Bull. Math. Soc. Sci. Math. Roumanie
Tome 53(101) No. 4, 2010, 363–372

Special Stanley Decompositions

by
Adrian Popescu

Abstract

Let I be an intersection of three monomial prime ideals of a polynomial algebra S over a field. We give a special Stanley decomposition of I which provides a lower bound of the Stanley depth of I, greater or equal with depth (I), that is Stanley's Conjecture holds for I.

Key Words: Monomial Ideals, Stanley decompositions, Stanley depth.
2010 Mathematics Subject Classification: Primary 13C15; Secondary 13F20, 13F55, 13P10.

Introduction

Let K be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial ring over K in n variables. Let $I \subset S$ be a squarefree monomial ideal of S, $u \in I$ a monomial and $uK[Z]$, $Z \subset \{x_1, \ldots, x_n\}$ the linear K-subspace of I of all elements uf, $f \in K[Z]$. A Stanley decomposition of I is a presentation of I as a finite direct sum of spaces $\mathcal{D}: \ I = \bigoplus_{i=1}^{r} u_i K[Z_i]$. Set $\text{sdepth}(\mathcal{D}) = \min\{|Z_i| : i = 1, \ldots, r\}$ and

$$\text{sdepth } (I) := \max\{\text{sdepth } (\mathcal{D}) : \ \mathcal{D} \text{ is a Stanley decomposition of } I\}.$$

By Stanley's Conjecture [10] the Stanley depth sdepth (I) of I is $\geq$ depth (I). This is proved if either $n \leq 5$ by [6], or I is the intersection of two monomial irreducible ideals by [7, Theorem 5.6]. It is the purpose of our paper to show that Stanley's Conjecture holds for intersections of three monomial prime ideals (see Theorem 2.6) and for arbitrary intersections of prime ideals generated by disjoint sets of variables (see Theorem 1.4). For the proof we give a special Stanley decomposition $\mathcal{D}$ of I and compute sdepth $(\mathcal{D})$ (see Lemma 2.2 and Proposition 2.3) which is $\geq$ depth (I).

1 Intersections of primes generated by disjoint sets of variables

Let $S = K[x_1, \dots, x_n]$, $I \subset K[x_1, \dots, x_r] = S'$ and $J \subset K[x_{r+1}, \dots, x_n] = S''$ be monomial ideals, where $1 < r < n$. The following two lemmas are elementary, their proofs being suggested by [11, Theorem 2.2.21] and [7, Lemma 4.1].

Lemma 1.1. *Then*

$$\text{depth}_S(IS \cap JS) = \text{depth}_{S'}(I) + \text{depth}_{S''}(J).$$

Proof: In the exact sequence of S-modules:

$$0 \to \frac{S}{IS \cap JS} \to \frac{S}{IS} \oplus \frac{S}{JS} \to \frac{S}{IS + JS} \to 0$$

we have $\text{depth}_S\left(\frac{S}{IS+JS}\right) = \text{depth}_{S'}(S'/I) + \text{depth}_{S''}(S''/J)$ by [11, Theorem 2.2.21]. Using Depth Lemma we get

$$\text{depth}_S\left(\frac{S}{IS \cap JS}\right) = \text{depth}_S\left(\frac{S}{IS + JS}\right) + 1 =$$

$$= \text{depth}_{S'}(S'/I) + \text{depth}_{S''}(S''/J) + 1,$$

which is enough. □

The following result is analogous to [8, Theorem 3.1].

Lemma 1.2. *With the hypotheses from the previous lemma, we have*

$$\text{sdepth}_S(IS \cap JS) \geq \text{sdepth}_{S'}(I) + \text{sdepth}_{S''}(J).$$

The proof of [7, Lemma 4.1] works also in our case.

Remark 1.3. The inequality of the above lemma can be strict as happens in the case when $n = 4$, $r = 2$, $I = (x_1, x_2)$, $J = (x_3, x_4)$. Indeed, then $\text{sdepth}_{S'}(I) = 1$, $\text{sdepth}_{S''}(J) = 1$, and $\text{sdepth}_S(IS \cap JS) = 3 > 2 = \text{sdepth}_{S'}(I) + \text{sdepth}_{S''}(J)$, as shows the Stanley decomposition

$$IS \cap JS = x_1x_3K[x_1, x_3, x_4] \oplus x_1x_4K[x_1, x_2, x_4] \oplus$$

$$x_2x_3K[x_1, x_2, x_3] \oplus x_2x_4K[x_2, x_3, x_4] \oplus x_1x_2x_3x_4S.$$

Theorem 1.4. *Let* $0 = r_0 < r_1 < r_2 < \ldots < r_s = n$, $S = K[x_1, \dots, x_n]$ *and set* $P_1 = (x_1, \dots, x_{r_1})$, $P_2 = (x_{r_1+1}, \dots, x_{r_2}), \dots, P_s = (x_{r_{s-1}+1}, \dots, x_{r_s})$ *and* $I = \bigcap_{i=1}^{s} P_i$. *Then,*

$$\text{sdepth } (I) \geq \text{depth } (I) = s,$$

and in particular Stanley's Conjecture holds in this case.

Proof: Let us denote $S^i = K[x_{r_{i-1}+1}, \ldots, x_{r_i}]$. Since $\operatorname{depth}_{S^i}\left(\frac{S^i}{P_i \cap S^i}\right) = 0$ we get $\operatorname{depth}_{S^i}(P_i \cap S^i) = 1$. By Lemma 1.1 and recurrence we obtain depth $(I) = s$. Applying Lemma 1.2, by recurrence and [9] we get that

$$\operatorname{sdepth}_S\ (I) \geq \sum_{i=1}^{s} \operatorname{sdepth}_{S^i}(P_i \cap S^i) = \sum_{i=1}^{s} \left\lceil \frac{r_i - r_{i-1}}{2} \right\rceil \geq s,$$

where $\lceil a \rceil$ is the lowest integer number greater or equal to $a \in \mathbb{R}$. $\square$

Remark 1.5. *If $s = 2$, then Ishaq [5, Corollary 2.9, 2.10] proved that* $\operatorname{sdepth}(I) = \left\lceil \frac{n+1}{2} \right\rceil$ *if either n is odd, or n is even but r_1 is odd, and* $\frac{n}{2} \leq \operatorname{sdepth}(I) \leq \frac{n}{2}+1$ *if n and r_1 are even.*

In the next section we need the following two lemmas:

Lemma 1.6. *[7, Lemma 4.3] Let $S = K[x_1, \ldots, x_n]$, $Q = (x_1, \ldots, x_t)$ and $Q' = (x_{r+1}, \ldots, x_n)$ where $1 \leq r \leq t < n$. Then*

$$\operatorname{sdepth}(Q \cap Q') \geq \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{n-t}{2} \right\rceil.$$

Lemma 1.7. *[4, Lemma 3.6] Let $I \subset S = K[x_1, \ldots, x_n]$ and $S' = S[x_{n+1}, \ldots, x_t]$, where $t > n$. Then,*

$$\operatorname{sdepth}_{S'}(IS') = \operatorname{sdepth}_S(I) + (t - n).$$

2 Intersections of three prime ideals

Let S=K$[x_1,\ldots,x_n]$ and P_1,P_2,P_3 be three non-zero monomial prime ideals not included one in the other such that $\sum_{i=1}^{3} P_i = (x_1, \ldots, x_n)$. Let I=$P_1 \cap P_2 \cap P_3$.

Proposition 2.1. *Then*

$$\operatorname{depth}\ (I) = \begin{cases} 3, \text{ if } P_i \not\subset P_j + P_k \text{ for any different } i,j,k \in \{1,2,3\} \\ n + 2 - \max\{\operatorname{ht}(P_i + P_j), \operatorname{ht}(P_i + P_k)\}, \text{if } P_i \subset P_j + P_k\ . \end{cases}$$

Proof: Consider the following exact sequence of S-modules:

$$(1) \quad 0 \to \frac{S}{P_1 \cap P_2} \to \frac{S}{P_1} \oplus \frac{S}{P_2} \to \frac{S}{P_1 + P_2} \to 0.$$

We have :

$$\operatorname{depth}\left(\frac{S}{P_1+P_2}\right)=n-\operatorname{ht}(P_1+P_2)$$

and

$$\operatorname{depth}\left(\frac{S}{P_1}\oplus\frac{S}{P_2}\right)=\min\left\{\operatorname{depth}(S/P_1),\operatorname{depth}(S/P_2)\right\}=n-\max\left\{\operatorname{ht}(P_1),\operatorname{ht}(P_2)\right\}.$$

By hypotheses $\operatorname{ht}(P_1+P_2)>\max\{\operatorname{ht}(P_1),\operatorname{ht}(P_2)\}$ thus $n-\operatorname{ht}(P_1+P_2)<n-\max\{\operatorname{ht}(P_1),\operatorname{ht}(P_2)\}$. That means $\operatorname{depth}(\frac{S}{P_1+P_2})<\operatorname{depth}(\frac{S}{P_1}\oplus\frac{S}{P_2})$ and applying Depth Lemma to (1) we obtain $\operatorname{depth}(\frac{S}{P_1\cap P_2})=n-\operatorname{ht}(P_1+P_2)+1$. There are two cases:

Case 1. $P_i\not\subset P_j+P_k$ for any different $i,j,k\in\{1,2,3\}$.

Let us consider the following two exact sequences of S-modules:

$$(2)\quad 0\to\frac{S}{I}\to\frac{S}{P_1\cap P_2}\oplus\frac{S}{P_3}\to\frac{S}{P_3+(P_1\cap P_2)}\to 0,$$

$$(3)\quad 0\to\frac{S}{(P_1+P_3)\cap(P_2+P_3)}\to\frac{S}{(P_1+P_3)}\oplus\frac{S}{(P_2+P_3)}\to\frac{S}{P_1+P_2+P_3}\to 0.$$

By the hypothesis of this case, we have $\operatorname{depth}(\frac{S}{P_1+P_3})>0$ and $\operatorname{depth}(\frac{S}{P_2+P_3})>0$. Applying Depth Lemma to (3) we get $\operatorname{depth}\left(\frac{S}{(P_1+P_3)\cap(P_2+P_3)}\right)=1$ because $P_1+P_2+P_3$ is the maximal ideal. But $(P_1+P_3)\cap(P_2+P_3)=P_3+(P_1\cap P_2)$, so we get $\operatorname{depth}\left(\frac{S}{P_3+(P_1\cap P_2)}\right)=1$. Using again the hypothesis of this case in (2) we can say that $\operatorname{depth}(S/P_3)>1$ and $\operatorname{depth}(\frac{S}{P_1\cap P_2})>1$. By Depth Lemma applied to (2) we have $\operatorname{depth}(S/I)=2$. Thus $\operatorname{depth}(I)=3$.

Case 2. There exist different $i,j,k\in\{1,2,3\}$ such that $P_i\subset P_j+P_k$.

After a possible renumbering of $(P_i)_{1\le i\le 3}$ we may suppose that $P_1\subset P_2+P_3$. Note that $P_1=P_1\cap P_2+P_1\cap P_3$. Let us consider the next exact sequence of S-modules:

$$(4)\quad 0\to\frac{S}{I}\to\frac{S}{(P_1\cap P_2)}\oplus\frac{S}{P_1\cap P_3}\to\frac{S}{P_1}\to 0.$$

Remark that $\operatorname{depth}(\frac{S}{P_1\cap P_2})$ and $\operatorname{depth}(\frac{S}{P_1\cap P_3})$ are smaller or equal than $\dim(S/P_1)$ (see [2]). We prove that

$$\operatorname{depth}(S/I)=\min\{\operatorname{depth}(\frac{S}{P_1\cap P_2}),\operatorname{depth}(\frac{S}{P_1\cap P_3})\}.$$

If $n-\operatorname{ht}(P_1)=\dim(S/P_1)>\min\{\operatorname{depth}(\frac{S}{P_1\cap P_2}),\operatorname{depth}(\frac{S}{P_1\cap P_3})\}=n+1-\max\{\operatorname{ht}(P_1+P_2),\operatorname{ht}(P_1+P_3)\}$ then we are done by Depth Lemma applied to (4).

Otherwise, $n-\operatorname{ht}(P_1)=n+1-\max\{\operatorname{ht}(P_1+P_2),\operatorname{ht}(P_1+P_3)\}$ and applying again Depth Lemma we get that $\operatorname{depth}(S/I)\geq\min\{\operatorname{depth}(\frac{S}{P_1\cap P_2}),\operatorname{depth}(\frac{S}{P_1\cap P_3})\}$, the inequality being equality because $\operatorname{depth}(S/I) \leq \dim(S/P_1) = n - \operatorname{ht}(P_1) = \min\{\operatorname{depth}(\frac{S}{P_1 \cap P_2}), \operatorname{depth}(\frac{S}{P_1 \cap P_3})\}$. Thus we get

$$\operatorname{depth}(S/I) = n + 1 - \max\{\operatorname{ht}(P_1 + P_2), \operatorname{ht}(P_1 + P_3)\}$$

and so

$$\operatorname{depth}\ (I) = n + 2 - \max\{\operatorname{ht}(P_1 + P_2), \operatorname{ht}(P_1 + P_3)\}.$$

□

The next lemma presents a decomposition of the above I as a direct sum of its linear subspaces. These subspaces are "simpler" monomial ideals, for which we already know "good" Stanley decompositions. Substituting them in the above direct sum we get some *special Stanley decompositions* where it is easier to lower bound their Stanley depth.

We may suppose after a possible renumbering of variables that

$$P_1 = (x_1, \ldots, x_r).$$

Let us denote the following:

$$b_2 - \text{the number of variables from } \{x_i | 1 \leq i \leq r\} \text{ for which } x_i \in P_2,$$

$$b_3 - \text{the number of variables from } \{x_i | 1 \leq i \leq r\} \text{ for which } x_i \in P_3,$$

$$b_1 - \text{the number of variables from } \{x_i | 1 \leq i \leq r\} \text{ for which } x_i \in P_2 \cup P_3,$$

$$a_{23} - \text{the number of variables from } \{x_i | 1 \leq i \leq r\} \text{ for which } x_i \in P_2 \setminus P_3,$$

$$a_{32} - \text{the number of variables from } \{x_i | 1 \leq i \leq r\} \text{ for which } x_i \in P_3 \setminus P_2,$$

$$c - \text{the number of variables from } \{x_i | r + 1 \leq i \leq n\} \text{ for which } x_i \in P_2 \cap P_3,$$

$$A = \left\lceil \frac{a_{32}}{2} \right\rceil + \left\lceil \frac{\operatorname{ht}(P_2) - b_2}{2} \right\rceil + n - a_{32} - \operatorname{ht}(P_2),$$

$$B = \left\lceil \frac{a_{23}}{2} \right\rceil + \left\lceil \frac{\operatorname{ht}(P_3) - b_3}{2} \right\rceil + n - a_{23} - \operatorname{ht}(P_3),$$

$$C = \left\lceil \frac{r - b_1}{2} \right\rceil + \left\lceil \frac{\operatorname{ht}(P_2) - b_2 - c}{2} \right\rceil + \left\lceil \frac{\operatorname{ht}(P_3) - b_3 - c}{2} \right\rceil,$$

$$S' = K[\{x_i | 1 \leq i \leq r, x_i \notin P_3\}],\ S'' = K[\{x_i | 1 \leq i \leq r, x_i \notin P_2\}],$$

$$\widetilde{S} = K[\{x_i | 1 \leq i \leq r, x_i \notin P_2 + P_3\}, x_{r+1}, \ldots, x_n].$$

Lemma 2.2. *Let $S=K[x_1,\ldots,x_n]$ and P_1,P_2,P_3 be three non-zero monomial prime ideals not included one in the other such that $\sum_{i=1}^{3} P_i = (x_1,\ldots,x_n)$. Let $I=P_1\cap P_2\cap P_3$. The next sum is a direct sum of linear subspaces of I:*

$$I = I_1 \oplus I_2 \oplus I_3 \oplus I_4,$$

where:

$$I_1 = (I\cap K[x_1,\ldots,x_r])S,\quad I_2 = (P_2\cap S')S'[x_{r+1},\ldots,x_n]\cap(P_3\cap S'[x_{r+1},\ldots,x_n]),$$

$$I_3 = (P_3\cap S'')S''[x_{r+1},\ldots,x_n]\cap(P_2\cap S''[x_{r+1},\ldots,x_n]),\quad I_4 = I\cap \widetilde{S}.$$

Proof: Note that $I \supseteq I_1+I_2+I_3+I_4$ is obvious because every $I_i \subseteq I$. Conversely, let a be a monomial from I. If $a \notin I_1$, then we have the next *three disjoint cases*:

Case 1. $a \notin (P_2\cap K[x_1,\ldots,x_r])S$ but $a \in (P_3\cap K[x_1,\ldots,x_r])S$.

Let $a = uv$, where $u \in K[x_1,\ldots,x_r]$ and $v \in K[x_{r+1},\ldots,x_n]$ monomials. From the hypothesis of this case we get that $u \notin (P_2\cap K[x_1,\ldots,x_r])$. But P_2 is a prime ideal, so it follows that $v \in P_2$, which leads us to $a \in I_3$.

Case 2. $a \in (P_2\cap K[x_1,\ldots,x_r])S$ but $a \notin (P_3\cap K[x_1,\ldots,x_r])S$.

This case is similar with *Case 1.*

Case 3. $a \notin (P_2\cap K[x_1,\ldots,x_r])S$ and $a \notin (P_3\cap K[x_1,\ldots,x_r])S$.

Let $a = uv$, where $u \in K[x_1,\ldots,x_r]$ and $v \in K[x_{r+1},\ldots,x_n]$ monomials. From the hypothesis of this case we get that $u \notin P_2\cap K[x_1,\ldots,x_r]$ and $u \notin P_3\cap K[x_1,\ldots,x_r]$. Thus $v \in P_2\cap P_3\cap K[x_{r+1},\ldots,x_n]$ because P_2 and P_3 are prime ideals. Hence $a \in I_4$ since $u \in P_1$.

Because the cases are disjoint we get that the sum $I = I_1+I_2+I_3+I_4$ is direct. □

Proposition 2.3. *Let P_1,P_2,P_3 be three non-zero prime monomial ideals of S such that there exists no inclusion between any two of them, $\sum_{i=1}^{3} P_i = (x_1,\ldots,x_n)$ and set $I = P_1\cap P_2\cap P_3$. With the above notations set $D = \operatorname{sdepth}((I\cap K[x_1,\ldots,x_r])S)$ if $I\cap K[x_1,\ldots,x_r] \neq 0$. Then*

$$\operatorname{sdepth}(I) \geq \begin{cases} \min\{A,B,C,D\} & \text{, if } P_i \not\subset P_j + P_k \text{ for any different } \\ & \quad i,j,k \in \{1,2,3\} \\ \min\{A,B,D\} & \text{, if } P_1 \subset P_2 + P_3. \end{cases}$$

The proof follows from Lemma 2.2, but first we see the idea in the following example:

Example 2.4. Let $S = K[x_1,x_2,x_3,x_4]$, $P_1 = (x_1,x_2)$, $P_2 = (x_2,x_3,x_4)$ and $P_3 = (x_1,x_3)$. Then $I = P_1\cap P_2\cap P_3 = (x_1x_2,x_1x_3,x_1x_4,x_2x_3)$ and the following Stanley decomposition of I is given by Lemma 2.2:

$$I = (x_1x_2)\ K[x_1,x_2,x_3,x_4] \oplus (x_2x_3)\ K[x_2,x_3,x_4] \oplus (x_1x_3,x_1x_4)\ K[x_1,x_3,x_4].$$

Note that the last term of Lemma 2.2 does not appear in this example since $P_1 \subset P_2 + P_3$. We see that the first and the third term in the sum are principal ideals. Therefore $\mathrm{sdepth}((x_1x_2)K[x_1, x_2, x_3, x_4]) = 4$ and $\mathrm{sdepth}((x_2x_3)K[x_2, x_3, x_4]) = 3$. As for the second term we use Lemma 1.6, so

$$\mathrm{sdepth}((x_1) \cap (x_3, x_4)\ K[x_1, x_3, x_4]) = \left\lceil \frac{1}{2} \right\rceil + \left\lceil \frac{2}{2} \right\rceil = 2.$$

Thus sdepth $(I) \geq \min\{4, 2, 3\} = 2$. The same thing follows from the Proposition 2.3 because in this case $b_2 = 1, b_3 = 1, b_1 = 2, a_{23} = 1, a_{32} = 1, c = 1, A = 2$ and $B = 3$. Therefore $\mathrm{sdepth}(I) \geq \min\{A, B\} = 2$. Note that depth $(I) = 2$ by Proposition 2.1 so Stanley's Conjecture holds for this example.

Proof of Proposition 2.3

From Lemma 2.2 we have the direct sum of spaces $I = I_1 \oplus I_2 \oplus I_3 \oplus I_4$ where:

$$I_1 = (I \cap K[x_1, \ldots, x_r])S,\ \ I_2 = (P_2 \cap S')S'[x_{r+1}, \ldots, x_n] \cap (P_3 \cap S'[x_{r+1}, \ldots, x_n]),$$

$$I_3 = (P_3 \cap S'')S''[x_{r+1}, \ldots, x_n] \cap (P_2 \cap S''[x_{r+1}, \ldots, x_n]),\quad I_4 = I \cap \widetilde{S}.$$

Then sdepth $(I) \geq \min_{1 \leq i \leq 4} \mathrm{sdepth}(I_i)$. Note that $D = \mathrm{sdepth}_S\ (I_1)$ is strictly greater than the number of free variables which is $n - r = \dim(S/P_1)$. Thus $D \geq$ depth (I) (so D can be later *omitted for* depth *computation*). By Lemma 1.2 we have

$$\mathrm{sdepth}_{S'[x_{r+1},\ldots,x_n]}\ (I_2) \geq \mathrm{sdepth}_{S'}(P_2 \cap S') +$$
$$+\,\mathrm{sdepth}_{K[x_{r+1},\ldots,x_n]}(P_3 \cap K[x_{r+1}, \ldots, x_n]) =$$
$$= \left(\left\lceil \frac{a_{23}}{2} \right\rceil + r - b_3 - a_{23} \right) + \left(\left\lceil \frac{\mathrm{ht}(P_3) - b_3}{2} \right\rceil + (n - r) - (\mathrm{ht}(P_3) - b_3) \right) =$$
$$= \left\lceil \frac{a_{23}}{2} \right\rceil + \left\lceil \frac{\mathrm{ht}(P_3) - b_3}{2} \right\rceil + n - a_{23} - \mathrm{ht}(P_3) = B,$$

applying also [9](see also [3]). Similarly we get

$$\mathrm{sdepth}_{S''[x_{r+1},\ldots,x_n]}\ (I_3) \geq \left\lceil \frac{a_{32}}{2} \right\rceil + \left\lceil \frac{\mathrm{ht}(P_2) - b_2}{2} \right\rceil + n - a_{32} - \mathrm{ht}(P_2) = A.$$

Finally,

$$\mathrm{sdepth}_{\widetilde{S}}\ (I_4) \geq \left\lceil \frac{r - b_1}{2} \right\rceil + \mathrm{sdepth}_{K[x_{r+1},\ldots,x_n]}(P_2 \cap P_3 \cap K[x_{r+1}, \ldots, x_n]).$$

There are two cases:

Case 1. If $P_i \not\subset P_j + P_k$ for any different $i, j, k \in \{1, 2, 3\}$

Note that $P_2 \cap K[x_{r+1}, \ldots, x_n] \not\subset P_3 \cap K[x_{r+1}, \ldots, x_n]$ (otherwise it would result that $P_2 \subset P_1 + P_3$, *contradicting* the hypothesis of this case). In the same idea $P_3 \cap K[x_{r+1}, \ldots, x_n] \not\subset P_2 \cap K[x_{r+1}, \ldots, x_n]$. By applying Lemma 1.6 for $r = \mathrm{ht}(P_2) - b_2 - c$ and $n - t = \mathrm{ht}(P_3) - b_3 - c$ we get

$$\mathrm{sdepth}_{K[x_{r+1},\ldots,x_n]}(P_2 \cap P_3 \cap K[x_{r+1}, \ldots, x_n]) \geq$$
$$\geq \left\lceil \frac{\mathrm{ht}(P_2) - b_2 - c}{2} \right\rceil + \left\lceil \frac{\mathrm{ht}(P_3) - b_3 - c}{2} \right\rceil .$$

Note that there are no free variables above. Therefore

$$\mathrm{sdepth}_{\widetilde{S}}\ (I_4) \geq \left\lceil \frac{r - b_1}{2} \right\rceil + \left\lceil \frac{\mathrm{ht}(P_2) - b_2 - c}{2} \right\rceil + \left\lceil \frac{\mathrm{ht}(P_3) - b_3 - c}{2} \right\rceil = C.$$

Consequently, it follows

$$\mathrm{sdepth}\ (I) \geq \min\{A, B, C, D\}.$$

Case 2. If $P_1 \subset P_2 + P_3$

Note that in this case $\widetilde{S} = K[x_{r+1}, \ldots, x_n]$ and $P_1 \cap \widetilde{S} = 0$. Thus I_4 does not appear in the Stanley decomposition of I given by Lemma 2.2. Hence sdepth $(I) \geq \min\{A, B, D\}$.

If one $I_i = 0$, we consider in both cases that its corresponding integer from $\{A, B, C, D\}$ will not appear in the sdepth formula. □

Remark 2.5. In the notations and hypotheses of Proposition 2.3, let $\widehat{S} = S[x_{n+1}, \ldots, x_t]$ for some $t > n$. Then

$$\mathrm{sdepth}_{\widehat{S}}\ (I\widehat{S}) \geq \begin{cases} \min\{A, B, C, D\} + (t - n) & \text{, if } P_i \not\subset P_j + P_k \\ & \quad \text{for any different } i, j, k \\ \min\{A, B, D\} + (t - n) & \text{, if } P_1 \subset P_2 + P_3, \end{cases}$$

by the Lemma 1.7 and the Proposition 2.3.

Theorem 2.6. *Let P_1, P_2 and P_3 be three non-zero prime monomial ideals of S not included one in the other and set $I = P_1 \cap P_2 \cap P_3$. Then,*

$$\mathrm{sdepth}\ (I) \geq \mathrm{depth}\ (I).$$

Proof: By [4] it is enough to suppose the case when $\sum_{i=1}^{3} P_i = (x_1, \ldots, x_n)$. As in Proposition 2.1 and Proposition 2.3 there are two cases:

Case 1. If $P_i \not\subset P_j + P_k$ for any different $i, j, k \in \{1, 2, 3\}$.

By Propositions 2.1 and 2.3 we have $\mathrm{depth}(I) = 3$ and we must prove that $A \geq 3$, $B \geq 3$ and $C \geq 3$.

By hypothesis of *Case 1* we have $n \geq 3$ and $A \geq \lceil \frac{1}{2} \rceil + \lceil \frac{1}{2} \rceil + 1 = 3$. Similarly we get $B \geq 3$. Also it follows $C \geq \lceil \frac{1}{2} \rceil + \lceil \frac{1}{2} \rceil + \lceil \frac{1}{2} \rceil = 3$. Therefore, sdepth $(I) \geq \mathrm{depth}(I)$.

Case 2. There exist different $1 \leq i \leq j \leq k \leq 3$ such that $P_i \subset P_j + P_k$.

After a possible renumbering of $(P_i)_{1 \leq i \leq 3}$ we may suppose that $P_1 \subset P_2 + P_3$. In this case we show that $A, B \geq \operatorname{depth}(I) = n+2-\max\{\operatorname{ht}(P_1+P_2), \operatorname{ht}(P_1+P_3)\}$ by Propositions 2.1 and 2.3. Since $P_1 \subset P_2 + P_3$ we have

$$a_{32} + \operatorname{ht}(P_2) = \operatorname{ht}(P_1 + P_2).$$

As the P_i's are not included one in the other we get

$$\left\lceil \frac{a_{32}}{2} \right\rceil + \left\lceil \frac{\operatorname{ht}(P_2) - b_2}{2} \right\rceil + n - (a_{32} + \operatorname{ht}(P_2)) \geq 1 + 1 + n - \operatorname{ht}(P_1 + P_2),$$

thus $A \geq \operatorname{depth}(I)$. Similarly it will result that $B \geq \operatorname{depth}(I)$.

In conclusion,

$$\operatorname{sdepth}(I) \geq \operatorname{depth}(I).$$

□

Next we express the integers A, B, C only in terms of heights of (P_i), thus independently of the numbering of the variables.

Proposition 2.7. *With the notations above, we get:*

$$A = \left\lceil \frac{3n - \operatorname{ht}(P_1 + P_2) - \operatorname{ht}(P_2 + P_3) - \operatorname{ht}(P_2)}{2} \right\rceil + \left\lceil \frac{\operatorname{ht}(P_1 + P_2) - \operatorname{ht}(P_1)}{2} \right\rceil,$$

$$B = \left\lceil \frac{3n - \operatorname{ht}(P_1 + P_3) - \operatorname{ht}(P_2 + P_3) - \operatorname{ht}(P_3)}{2} \right\rceil + \left\lceil \frac{\operatorname{ht}(P_1 + P_3) - \operatorname{ht}(P_1)}{2} \right\rceil,$$

$$C = \left\lceil \frac{n - \operatorname{ht}(P_2 + P_3)}{2} \right\rceil + \left\lceil \frac{n - \operatorname{ht}(P_1 + P_3)}{2} \right\rceil + \left\lceil \frac{n - \operatorname{ht}(P_1 + P_2)}{2} \right\rceil.$$

Proof: By definition we have $r = \operatorname{ht}(P_1)$,

$$b_2 = \operatorname{ht}(P_1) + \operatorname{ht}(P_2) - \operatorname{ht}(P_1 + P_2),\ b_3 = \operatorname{ht}(P_1) + \operatorname{ht}(P_3) - \operatorname{ht}(P_1 + P_3),$$

$$b_1 = \operatorname{ht}(P_1) + \operatorname{ht}(P_2 + P_3) - n,$$

$$a_{23} = \operatorname{ht}(P_1 + P_3) + \operatorname{ht}(P_2 + P_3) - \operatorname{ht}(P_3) - n,$$

$$a_{32} = \operatorname{ht}(P_1 + P_2) + \operatorname{ht}(P_2 + P_3) - \operatorname{ht}(P_2) - n,$$

$$c = \operatorname{ht}(P_1 + P_2) + \operatorname{ht}(P_1 + P_3) - \operatorname{ht}(P_1) - n,$$

and it is enough to replace them into the definition of A, B and C. □

References

[1] J. Apel, *On a conjecture of R.P. Stanley; Part I Monomial ideals,* J. Algebraic Combin. 17 (2003), 39-56.

[2] W. Bruns, J. Herzog, *Cohen Macaulay Rings, Revised edition, Cambridge,* Cambridge University Press, 1996.

[3] M. Cimpoeas, *Stanley depth of complete intersection monomial ideals,* Bull. Math. Soc. Sc. Math. Roumanie 51(99)(2008), 205-211.

[4] J. Herzog, M. Vladoiu, X. Zheng, *How to compute the Stanley depth of a monomial ideal,* J. Algebra, 322 (2009), 3151-3169.

[5] M. Ishaq, *Upper bounds for the Stanley depth*, to appear in Comm. Algebra.

[6] D. Popescu, *An inequality between depth and Stanley depth*, Bull. Math. Soc. Sc. Math. Roumanie 52(100), (2009), 377-382.

[7] D. Popescu, I. Qureshi, *Computing the Stanley depth*, J. Algebra, 323 (2010), 2943-2959.

[8] A. Rauf, *Depth and Stanley depth of multigraded modules*, Comm. Algebra, 38 (2010),773-784.

[9] Y. Shen, *Stanley depth of complete intersection monomial ideals and upper-discrete partitions,* J. Algebra 321 (2009), 1285-1292.

[10] R.P. Stanley, *Linear Diophantine equations and local cohomology,* Invent. Math. 68 (1982) 175-193.

[11] R. H. Villarreal, *Monomial Algebras*, Marcel Dekker Inc., New York, 2001.

Received: 10.07.2010
Accepted: 12.09.2010

University of Bucharest,
Department of Mathematics,
Str. Academiei 14, Bucharest, Romania,
E-mail: `adi_popescum@yahoo.de`