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On the $\mathbb{C}_p$-Banach algebra of the r-Lipschitz functions

by
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Dedicated to the memory of Laurenţiu Panaitopol (1940-2008) on the occasion of his 70th anniversary

Abstract

Given a prime number p, we study the $\mathbb{C}_p$-Banach algebra of the r-Lipschitz functions defined on compact subsets of $\mathbb{C}_p$ by introducing a new seminorm on this space. Also, we give an estimation of the integral of a r-Lipschitz function with respect to a s-distribution and then we obtain a analogous of Hölder's inequality.

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1 Introduction

Let p be a prime number, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers and let $|\cdot|$ be the usual p-adic module. This module can be uniquely extended to a module (denoted also by $|\cdot|$) on $\overline{\mathbb{Q}}_p$, a fixed algebraic closure of $\mathbb{Q}_p$. Further, denote by $\mathbb{C}_p$, which is called the Tate field, the completion of $(\overline{\mathbb{Q}}_p, |\cdot|)$, and we use the same notation $|\cdot|$ for the unique p-adic module that extends the p-adic module $|\cdot|$ on $\overline{\mathbb{Q}}_p$. Denote $G = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, which is the absolute Galois group, and topologise it with the so called Krull topology. Then G acts continuously on $\overline{\mathbb{Q}}_p$ and, it is easy to see that G is canonically isomorphic with the group $Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$ of all continuous automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$. Let $O(T)$ be the orbit of an element T of $\mathbb{C}_p$ with respect to the Galois group G.

The paper consists of two sections. The first section contains some basic results and preliminaries. In the second section we study the $\mathbb{C}_p$-Banach algebra of the r-Lipschitz functions defined on G-equivariant compacts of $\mathbb{C}_p$ by introducing a new seminorm on this space, see Proposition 1. Here, by G-equivariant compacts of $\mathbb{C}_p$ we mean compacts of $\mathbb{C}_p$ which are equivariant with respect to the absolute Galois group G, like finite union of orbits of elements of $\mathbb{C}_p$. We have a few estimations for p-adic integrals, see Theorem 2 and Theorem 3 where we obtain a analogous of Hölder's inequality. An estimation for the norm of a Lipschitz function is also given in Theorem 4 for compacts like orbits of elements of $\mathbb{C}_p$.

2 Background material

Let p be a prime number and $\mathbb{Q}_p$ the field of p-adic numbers endowed with the p-adic absolute value $|\cdot|$, normalized such that $|p| = 1/p$. Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and denote by the same symbol $|\cdot|$ the unique extension of $|\cdot|$ to $\overline{\mathbb{Q}}_p$. Further, denote by $(\mathbb{C}_p, |\cdot|)$ the completion of $(\overline{\mathbb{Q}}_p, |\cdot|)$ (see [1], [3]). Let $G = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ endowed with the Krull topology. The group G is canonically isomorphic with the group $Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$, of all continuous automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$. We shall identify these two groups. For any $T \in \mathbb{C}_p$ denote $O(T) = \{\sigma(T) : \sigma \in G\}$ the orbit of T, and let $\widetilde{\mathbb{Q}_p[T]}$ be the closure of the ring $\mathbb{Q}_p[T]$ in $\mathbb{C}_p$.

For any closed subgroup H of G denote $Fix(H) = \{x \in \mathbb{C}_p : \sigma(x) = x \text{ for all } \sigma \in H\}$. Then $Fix(H)$ is a closed subfield of $\mathbb{C}_p$. Denote $H(T) = \{\sigma \in G : \sigma(T) = T\}$. Then $H(T)$ is a subgroup of G, and $Fix(H(T)) = \widetilde{\mathbb{Q}_p[T]}$. Moreover, for any $\varepsilon > 0$ and $T \in \mathbb{C}_p$ denote $H(T, \varepsilon) = \{\sigma \in G : |\sigma(T) - T| < \varepsilon\}$. Let $\mathcal{S}_\varepsilon$ be a complete system of representatives for the left cosets of G with respect to $H(T, \varepsilon)$.

The map $\sigma \rightsquigarrow \sigma(T)$ from G to $O(T)$ is continuous, and it defines a homeomorphism from $G/H(T)$ (endowed with the quotient topology) to $O(T)$ (endowed with the induced topology from $\mathbb{C}_p$) (see [2]). In such a way $O(T)$ is a closed compact and totally disconnected subspace of $\mathbb{C}_p$, and the group G acts continuously on $O(T)$: if $\sigma \in G$ and $\tau(T) \in O(T)$ then $\sigma \star \tau(T) := (\sigma\tau)(T)$.

Now, if $\mathcal{X}$ is a compact subset of $\mathbb{C}_p$ then by an open ball in $\mathcal{X}$ we means a subset of the form $B(x, \varepsilon) \cap \mathcal{X}$ where $x \in \mathbb{C}_p$ and $\varepsilon > 0$. Let us denote by $\Omega(\mathcal{X})$ the set of subsets of $\mathcal{X}$ which are open and compact. It is easy to see that any $D \in \Omega(\mathcal{X})$ can be written as a finite union of open balls in $\mathcal{X}$, any two disjoint.

Definition 1. *By a distribution on $\mathcal{X}$ with values in $\mathbb{C}_p$ we mean a map $\mu : \Omega(\mathcal{X}) \to \mathbb{C}_p$ which is finitely additive, that is, if $D = \cup_{i=1}^n D_i$ with $D_i \in \Omega(\mathcal{X})$ for $1 \le i \le n$ and $D_i \cap D_j = \emptyset$ for $1 \le i \ne j \le n$, then $\mu(D) = \sum_{i=1}^n \mu(D_i)$. The space $\mathcal{D}(\mathcal{X}, \mathbb{C}_p)$ of all distributions on $\mathcal{X}$ with values in $\mathbb{C}_p$ becomes naturally a $\mathbb{C}_p$-vector space (See [4]).*

The norm of μ is defined by $\|\mu\| := \sup\{|\mu(D)| : D \in \Omega(\mathcal{X})\}$. If $\|\mu\| < \infty$ we say that μ is a measure on $\mathcal{X}$. With this norm, the space $\mathcal{M}(\mathcal{X}, \mathbb{C}_p)$ of all measures on $\mathcal{X}$ with values in $\mathbb{C}_p$ becomes a $\mathbb{C}_p$-Banach space.

The set $\mathcal{X} \subset \mathbb{C}_p$ is said G-equivariant provided that $\sigma(x) \in \mathcal{X}$ for any $x \in \mathcal{X}$ and any $\sigma \in G$. ($\mathcal{X} = O(T)$ is such an example.)

Definition 2. *Let $\mathcal{X}$ be a G-equivariant compact subset of $\mathbb{C}_p$ and μ a distribution on $\mathcal{X}$ with values in $\mathbb{C}_p$. We say that μ is G-equivariant if $\mu(\sigma(B)) = \sigma(\mu(B))$ for any ball B in $\mathcal{X}$ and any $\sigma \in G$. Denote by $\mathcal{D}^G(\mathcal{X}, \mathbb{C}_p)$ the set of G-equivariant distributions on $\mathcal{X}$.*

Remark. On a Galois orbit $O(T)$ there exists a unique G-equivariant probability distribution (with values in $\mathbb{Q}_p$), namely the Haar distribution π_T.

Definition 3. *Let s be a positive real number. We say that a distribution $\mu \in \mathcal{D}(\mathcal{X}, \mathbb{C}_p)$ is s-boundedly increasing distribution (or simply a s-distribution) if*

$$\lim_{\varepsilon \to 0} \varepsilon^s \max |\mu(B(a, \varepsilon))| = 0.$$

(Here the "max" is taken over all the balls $B(a, \varepsilon)$ from $\Omega(\mathcal{X})$, the set of all open compact subsets of $\mathcal{X}$.) The space $\mathcal{D}_s(\mathcal{X}, \mathbb{C}_p)$ of all s-distributions becomes $\mathbb{C}_p$-vector space. When $\mathcal{X}$ is G-equivariant denote by $\mathcal{D}_s^G(\mathcal{X}, \mathbb{C}_p)$ the subspace of G-equivariant distributions.

Remark 1) Any measure on $\mathcal{X}$ is s-boundedly increasing distribution.

2) There is no other distribution, except for the identically zero distribution with the property that

$$\lim_{\varepsilon \to 0} \max_{B(a,\varepsilon) \subset \mathcal{X}} |\mu(B(a, \varepsilon)| = 0.$$

Indeed, every $A \in \Omega(\mathcal{X})$, which is open compact set, is a union of sets of the form $B(a, \varepsilon)$ with ε arbitrarily small. Clearly,

$$|\mu(A)| \le \max_{B(a,\varepsilon) \subset \mathcal{X}} |\mu(B(a, \varepsilon))| \to 0,$$

which implies $\mu(A) = 0$.

3) The s-boundedly increasing distributions increase strictly slower than the Haar distribution.

4) An element $T \in \mathbb{C}_p$ is called s-boundedly iff the Haar distribution π_T is s-distribution, which means $\lim_{\varepsilon \to o} \frac{\varepsilon^s}{|N(T,\varepsilon)|} = 0$, where $N(T, \varepsilon)$ is the number of balls of radius ε that cover the orbit of T.

5) An element $T \in \mathbb{C}_p$ is called p-bounded if there exists a positive integer k such that for any $\varepsilon > 0$ one has p^k is not a divisor of $N(T, \varepsilon)$. In this situation the Haar distribution π_T is a measure.

We have $\mathcal{M}(\mathcal{X}, \mathbb{C}_p) \subset \mathcal{D}_s(\mathcal{X}, \mathbb{C}_p) \subset \mathcal{D}(\mathcal{X}, \mathbb{C}_p)$. In the case when $s = 1$, we have 1-distributions $\mathcal{D}_1(\mathcal{X}, \mathbb{C}_p)$ that are called *Lipschitz* distributions, and these distributions play an important role in nonarchimedean integration theory, see [6].

Definition 4. *Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$ and let r be a positive real number. A function $f : \mathcal{X} \to \mathbb{C}_p$ is called of type r, or r-Lipschitz function, iff there exists a positive constant c such that*

$$|f(x) - f(y)| \le c|x - y|^r, \tag{1}$$

for any $x, y \in \mathcal{X}$.

Now, for a r-Lipschitz function f as above, the best constant in (1) is

$$c_f = \sup_{\substack{x,y\in\mathcal{X}\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^r}. \tag{2}$$

Denote $\|f\|_r := \max\{c_f, \|f\|\}$, where $\|f\| = \sup_{x\in\mathcal{X}} |f(x)|$.

Finally, if $Lip_r(\mathcal{X}, \mathbb{C}_p)$ is the set of r-Lipschitz functions as above, then it becomes naturally a $\mathbb{C}_p$-Banach algebra with the norm $\|\cdot\|_r$ defined above.

3 Main results

In what follows we keep the same notations and definitions as in the previous paragraph. Here and henceforth we suppose that $\mathcal{X}$ is an open compact of $\mathbb{C}_p$. Let us recall the following theorem.

Theorem 1. *(See [6]) Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$. Then any function $f : \mathcal{X} \to \mathbb{C}_p$ of type $r > 0$ is integrable with respect to any s-distribution $\mu : \Omega(\mathcal{X}) \to \mathbb{C}_p$, whenever $0 < s \leq r$.*

We use this theorem to prove the following result.

Theorem 2. *Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$ and let s be a fixed positive real number. Then, for any $r \geq s$ any $f \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ and any $\mu \in \mathcal{D}_s(\mathcal{X}, \mathbb{C}_p)$, there exists a positive real number A, which is independent of f, such that*

$$\left| \int_{\mathcal{X}} f \mathrm{d}\mu \right| \leq A \cdot \|f\|_r . \tag{3}$$

Proof: From Theorem 1, any function $f : \mathcal{X} \to A$ of type r is integrable with respect to any s-distribution $\mu : \Omega(X) \to \mathbb{C}_p$, whenever $r \geq s$. We first construct a sequence $(S_m)_{m\in\mathbb{N}}$ of Riemann sums as follows. For each positive integer m, write $\mathcal{X}$ as a finite union of open balls of radius $\frac{1}{2^m}$, any two disjoint, denote them by $B_{m,1}, B_{m,2}, \ldots, B_{m,N_m}$. Next, choose points $x_{m,j} \in B_{m,j}$, for $1 \leq j \leq N_m$, and denote by S_m the corresponding Riemann sums, $S_m = S(\mu, f, B_{m,1}, \ldots, B_{m,N_m}, x_{m,1}, \ldots, x_{m,N_m})$. We know that $(S_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}_p$ that converges to $\int_{\mathcal{X}} f\mathrm{d}\mu$. Each open ball of radius $\frac{1}{2^m}$ in $\mathcal{X}$ can be written as a finite union of open balls of radius $\frac{1}{2^{m+1}}$, any two disjoint. Therefore there are disjoint nonempty sets $J_1, \ldots, J_{N_m}$, with $J_1 \cup \cdots \cup J_{N_m} = \{1, 2, \ldots, N_{m+1}\}$, such that for any $i \in \{1, 2, \ldots, N_m\}$, $B_{m,i} = \bigcup_{j\in J_i} B_{m+1,j}$. We now put $S_m - S_{m+1}$ in the form

$$\begin{aligned} S_m - S_{m+1} &= \sum_{i=1}^{N_m} \mu(B_{m,i}) f(x_{m,i}) - \sum_{i=1}^{N_{m+1}} \mu(B_{m+1,i}) f(x_{m+1,i}) \\ &= \sum_{i=1}^{N_m} \Big(\mu(B_{m,i}) f(x_{m,i}) - \sum_{j\in J_i} \mu(B_{m+1,j}) f(x_{m+1,j}) \Big). \end{aligned}$$

Here we may rewrite $\mu(B_{m,i})$ as $\sum_{j\in J_i}\mu(B_{m+1,j})$ by the additivity of μ. Hence

$$S_m - S_{m+1} = \sum_{i=1}^{N_m}\sum_{j\in J_i}\mu(B_{m+1,j})\big(f(x_{m,i}) - f(x_{m+1,j})\big).$$

It follows that

$$|S_m - S_{m+1}| \le \max_{1\le i\le N_m}\max_{j\in J_i}|\mu(B_{m+1,j})|\cdot|f(x_{m,i}) - f(x_{m+1,j})|.$$

Since f is of type r for some $r>0$, we derive

$$|S_m - S_{m+1}| \le c_f \max_{1\le i\le N_m}\max_{j\in J_i}|\mu(B_{m+1,j})|\cdot|x_{m,i} - x_{m+1,j})|^r,$$

where c_f is a constant that depends only on f. Here both $x_{m,i}$ and $x_{m+1,j}$ belong to the open ball $B_{m,i}$ of radius $\frac{1}{2^m}$, so

$$\begin{aligned}|S_m - S_{m+1}| &\le \frac{c_f}{2^{rm}}\max_{1\le i\le N_m}\max_{j\in J_i}|\mu(B_{m+1,j})|\\ &\le 2^r c_f \max_{1\le i\le N_m}\max_{j\in J_i}\frac{|\mu(B_{m+1,j})|}{2^{s(m+1)}}.\end{aligned}$$

Here on the far right side $B_{m+1,j}$ is an open ball of radius $\frac{1}{2^{m+1}}$, therefore the ratio $\frac{|\mu(B_{m+1,j})|}{2^{s(m+1)}}$ goes to zero as $m\to\infty$, uniformly for $j\in J_i$, $1\le i\le N_m$. Precisely, for any $\varepsilon>0$ there is a $\delta_\varepsilon>0$ such that for any $0<\delta\le\delta_\varepsilon$ and any open ball B of radius δ one has $\delta^s|\mu(B)|\le\varepsilon$. Then for any $m\ge[\log_2(1/\delta_\varepsilon)]$, we have $\frac{|\mu(B_{m+1,j})|}{2^{s(m+1)}}\le\varepsilon$ for any j, and so $|S_m - S_{m+1}|\le 2^r c_f\varepsilon$.

Let m be large enough such that

$$\begin{aligned}\Big|\int_{\mathcal{X}} f(x)\mathrm{d}\mu(x)\Big| &= |S_{m+1}|\\ &= |S_{m+1} - S_m + S_m - S_{m-1} + \cdots + S_2 - S_1 + S_1|\\ &\le \max_{1\le i\le m}\{|S_1|, |S_{i+1} - S_i|\} \le A\cdot\|f\|_r\,,\end{aligned}$$

where

$$A = \max\{2^r \sup_{m\ge1}\max_{1\le i\le N_m}\max_{j\in J_i}\frac{|\mu(B_{m+1,j})|}{2^{s(m+1)}}, \max_{1\le i\le N_1}\{|\mu(B_{1,i})|\}\}$$

and the proof is done. □

Now let $f, g\in Lip_r(\mathcal{X},\mathbb{C}_p)$ and c_f, c_g defined as above. One has

$$\begin{aligned}|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)|\\ &= |(f(x) - f(y))g(x) + f(y)(g(x) - g(y))|\\ &\le \max\{\|g\|\,c_f|x-y|^r, \|f\|\,c_g|x-y|^r\}\\ &= \max\{\|g\|\,c_f, \|f\|\,c_g\}|x-y|^r,\end{aligned}$$

so we have $fg \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ and

$$c_{fg} \le \max\{\|g\|\, c_f, \|f\|\, c_g\}. \tag{4}$$

Moreover, if $f \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ and $g \in Lip_s(\mathcal{X}, \mathbb{C}_p)$ then $fg \in Lip_{\min\{r,s\}}(\mathcal{X}, \mathbb{C}_p)$. In the same manner it is easy to see that $f + g \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ and

$$c_{f+g} \le \max\{c_f, c_g\}. \tag{5}$$

From Theorem 2, if $f, g \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ we infer

$$\left| \int_{\mathcal{X}} f(t)g(t)\mathrm{d}\mu(t) \right| \le A\, \|fg\|_r \le A \max\{\|g\|\, c_f, \|f\|\, c_g, \|f\|\, \|g\|\}. \tag{6}$$

By (6) one can define

$$\|f\|_\mu := \sup_{\substack{g \in Lip_r(\mathcal{X}, \mathbb{C}_p) \\ g \ne 0}} \frac{\left| \int_{\mathcal{X}} f(t)g(t)\mathrm{d}\mu(t) \right|}{\|g\|_r} < \infty. \tag{7}$$

It is easy to see that $\|f\|_\mu \ge 0$ and $\|\lambda f\|_\mu = |\lambda|\, \|f\|_\mu$, for any $f \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ and any $\lambda \in \mathbb{C}_p$. If $f_1, f_2 \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ we have $\|f_1 + f_2\|_\mu \le \max\{\|f_1\|_\mu, \|f_2\|_\mu\}$. Because of μ-neglected functions from $\|f\|_\mu = 0$ we do not have $f = 0$. We collect the above result in the following

Proposition 1. *Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$ and let s be a positive real number. Then, for any $r \ge s$ and any $\mu \in \mathcal{D}_s(\mathcal{X}, \mathbb{C}_p)$, one has that $\|\cdot\|_\mu$ is a seminorm on $Lip_r(\mathcal{X}, \mathbb{C}_p)$.*

We have the following result.

Proposition 2. *For any $D \in \Omega(\mathcal{X})$, $\emptyset \ne D \ne \mathcal{X}$ we have $d(D, \mathcal{X} \setminus D) > 0$, so the characteristic function χ_D of D is an element of $Lip_r(\mathcal{X}, \mathbb{C}_p)$ for any $r > 0$.*

Proof: It is enough to prove the proposition for $D = B(a, \varepsilon) \in \Omega(\mathcal{X})$, $\varepsilon > 0$. Then we cover $\mathcal{X}$ with balls of radius ε and it is easy to see that $d(D, \mathcal{X} \setminus D) = \inf_{x \in D,\, y \in \mathcal{X} \setminus D} |x - y| = \varepsilon > 0$. By a simple computation one has $c_{\chi_D} = \frac{1}{\varepsilon^r}$ so $\chi_D \in Lip_r(\mathcal{X}, \mathbb{C}_p)$. For more details see Proposition 19.2, page 51 from [5]. □

Remark 1) For any $D \in \Omega(\mathcal{X})$ we have $\|\chi_D\|_\mu$ is well defined and $\|\chi_D\|_\mu \ge |\mu(D)|$.

2) Let $\{B_n\}_{n \ge 0}$ be a sequence of balls of radius ε_n such that $\lim_{n \to \infty} \varepsilon_n = 0$ and let $\{a_n\}_{n \ge 0}$ be a sequence of elements of $\mathbb{C}_p$ such that $\sup_{n \ge 0} \frac{|a_n|}{\varepsilon_n^r} < \infty$. Then $f = \sum_{n \ge 0} a_n \chi_{B_n} \in Lip_r(\mathcal{X}, \mathbb{C}_p)$. Indeed, one has

$$|f(x) - f(y)| = \Big| \sum_{n \ge 0} a_n [\chi_{B_n}(x) - \chi_{B_n}(y)] \Big| \le \sup_{n \ge 0} |a_n| |\chi_{B_n}(x) - \chi_{B_n}(y)| \le$$

$$\leq \sup_{n\geq 0} \frac{|a_n|}{\varepsilon_n^r}|x-y|^r,$$

by Proposition 2.

Theorem 3. *Let $\mathcal{X}$ be a compact subset of $\mathbb{C}_p$ and let s be a positive real number. Then, for any $r \geq s$ any $f, g \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ and any $\mu \in \mathcal{D}_s(\mathcal{X}, \mathbb{C}_p)$, we have $\left|\int_{\mathcal{X}} f \mathrm{d}\mu\right| \leq \|f\|_\mu$ and $\|fg\|_\mu \leq \|f\|_\mu \|g\|_r$ (Hölder's inequality).*

Proof: The first inequality is clear from definition definition of $\|f\|_\mu$ by taking $g = 1$. For the second, we have $\|gh\|_r \leq \|g\|_r \|h\|_r$ so

$$\|f\|_\mu \geq \frac{\left|\int_{\mathcal{X}} fgh\mathrm{d}\mu\right|}{\|gh\|_r} \geq \frac{\left|\int_{\mathcal{X}} fgh\mathrm{d}\mu\right|}{\|g\|_r \|h\|_r}.$$

We infer that

$$\|g\|_r \|f\|_\mu \geq \frac{\left|\int_{\mathcal{X}} fgh\mathrm{d}\mu\right|}{\|h\|_r},$$

for any $h \in Lip_r(\mathcal{X}, \mathbb{C}_p)$, $h \neq 0$ so by definition of $\|fg\|_\mu$ one has the Hölder's inequality. □

Let us define $\|U\|_\mu := \|\chi_U\|_\mu$, for any $U \in \Omega(\mathcal{X})$. By a simple calculation we have $\|B_1 \cup B_2\|_\mu \leq \max\{\|B_1\|_\mu, \|B_2\|_\mu\}$ for any balls $B_1, B_2 \in \Omega(\mathcal{X})$ with $B_1 \cap B_2 = \emptyset$.

Now, let us suppose that $\mathcal{X}$ is G-equivariant and $\mu \in \mathcal{D}_s^G(\mathcal{X}, \mathbb{C}_p)$. For any $\sigma \in G$ we have

$$\|B\|_\mu = \|B^\sigma\|_\mu, \tag{8}$$

where $B = B(x, \varepsilon) \in \Omega(\mathcal{X})$ and $B^\sigma = B(\sigma(x), \varepsilon)$. Indeed, let $g \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ be a Lipschitz function of type r such that $g \neq 0$ and let $\sigma \in G$ be a continuous automorphism. Denote $h := \sigma \circ g \circ \sigma^{-1}$. Then $|h(x) - h(y)| = |g \circ \sigma^{-1}(x) - g \circ \sigma^{-1}(y)| \leq c_g |\sigma^{-1}(x) - \sigma^{-1}(y)|^r = c_g |x-y|^r$, so $h \in Lip_r(\mathcal{X}, \mathbb{C}_p)$, $c_h = c_g$ and $\|g\| = \|h\|$. Moreover, if $B = \cup_{i=1}^{N(\delta)} B_i$ is a decomposition of B in balls of radius δ, $0 < \delta < \varepsilon$, where $B_i = B(x_i, \delta)$ then

$$\begin{aligned}\int_{B^\sigma} h\mathrm{d}\mu &= \lim_{\delta\to 0} \sum_{i=1}^{N(\delta)} \mu(B_i^\sigma) h(\sigma x_i)\\ &= \lim_{\delta\to 0} \sum_{i=1}^{N(\delta)} \sigma\mu(B_i)\sigma g(x_i) = \sigma\Big(\int_B g\mathrm{d}\mu\Big).\end{aligned} \tag{9}$$

By (9) we infer that

$$\|B_\sigma\|_\mu \geq \frac{\left|\int_{B^\sigma} h\mathrm{d}\mu\right|}{\|h\|_r} = \frac{\left|\int_B g\mathrm{d}\mu\right|}{\|g\|_r},$$

so $\|B\|_\mu \leq \|B_\sigma\|_\mu$. Now, if we consider $h \in Lip_r(\mathcal{X}, \mathbb{C}_p)$, $h \neq 0$, $\sigma \in G$, then by defining $g = \sigma^{-1} \circ h \circ \sigma$ the reverse inequality goes in the same way as above. One has the following result.

Proposition 3. *Let $\mathcal{X}$ be a G-equivariant compact of $\mathbb{C}_p$ and let μ be a G-equivariant distribution on $\mathcal{X}$ with values in $\mathbb{C}_p$. Then, for any $B \in \Omega(\mathcal{X})$ and any $\sigma \in G$, we have*

$$\|B\|_\mu = \|B^\sigma\|_\mu .$$

In what follows let $\mathcal{X} = O(T)$ be the orbit of a p-bounded element T of $\mathbb{C}_p$. We consider $r \geq s > 0$ two positive real numbers. Let $\mu \in D_s^G(\mathcal{X}, \mathbb{Q}_p)$ be a s-distribution with values in $\mathbb{Q}_p$ that is G-equivariant. For any $\varepsilon > \varepsilon' > 0$ one considers $B(\varepsilon) = B(T, \varepsilon) = \cup_{i=1}^N B(\sigma_i(T), \varepsilon')$ be a decomposition of the ball $B(\varepsilon)$ in balls of radius ε'. One has $N = \frac{N(T,\varepsilon')}{N(T,\varepsilon)}$ and, because T is p-bounded p is not a divisor of N for any $\varepsilon > \varepsilon' > 0$, with ε small enough. Now, for any $g \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ one defines $h \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ as follows:

$$h(x) = (g \circ \sigma^{-1})(x), \text{ when } x \in B(\sigma(T), \varepsilon'),\ \sigma \in \mathcal{S}_{\varepsilon'}. \tag{10}$$

By a simple computation we infer that $h \in Lip_r(\mathcal{X}, \mathbb{C}_p)$ and $\|h\|_r \leq \|g\|_r$. Moreover $\int_{B(\varepsilon)} h \mathrm{d}\mu = N \int_{B(\varepsilon')} g \mathrm{d}\mu$ and then $\left| \int_{B(\varepsilon)} h \mathrm{d}\mu \right| = \left| \int_{B(\varepsilon')} g \mathrm{d}\mu \right|$. By this one has

$$\frac{\left| \int_{B(\varepsilon')} g \mathrm{d}\mu \right|}{\|g\|_r} \leq \frac{\left| \int_{B(\varepsilon)} h \mathrm{d}\mu \right|}{\|h\|_r}, \tag{11}$$

so $\|B(\varepsilon')\|_\mu \leq \|B(\varepsilon)\|_\mu$ for any $\varepsilon > \varepsilon' > 0$, with ε small enough. In such a way if we define

$$\|T\|_\mu = \inf_{\varepsilon > 0} \|B(T, \varepsilon)\|_\mu$$

the following result holds.

Theorem 4. *Let T be a p-bounded element of $\mathbb{C}_p$ and let G be the absolute Galois group. Let s be positive real number and let $\mu \in D_s^G(O(T), \mathbb{Q}_p)$ be a s-distribution that is G-equivariant. Then, for any $r \geq s$ and any $f \in Lip_r(O(T), \mathbb{C}_p)$, one has*

$$\|f\|_\mu \leq \|T\|_\mu \cdot \|f\|_r .$$

Proof: Let $O(T) = \cup_{\sigma \in \mathcal{S}_\varepsilon} B(\sigma(T), \varepsilon)$ be a decomposition of the orbit of T in balls of radius ε. Then we decompose $f = \sum_{\sigma \in \mathcal{S}_\varepsilon} f \chi_{B(\sigma(T), \varepsilon)}$, where $\chi_{B(\sigma(T), \varepsilon)}$ is the characteristic function of $B(\sigma(T), \varepsilon)$. By Hölder's inequality and (8) we have

$$\|f\|_\mu \leq \max_{\sigma \in \mathcal{S}_\varepsilon} \left\| f \chi_{B(\sigma(T), \varepsilon)} \right\|_\mu \leq \|f\|_r \max_{\sigma \in \mathcal{S}_\varepsilon} \left\| \chi_{B(\sigma(T), \varepsilon)} \right\|_\mu = \|f\|_r \cdot \|B(T, \varepsilon)\|_\mu$$

for any $\varepsilon > 0$, so the proof is done. □

Remark 1) Under the hypothesis of Theorem 4 it is clear that $\|T\|_\mu = 0$ if and only if $\mu = 0$.

2) If f, g are r-Lipschtz functions and g is neglected i.e. $\|g\|_\mu = 0$, then $\|f + g\|_\mu = \|f\|_\mu$. Indeed, on one hand $\|f + g\|_\mu \leq \max\{\|f\|_\mu, \|g\|_\mu\} = \|f\|_\mu$ and, on the other hand $\|f\|_\mu = \|f + g - g\|_\mu \leq \max\{\|f + g\|_\mu, \|g\|_\mu\} = \|f + g\|_\mu$.

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