a_3tv+75a_7ut-129v^4-2538v^2+8v^5+816v^3\\
&+156a_7v^5-20tv^3-26a_6v^2+10a_6t+a_8t^3+914a_2v\\
&-63a_8v^4-850a_1t-10a_7v^6-223a_7t^2+3372a_7v^3+1670a_7t\\
&-10a_1v^5+138a_7t^2v+20a_1tv^3+732a_1tv+435a_5+1334a_7tv^2\\
&-15a_7ut^2-315a_7uv^2-5a_4+184a_7uv^3-2971a_7+91a_3-t^2\\
&-5a_5v^5-663a_2-24a_7v^3ut-99a_3v-202a_5t-492a_2v^2+75a_8t\\
&-15a_8t^2+183a_2t+6680a_7v-10a_2v^4+39a_3v^2+300a_8v+12a_8v^5\\
&-4063a_1v-997a_7v^4+23a_5t^2-a_6t^2+59a_5v^4-315a_8v^2-788tv\\
&-a_4v^2+116a_2v^3+2771a_1v^2+670a_5v^2+a_4t+154a_1v^4-a_8v^6\\
&+184a_8v^3-5a_3v^3-278a_5v^3+8a_6v^3+40a_6v+4a_4v-125a_7u\\
&-a_6v^4-21a_3t-833a_5v-932a_1v^3-120a_7utv-3a_7t^2uv^2\\
&+12a_7vut^2+78a_7utv^2-2393\big)c_2^7+\big[a_1(120tv^2+1402t-820tv\\
&+6526v-3525v^2-72v^4+832v^3-4764)+a_2(60tv-966v\\
&+376v^2-205t-48v^3+918)+a_3(51v-12v^2+12t-73)\\
&+a_5(24tv^2-104tv+150t-12t^2-12v^4-362v^2+102v^3\\
&+610v-422)+a_7(3v^4u+2274tv-120uv+6662v^2+120tv^3\\
&-30ut-48v^5+228t^2-2716v^3-2114t-72t^2v-900tv^2\\
&+3ut^2+78uv^2-24uv^3-8456v+570v^4+75u+24utv\\
&-6utv^2+4534)-138tv^2-7104v-1776t+10a_6+69v^4\\
&+3738v^2-838v^3+2a_6v^2-2a_6t+a_4+981tv-8a_6v
\end{aligned}
$$

$$
\begin{aligned}
&+a_8(24tv-6tv^2+3v^4-30t+3t^2-120v+78v^2\\
&-24v^3+75)+5332\big]c_2^5c_3+(1232v-132a_7tv+308t+880a_1\\
&-460a_7v^2-476v^2+56v^3+64a_2v-207a_1t-15a_7t^2+128a_7v^3\\
&+196a_7t+60a_1tv+20a_5+30a_7tv^2-549a_7+2a_3-94a_2\\
&-4a_5t-15a_2v^2+15a_2t+784a_7v-922a_1v-15a_7v^4-84tv\\
&+359a_1v^2+4a_5v^2-16a_5v-45a_1v^3-1184)c_2^4c_4\\
&+[29v^2-29t+a_7(30+6v^2-6t-24v)-116v+3a_2\\
&+a_1(16t+67v-16v^2-99)+165]c_2^3c_5+(2304a_7v-2940\\
&-120a_1v^3+352a_7v^3+87a_5-3a_8v^2-1667a_7+3036v\\
&-a_6-16a_5t+12a_8v-2385a_1v-65a_5v-15a_8+144v^3\\
&-15a_7u+936a_1v^2+16a_5v^2-216tv+40a_2t+3a_8t+576a_7t\\
&+2294a_1-368a_7tv+176a_2v+8a_3-40a_7t^2+160a_1tv+3a_7ut\\
&-3a_7uv^2+12a_7uv+80a_7v^2t-40a_7v^4-40a_2v^2+759t-273a_2\\
&-1312a_7v^2-528a_1t-1191v^2)c_2^3c_3^2+(4a_1-8)c_2^2c_6\\
&+(123v^2-72a_1v^2+42a_7v^2-42a_7t-481a_1-4a_5+21a_2\\
&-168a_7v+72a_1t+309a_1v-492v-123t+224a7+732)c_2^2c_3c_4\\
&+(20a_1-11a_7-29)c_3^2c_4+(12a_1-4a_7-20)c_2c_4^2\\
&+(20a_1-6a_7-34)c_2c_3c_5+(52v^2-208v+a_8+32a_1t-96a_7v\\
&+132a_7-52t+12a_2-24a_7t+a_7u-224a_1-4a_5\\
&+24a_7v^2-32a_1v^2+140a_1v+319)c_2c_3^3.
\end{aligned}
$$

This means that the methods defined by (11) are at least fifth-order convergent when $a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,\ u,v,t\in\mathbb{R}$.

If we take $a_1=2,a_2,a_3,a_4,a_5,a_6,a_7,a_8,u,v,t\in\mathbb{R}$, then

$$
\begin{aligned}
e_{n+1} \;=\; &[(30+2tv-5a_2-25a_7-34v+14v^2-2v^3-6t+40a_7v-26a_7v^2\\
&+10a_7t+8a_7v^3-a_7v^4-a_7t^2+4a_2v+a_2t-a_2v^2-8a_7vt+2a_7v^2t)c_2^5\\
&+(1-a_7)c_2c_3^2+(6v-11+t-v^2+10a_7+a_2+2a_7v^2\\
&-2a_7t-8a_7v)c_2^3c_3]e_n^6+O(e_n^7). \qquad (23)
\end{aligned}
$$

This means that the methods defined by (11) are at least sixth-order convergent in this case.

Furthermore, if we take $a_1=2,a_2=1+2v-v^2+t,a_7=1,a_3,a_4,a_5,a_6,a_8,u,v,t\in\mathbb{R}$, then

$$
\begin{aligned}
e_{n+1} \;=\; &[(80+tv^3-8tv^2+23tv-114v+66v^2-18v^3+2v^4-t^2v-26t\\
&+2t^2-5a_3-25a_5+4a_3v-a_3v^2+a_3t+40a_5v-26a_5v^2+10a_5t\\
&-8a_5vt+2a_5v^2t+8a_5v^3-a_5v^4-a_5t^2)c_2^6(6t-36+26v-6v^2-tv\\
&+a_3+10a_5-8a_5v+2a_5v^2-2a_5t)c_2^4c_3+(4-a_5)c_2^2c_3^2]e_n^7+O(e_n^8). \qquad (24)
\end{aligned}
$$

This means that the methods defined by (11) are at least seventh-order convergent in this case.

From (22) and (24), it is easy to know that if we take $a_1 = 2, a_2 = 1+2v-v^2+t, a_3 = 6v-2v^2-4+tv+2t,\ a_5 = 4, a_7 = 1, a_4, a_6, a_8, u, v, t \in \mathbb{R}$, then the methods defined by (11) are at least eighth-order convergent. And especially when we take $a_1 = 2, a_2 = 1+2v-v^2+t, a_3 = 6v-2v^2-4+tv+2t, a_4 = 2v^3-14v^2+32v+5t-25, a_5 = 4, a_6 = t-v^2+9,\ a_7 = 1, a_8 = 1-u, u, v, t \in \mathbb{R}$, then we can obtain the methods defined by (11) satisfy

$$e_{n+1} = \left[(5-4v+v^2-t)c_4c_2^4 - c_4c_3c_2^2\right] e_n^8 + O(e_n^9). \tag{25}$$

This ends the proof. □

Note that if we take $P = (1,2,1,0,0,4,0,1,0)^T, v = 2, t = 0, u \in \mathbb{R}$, we get a family of eighth-order methods given by (6), so we know that one families of methods stated in [11] is a special one of the family defined by (11).

We assume that the computational cost of the first derivative evaluation is the same to the function evaluation, then per iteration the number of function evaluations of the methods given by (11) is four. We consider the definition of efficiency index [12] as $p^{\frac{1}{w}}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method. We have that the methods given by (11) have the max efficiency index equal to $\sqrt[4]{8} \backsimeq 1.682$, which is better than the ones of Ostrowski's method ($\sqrt[3]{4} \backsimeq 1.587$), the sixth-order variants given by (2)-(4) ($\sqrt[4]{6} \backsimeq 1.565$) and the seventh-order methods given by (5) ($\sqrt[4]{7} \backsimeq 1.627$).

3. Numerical tests

Now we employ the new methods given by (11) to solve some non-linear equations ,where $P = (1,2,1+2v-v^2+t, 6v-2v^2-4+tv+2t, 2v^3-14v^2+32v+5t-25, 4, t-v^2+9, 1, 1-u)^T, v, t, u$ are given in table 1. We compare them with Ostrowski's method (OM), the method [2] given by (2) (GDM), the method [5] given by (5) (KM1) with $\alpha = 3$, the method [11] given by (6) (KWM1) and (7) (KWM2) with $\beta = 3$.

Table 1 shows the expression of the test functions, the root with fourteen significant digits. In these methods it is necessary to begin with one initial approximation x_0. In the second column of Table 2, we present the initial approximations which are the same for all methods.

Displayed in Table 2 are the absolute values of the function ($|f(x_n)|$) calculated by costing the same total number of function evaluations (TNFE) required by all methods. Here, the TNFE for all methods is equal to 12.

Table 1. Test functions and their roots

Test functions	Root	v	u	t
$f_1(x) = x^3 + 4x^2 - 15$	1.6319808055661	2	2.2	1
$f_2(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$	-1.2076478271309	2	-2	1
$f_3(x) = \sin(x) - \frac{1}{2}x$	1.8954942670340	2	-2	1
$f_4(x) = 10xe^{-x^2} - 1$	1.6796306104284	2	-2.5	1
$f_5(x) = \cos(x) - x$	0.73908513321516	2	1	1
$f_6(x) = \sin^2(x) - x^2 + 1$	1.4044916482153	2	-10	1
$f_7(x) = e^{-x} + \cos(x)$	1.7461395304080	2	3.5	1

Table 2. Comparison of various iterative methods

	x_0	OM	GDM	KM1	KWM1	KWM2	Eq. (11)
f_1	2	1.03e-228	4.46e-179	3.93e-276	1.33e-438	1.04e-440	6.02e-840
f_2	-1	8.82e-223	2.54e-155	3.08e-264	4.07e-425	6.60e-429	1.86e-490
f_3	1.9	8.18e-656	5.71e-541	2.93e-844	4.41e-1299	3.38e-1295	5.54e-1422
f_4	1.5	1.91e-210	7.81e-165	1.33e-252	5.99e-463	2.64e-442	1.54e-595
f_5	1	7.05e-296	4.12e-237	5.87e-366	1.95e-571	2.96e-619	9.16e-713
f_6	1.5	6.99e-300	1.05e-239	2.21e-369	3.24e-586	3.21e-586	2.06e-729
f_7	2	1.05e-279	1.58e-223	1.86e-335	4.50e-545	2.92e-607	9.44e-860

Many numerical applications use high precision in their computations. In these types of applications, high order numerical methods are important [2]. As far as the tests we consider, the new eighth-order methods show higher precision than Ostrowski's method with the same TNFE. This superiority of the new eighth-order methods, illustrated by these results, is in accordance with the theory of efficiency analysis developed in the previous section.

4. Conclusions

We have obtained a class of new iterative methods with some parameters for the non-linear equation. By choosing the proper parameters, we get different families of methods with convergence order five, six, seven and eight. Per iteration the new methods require three evaluations of the function and one evaluation of its first derivative and therefore have the max efficiency index equal to $\sqrt[4]{8} \backsimeq 1.682$. Numerical tests demonstrate that the new eighth-order methods are preferable to the classical Ostrowski's method in high precision computations.

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