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Arithmetical rank of lexsegment edge ideals

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Abstract

Let $I \subset S = K[x_1, \dots, x_n]$ be a lexsegment edge ideal or the Alexander dual of such an ideal. In both cases it turns out that the arithmetical rank of I is equal to the projective dimension of S/I.

Key Words: Arithmetical rank, projective dimension, regularity, edge ideals, squarefree lexsegment ideals, Alexander dual.
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Introduction

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K. Let $I \subset S$ be a homogeneous ideal and $\sqrt{I}$ its radical. The *arithmetical rank* of I is defined as

$$\operatorname{ara}(I) = \min\{r \in \mathbb{N}\colon \text{ there exist } a_1, \dots, a_r \in I \text{ such that } \sqrt{I} = \sqrt{(a_1, \dots, a_r)}\}.$$

Geometrically, $\operatorname{ara}(I)$ is the smallest number of hypersurfaces whose intersection is set-theoretically equal to the algebraic set defined by I, if K is algebraically closed.

For a squarefree monomial ideal $I \subset S$ the following upper bound of $\operatorname{ara}(I)$ is known [8]. Namely,

$$\operatorname{ara}(I) \leq n - \operatorname{indeg}(I) + 1,$$

where $\operatorname{indeg}(I)$ is the *initial degree* of I, that is, $\operatorname{indeg}(I) = \min\{q\colon I_q \neq 0\}$.

Let $\operatorname{cd}(I) = \max\{i \in \mathbb{Z}\colon H_I^i(S) \neq 0\}$, where $H_I^i(S)$ denotes the i-th local cohomology module of S with support at $V(I)$. The number $\operatorname{cd}(I)$ is called the *cohomological dimension* of I. By expressing the local cohomology modules in terms of Cech complex, one can see that $\operatorname{ara}(I)$ is bounded below by $\operatorname{cd}(I)$. For

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a squarefree monomial ideal I of S, it is known that $\mathrm{cd}(I) = \mathrm{proj\,dim}_S(S/I)$ (see [15]). From these inequalities we get

$$\mathrm{proj\,dim}_S(S/I) = \mathrm{cd}(I) \le \mathrm{ara}(I) \le n - \mathrm{indeg}(I) + 1, \tag{1}$$

for any squarefree monomial ideal $I \subset S$.

There are many instances when the equality $\mathrm{proj\,dim}_S(S/I) = \mathrm{ara}(I)$ holds. We refer the reader to [2], [3], [4], [11], [12], [13], [14], [16], [17], [18] for classes of ideals $I \subset S$ whose arithmetical rank is equal to the projective dimension of S/I.

We show in Section 2 that the equality also holds for a lexsegment edge ideal. By a *lexsegment edge ideal* we mean a squarefree monomial ideal generated in degree two by a lexsegment set, that is, a set of the form

$$L(u,v) = \{w\colon w \text{ is a squarefree monomial of degree 2, } u \ge_{\mathrm{lex}} w \ge_{\mathrm{lex}} v\},$$

where $u \ge_{\mathrm{lex}} v$ are two squarefree monomials of degree 2 in S. In order to prove the equality $\mathrm{ara}(I) = \mathrm{proj\,dim}_S(S/I)$ for any lexsegment edge ideal we need first to compute some invariants of these classes of ideals. We make these computations in Section 1. Having the formulas for dimension and depth, we recover the characterization of the Cohen-Macaulay lexsegment edge ideals from [7]. Moreover, since by Theorem 2.2, $\mathrm{ara}(I) = \mathrm{proj\,dim}_S(S/I)$ for any lexsegment edge ideal I, it turns out that any Cohen-Macaulay lexsegment edge ideal is a set-theoretically complete intersection too. In the last section we show that, given a lexsegment edge ideal I, we have the equality $\mathrm{ara}(I^*) = \mathrm{proj\,dim}_S(S/I^*)$ for its Alexander dual I^* as well.

1 Invariants of lexsegment edge ideals

Given a lexsegment edge ideal I, we are going to determine $\dim(S/I)$, $\mathrm{depth}(S/I)$, and $\mathrm{reg}(I)$. Let $u = x_1x_i, v = x_jx_k, j < k$, be two squarefree monomials of degree 2 such that $u \ge_{\mathrm{lex}} v$ and $I = (L(u,v))$ the lexsegment edge ideal generated by the set $L(u,v)$.

We always assume that the set $L(u,v)$ contains at least two elements, that is, $u >_{\mathrm{lex}} v$.

Moreover, for our study, we may consider that $x_1|u$. Indeed, if $u = x_lx_q$ for some $l \ge 2$, then $x_1, \ldots, x_{l-1}$ is a regular sequence on S/I, and we may reduce to the computation of all the invariants in the ring of polynomials in the variables $x_l, \ldots, x_n$.

We first recall the well-known fact that if $u = x_1x_2$ and $v = x_{n-1}x_n$, that is I is equal to the ideal $I_{n,2}$ generated by all the squarefree monomials of degree two in n variables, then we have $\dim(S/I) = \mathrm{depth}(S/I) = 1$ and I has a linear resolution, that is, $\mathrm{reg}(I) = 2$. Therefore, further on, we always consider that $I \ne I_{n,2}$. Moreover, we notice that one may also assume that $j \ge 2$. Indeed, if $j = 1$, that is, $v = x_1x_k$ for some $k \ge i$, then all the invariants can be easily computed.

We begin our study with the computation of $\dim(S/I)$.

If I is an initial lexsegment edge ideal, that is, I is generated by a lexsegment set $L^i(v) = L(u, v)$ where $u = x_1x_2$, then, by [1, Proposition 1.1], we get $\dim(S/I) = n - j$. In the next lemma we compute $\dim(S/I)$ for a final lexsegment edge ideal, that is, generated by a lexsegment set $L^f(u) = L(u, v)$ where $v = x_{n-1}x_n$.

Lemma 1.1. *Let $I = (L^f(u))$, where $u = x_1x_i, i \geq 3$. Then $\dim(S/I) = 2$.*

Proof: Let $\mathfrak{p}$ be a minimal prime ideal of I. Then $\mathfrak{p}$ contains the ideal generated by all the squarefree monomials of degree 2 in the variables $x_2, \ldots, x_n$ whose height is $n - 2$, hence $\mathrm{ht}(\mathfrak{p}) \geq n - 2$, which implies that $\mathrm{ht}(I) \geq n - 2$. On the other hand, since $(x_3, \ldots, x_n) \supset I$, we get $\mathrm{ht}(I) \leq n - 2$. Consequently, $\mathrm{ht}(I) = n - 2$ and $\dim(S/I) = 2$. □

Proposition 1.2. *Let $I = (L(u, v))$ be a lexsegment edge ideal which is neither initial nor final and is determined by $u = x_1x_i$ and $v = x_jx_k$. Then $\dim(S/I) = n - j$.*

Proof: We clearly have $i \geq 3$ and we may assume that $2 \leq j \leq n - 2$. It is obviously that $(x_1, \ldots, x_j) \supset I$, hence $\mathrm{ht}(I) \leq j$, and, therefore, $\dim(S/I) \geq n-j$. We show that the other inequality holds as well. This is obvious if $j = 2$ since (x_1, x_2) is a minimal prime ideal of I. Thus we take $j \geq 3$.

Let us consider $\mathfrak{p}$ a prime ideal which contains I. We distinguish two cases.

Case (a). Let $x_1 \in \mathfrak{p}$, that is, $\mathfrak{p} = (x_1) + \mathfrak{p}'$ where $\mathfrak{p}'$ is generated by a subset of the set $\{x_2, \ldots, x_n\}$. As $I \subset \mathfrak{p}$, it follows that $\mathfrak{p}'$ contains the initial lexsegment defined by v in the ring $K[x_2, \ldots, x_n]$. Therefore, $\mathrm{ht}(\mathfrak{p}') \geq j-1$, by [1, Proposition 1.1.], whence $\mathrm{ht}(\mathfrak{p}) \geq j$.

Case (b). Let $x_1 \notin \mathfrak{p}$. Then $x_i, \ldots, x_n \in \mathfrak{p}$, that is, $\mathfrak{p}$ has the form $\mathfrak{p} = (x_i, \ldots, x_n) + \mathfrak{p}'$, where $\mathfrak{p}'$ is generated by a subset of $\{x_2, \ldots, x_{i-1}\}$. We need to consider the following subcases.

Subcase (b1). $x_{i-2}x_{i-1} \geq_{\mathrm{lex}} v$. Then the ideal generated by all the squarefree monomials of degree 2 in the variables $x_2, \ldots, x_{i-1}$ is contained in $\mathfrak{p}'$ which implies that $\mathrm{ht}(\mathfrak{p}') \geq i - 3$, thus $\mathrm{ht}(\mathfrak{p}) \geq n - 2 \geq j$.

Subcase (b2). Let $x_{i-2}x_{i-1} <_{\mathrm{lex}} v$. Then $\mathfrak{p}'$ contains the initial ideal $(L^i(v)) \subset K[x_2, \ldots, x_{i-1}]$. It follows that $\mathrm{ht}(\mathfrak{p}') \geq j-1$ and, therefore, $\mathrm{ht}(\mathfrak{p}) \geq n-i+j \geq j$.

Consequently, in all cases, we get $\mathrm{ht}(\mathfrak{p}) \geq j$ for any prime ideal $\mathfrak{p} \supset I$ which implies the inequality $\dim(S/I) \leq n - j$. □

In the second part of this section we compute the depth of S/I for an arbitrary lexsegment edge ideal I.

Proposition 1.3. *Let $u = x_1x_i, v = x_jx_k$ with $j \geq 2$, and $I = (L(u, v))$. Then $\mathrm{depth}(S/I) = 1$ if and only if $x_{i-1}x_n \geq_{\mathrm{lex}} v$.*

Proof: Let Δ be the simplicial complex on the vertex set $[n]$ whose Stanley-Reisner ideal is I. It is known that $\operatorname{depth}(S/I) = 1$ if and only if Δ is disconnected, which, in turn, is equivalent to the fact that the skeleton $\Delta^{(1)} = \{F \in \Delta\colon \dim F \leq 1\}$ of Δ is disconnected.

In the first place we consider $\Delta^{(1)}$ disconnected. Let $V_1, V_2 \neq \emptyset$, $V_1 \cup V_2 = [n]$, $V_1 \cap V_2 = \emptyset$, and such that no face of $\Delta^{(1)}$ has vertices in both V_1 and V_2. One may assume that $1 \in V_1$. Then, since $\{1,2\}, \ldots, \{1, i-1\} \in \Delta^{(1)}$, we must have $2, \ldots, i-1 \in V_1$. Let us assume that $v >_{\text{lex}} x_{i-1}x_n$. Then $\{\ell, n\} \in \Delta^{(1)}$ for all $\ell \geq i-1$ which implies that $i, \ldots, n \in V_1$ as well. This leads to $V_1 = [n]$ which is a contradiction to our hypothesis.

For the converse, let $x_{i-1}x_n \geq_{\text{lex}} v$. We claim that $\Delta^{(1)}$ is disconnected. Indeed, one may choose $V_1 = \{1, \ldots, i-1\}$ and $V_2 = \{i, \ldots, n\}$ and observe that for any $1 \leq r \leq i-1$ and $i \leq s \leq n$ we have $x_r x_s \in I$, hence $\{r, s\} \notin \Delta^{(1)}$. $\square$

Corollary 1.4. *Let u and v as in the above proposition. Then* $\operatorname{proj\,dim}_S(S/I) = n-1$ *if and only if* $x_{i-1}x_n \geq_{\text{lex}} v$.

Next we compute the depth of S/I in the case when $v = x_j x_k$ with $j \geq 2$ and $v >_{\text{lex}} x_{i-1}x_n$. In the next lemma we investigate the case $j \geq 3$.

Lemma 1.5. *Let $I = (L(u,v))$ where $u = x_1 x_i, v = x_j x_k$, $j \geq 3$, and $v >_{\text{lex}} x_{i-1}x_n$. Then* $\operatorname{depth}(S/I) = 2$.

Proof: By the hypothesis on v we have $\operatorname{depth}(S/I) \geq 2$. Let Δ be the simplicial complex on $[n]$ such that $I = I_\Delta$. We claim that $\{1, 2\}$ is a facet of Δ. Indeed, if $3 \leq p \leq n$, then $\{1, 2, p\} \notin \Delta$ since $x_2 x_p \in I_\Delta$. Thus $(x_3, \ldots, x_n)$ is a minimal prime of I and so $\operatorname{depth}(S/I) \leq 2$. $\square$

It remains to consider the case $v = x_2 x_k$ for some $k \geq 3$.

Lemma 1.6. *Let $u = x_1 x_i, v = x_2 x_k >_{\text{lex}} x_{i-1}x_n$ and $I = (L(u,v))$. Then*

$$\operatorname{depth}(S/I) = \begin{cases} 2, & \text{if } k \geq i, \\ i+1-k, & \text{if } i > k. \end{cases}$$

Proof: Let us first consider $k \geq i$. One may easily see that I has the following primary decomposition

$$I = (x_1, x_2) \cap (x_1, x_3, \ldots, x_k) \cap (x_2, x_i, \ldots, x_n) \cap (x_3, \ldots, x_n).$$

Hence $\operatorname{depth}(S/I) \leq 2$, which is enough by Proposition 1.3.

For $i > k$ one checks that the minimal monomial generators of I, let us say, $m_1, \ldots, m_r$, satisfy the following condition: for any $1 \leq i \leq r$, there exists $1 \leq j \leq n$ such that $x_j | m_i$ and $x_j \nmid m_\ell$ for all $\ell \neq i$. This implies that the Taylor resolution of S/I is minimal and, therefore, $\operatorname{proj}\dim_S(S/I)$ is equal to the number of the minimal monomial generators of I, that is, $\operatorname{proj}\dim_S(S/I) = n+k-i-1$. Consequently, $\operatorname{depth}(S/I) = i+1-k$. □

Based on the above formulas for dimension and depth we can easily recover the characterization of the Cohen-Macaulay lexsegment edge ideals given in [7].

Corollary 1.7. *Let $I = (L(u,v))$ be a lexsegment edge ideal with $x_1|u$ and $u \neq v$. Then I is Cohen-Macaulay if and only if one of the following conditions holds:*

(i) $I = I_{n,2}$.

(ii) $u = x_1x_n$ *and* $v \in \{x_2x_3, x_{n-2}x_{n-1}, x_{n-2}x_n\}$ *for* $n \geq 4$.

(iii) $u = x_1x_{n-1}$, $v = x_{n-2}x_{n-1}$ *for* $n \geq 3$.

In the last part of this section we compute the regularity of a lexsegment edge ideal.

We first notice that if I is an initial or final lexsegment edge ideal, then $\operatorname{reg}(I) = 2$ since I has a linear resolution. Therefore we may consider that $u \neq x_1x_2$, that is, $i \geq 3$, and $v \neq x_{n-1}x_n$, in other words, $2 \leq j \leq n-2$.

Lemma 1.8. *Let $I = (L(u,v))$ be a lexsegment edge ideal. Then* $\operatorname{reg}(I) \in \{2,3\}$.

Proof: The ideal I can be decomposed as $I = J + J'$ where J is generated by the lexsegment $L(u, x_1x_n)$ and J' by $L(x_2x_3, v)$. Both ideals J and J' have a linear resolution, hence $\operatorname{reg}(J) = \operatorname{reg}(J') = 2$. By [10] (see also [9] and [19]), it follows that $\operatorname{reg}(I) \leq \operatorname{reg}(J) + \operatorname{reg}(J') - 1 = 3$. □

This easy lemma shows that we have to distinguish only between two possible values of the regularity of I.

In the first place we recall the characterization of the squarefree lexsegment ideals of arbitrary degree which have a linear resolution (see [6] or [5]). The characterization depends on whether or not the lexsegment is complete. For the next two results we recall the following well-known notation. If $w \in S$ is a monomial we denote $\max(w) = \max\{j \colon x_j|w\}$ and $\min(w) = \min\{j \colon x_j|w\}$.

Theorem 1.9 ([6],[5]). *Let $u = x_1x_{i_2}\cdots x_{i_d}$ and $v = x_{j_1}\cdots x_{j_d}$ be two squarefree monomials of degree $d \geq 2$ and $I = (L(u,v))$ the squarefree lexsegment ideal generated by the lexsegment set $L(u,v)$. The following statements are equivalent:*

(a) I is a completely squarefree lexsegment ideal, that is, the squarefree shadow of $L(u,v)$ is a lexsegment set too.

(b) For any squarefree monomial w of degree d, $w <_{\mathrm{lex}} v$, there exists $i > 1$ such that $x_i|w$ and $x_1w/x_i \leq_{\mathrm{lex}} u$.

For this class of ideals we have the following result.

Theorem 1.10 ([6],[5]). *Let $u = x_1 x_{i_2} \cdots x_{i_d}$ and $v = x_{j_1} \cdots x_{j_d}$ be two squarefree monomials of degree $d \geq 2$ and $I = (L(u,v))$ the squarefree lexsegment ideal generated by the lexsegment set $L(u,v)$. Assume that I is a completely squarefree lexsegment ideal. Let B be the set of all the squarefree monomials w of degree d such that $w <_{lex} v$ and $x_1 w / x_{\max(w)} > u$. Then I has a linear resolution if and only if I is a final squarefree lexsegment ideal or the following condition holds: for all $(w_1, w_2) \in B \times B$ such that $w_1 \neq w_2$ and $x_1 w_1 / x_{\min(w_1)} \leq u$, there exists an index ℓ such that $\min(w_1) \leq \ell < \max(w_2)$, $x_\ell | w_2$, $x_1 w_2 / x_\ell \leq u$ and $w_1 / x_{\min(w_1)} \neq w_2 / x_\ell$.*

We now consider the particular settings which we are interested in.

Let $u = x_1 x_i$ and $v = x_j x_k$ with $i \geq 3$ and $2 \leq j \leq n-2$. According to Theorem 1.9 we get the following characterization of the completely lexsegment edge ideals.

Corollary 1.11. *Let u, v be as above and $I = (L(u,v))$. Then I is a completely lexsegment edge ideal if and only if $j \geq i - 2$.*

Proof: For $w = x_{j+1} x_{j+2}$ we see that $x_1 w / x_{j+1} \leq u$ if and only if $j + 2 \geq i$. □

Next we apply Theorem 1.10 and get the following

Corollary 1.12. *Let u, v be as above and $I = (L(u,v))$ a completely lexsegment edge ideal, that is, $j \geq i-2$. Then I has a linear resolution if and only if $i \leq j+1$ or $i = j+2$ and $v = x_j x_n$.*

Proof: In the case $i \leq j+1$ one may apply Theorem 1.10 or simply observe that if we order the minimal monomial generators of I as

$$x_2x_3, x_2x_4, \ldots, x_2x_n, x_3x_4, \ldots, x_jx_k, x_1x_i, x_1x_{i+1}, \ldots, x_1x_n,$$

then we get linear quotients, hence I has a linear resolution.

Let $i = j+2$. If $v = x_j x_n$, then we have $B = \{x_{j+1}x_{j+2}, \ldots, x_{j+1}x_n\}$. In this case one may choose $\ell = j+1$ in order to verify the condition from Theorem 1.10. Let $v >_{\text{lex}} x_j x_n$. Then one may choose $w_1 = x_j x_n, w_2 = x_{j+1} x_n \in B$. It follows that w_1, w_2 do not satisfy the condition from Theorem 1.10 since the only possible choice for ℓ is $\ell = j+1$, and, in this case, $w_1/x_j = w_2/x_{j+1}$. □

Next we consider lexsegment edge ideals which are not complete. To this aim we recall the following

Theorem 1.13 ([6],[5]). *Let $I = (L(u, v))$ be a squarefree lexsegment ideal determined by $u = x_1 x_{i_2} \cdots x_{i_d}$ and $v = x_{j_1} \cdots x_{j_d}$, $j_1 \geq 2$. Assume that I is not a completely squarefree lexsegment ideal. Then I has a linear resolution if and only if v is of the form $v = x_\ell x_{n-d+2} \cdots x_n$ for some $2 \leq \ell < n - d + 1$.*

Applying the above theorem to our particular setting we get the following

Corollary 1.14. *Let $I = (L(u, v))$ be a lexsegment edge ideal, where $u = x_1 x_i, i \geq 3$, and $v = x_j x_k$, $j < i - 2$. Then I has a linear resolution if and only if $v = x_j x_n$.*

By using the above results we can compute the regularity of the lexsegment edge ideals.

Proposition 1.15. *Let $I = (L(u, v))$ be a lexsegment edge ideal where $u = x_1 x_i, v = x_j x_k, j \geq 2$. Then*

$$\operatorname{reg} I = \begin{cases} 3, & \text{if } i \geq j + 2 \text{ and } x_n \not| v \\ 2, & \text{otherwise.} \end{cases}$$

Proof: The proof follows immediately from Corollaries 1.12 and 1.14. □

2 Arithmetical rank of lexsegment edge ideals

In this section we aim to prove Theorem 2.2 on the arithmetical rank of lexsegment edge ideals. A useful tool will be Schmitt-Vogel Lemma (see [17]).

Lemma 2.1. *[17] Let $I \subset S$ be a squarefree monomial and $A_1, \ldots, A_r$ be some subsets of the set of monomials of I. Suppose that the following conditions hold:*

(SV1) $|A_1| = 1$ *and A_i is a finite set for any $2 \leq i \leq r$;*

(SV2) The union of all the sets $A_i, i = \overline{1, r}$, contains the set of the minimal monomial generators of I;

(SV3) For any $i \geq 2$ and for any two different monomials $m_1, m_2 \in A_i$ there exists $j < i$ and a monomial $m' \in A_j$ such that $m' | m_1 m_2$.

Let $g_i = \sum_{m_i \in A_i} m_i$ for $1 \leq i \leq r$. Then $\sqrt{(g_1, \ldots, g_r)} = I$. In particular, $\operatorname{ara}(I) \leq r$.

Theorem 2.2. *Let $I = (L(u, v))$ be a lexsegment edge ideal. Then*

$$\operatorname{ara}(I) = \operatorname{proj}\dim_S(S/I).$$

Proof: Let $u = x_1x_i$ and $v = x_jx_k$ such that $u \geq_{\text{lex}} v$. In the first place we observe that the statement is obviously true if $j = 1$ since, for instance, I is isomorphic as an S-module to the ideal generated by the variables $x_i, \ldots, x_k$. Hence we may assume that $j \geq 2$. We will consider separately the case $j = 2$.

Let $j \geq 3$. By Corollary 1.4, we have $\operatorname{proj}\dim_S(S/I) = n - 1$ if and only if $x_{i-1}x_n \geq_{\text{lex}} v$. If this is the case, then, by using inequalities (1), it follows that $n - 1 = \operatorname{proj}\dim_S(S/I) \leq \operatorname{ara}(I) \leq n - 1$, and, consequently, the required equality.

Now let $v >_{\text{lex}} x_{i-1}x_n$ in the same hypothesis on j, namely $j \geq 3$. We have $\operatorname{proj}\dim_S(S/I) = n - 2$. We are going to distinguish two cases to study. In both cases we show that $\operatorname{ara}(I) = n - 2 = \operatorname{proj}\dim_S(S/I)$ by using Schmitt-Vogel Lemma.

Case (1). Let $i = 4$ or $x_{i-1}x_i \geq_{\text{lex}} v >_{\text{lex}} x_{i-1}x_n$. In particular, by our assumption $j \geq 3$, we have $i \geq 4$. We display the minimal monomial generators of I in an upper triangular tableau as follows. In the first row we put the generators divisible by x_2 ordered decreasingly with respect to the lexicographic order except the monomial x_2x_n which is intercalated between the monomials x_2x_{i-1} and x_2x_i. In the same way we order on the second row the monomials divisible by x_3, intercalating the monomial x_3x_n between x_3x_{i-1} and x_3x_i. We continue in this way up to the row containing the monomials divisible by x_{i-2}. On the next row we put the monomials $x_1x_n, x_1x_i, x_1x_{i+1}, \ldots, x_1x_{n-1}$, and, finally, on the last row, we put the remaining generators, namely $x_{i-1}x_i, \ldots, v$. Then our tableau looks as follows.

$$\begin{array}{ccccccccc}
x_2x_3 & x_2x_4 & \ldots & x_2x_{i-1} & \underline{x_2x_n} & x_2x_i & \ldots & x_2x_{n-2} & x_2x_{n-1} \\
 & x_3x_4 & \ldots & x_3x_{i-1} & \underline{x_3x_n} & x_3x_i & \ldots & x_3x_{n-2} & x_3x_{n-1} \\
 & & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
 & & & x_{i-2}x_{i-1} & \underline{x_{i-2}x_n} & x_{i-2}x_i & \ldots & x_{i-2}x_{n-2} & x_{i-2}x_{n-1} \\
 & & & & \underline{x_1x_n} & \underline{x_1x_i} & \ldots & \underline{x_1x_{n-2}} & \underline{x_1x_{n-1}} \\
 & & & & & x_{i-1}x_i & \ldots\ v & &
\end{array}$$

Next we define the sets $A_1, A_2, \ldots, A_{n-2}$ in the following way. In the first set we put the monomial from the left-up corner of the tableau. In the second set we put the two monomials from the left up parallel to the diagonal of the triangular tableau. In the third set we collect the three monomials from the next parallel to the diagonal, and so on. Explicitly, the sets are the following ones.

$$\begin{array}{ll}
A_1 = & \{x_2x_{n-1}\}, \\
A_2 = & \{x_2x_{n-2}, x_3x_{n-1}\}, \\
A_3 = & \{x_2x_{n-3}, x_3x_{n-2}, x_4x_{n-1}\}, \\
\vdots & \\
A_{n-i+1} = & \{x_2x_n, x_3x_i, x_4x_{i+1}, \ldots\}, \\
\vdots & \\
A_{n-2} = & \{x_2x_3, x_3x_4, \ldots, x_{i-2}x_{i-1}, x_1x_n, x_{i-1}x_i\}.
\end{array}$$

One may easy check that the sets $A_1, \ldots, A_{n-2}$ verify all the conditions from Lemma 2.1. We give only a brief explanation concerning the third condition. Indeed if one picks up two different monomials in the set A_j for some $j \geq 2$, let us say m_1 from the r-th row and m_2 from the s-th row of the tableau with $r < s$, then the monomial m' at the intersection of the r-th row and the column of m_2 divides the product $m_1 m_2$ and $m' \in A_\ell$ for some $\ell < j$.

Case (2). Let $x_3x_4 \geq_{\text{lex}} v = x_jx_k >_{\text{lex}} x_{i-1}x_i$ and $i \geq 5$. Then we construct a similar triangular tableau to that one from the previous case, but we preserve the decreasing lexicographic order in each row. In this tableau we will add the underlined monomials in the $(j-1)$-th row.

$$\begin{array}{ccccccccccc}
x_2x_3 & x_2x_4 & \ldots & x_2x_j & x_2x_{j+1} & \ldots & x_2x_k & x_2x_{k+1} & \ldots & x_2x_n \\
 & x_3x_4 & \ldots & x_3x_j & x_3x_{j+1} & \ldots & x_3x_k & x_3x_{k+1} & \ldots & x_3x_n \\
 & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
 & & & x_{j-1}x_j & x_{j-1}x_{j+1} & \ldots & x_{j-1}x_k & x_{j-1}x_{k+1} & \ldots & x_{j-1}x_n \\
 & & & & x_jx_{j+1} & \ldots & x_jx_k = v & \underline{x_1x_jx_{k+1}} & \ldots & \underline{x_1x_jx_n} \\
 & & & & & x_1x_i & & \ldots & & x_1x_n
\end{array}$$

Note that in this case it is impossible to have $i = j+1$. Indeed, if $i = j+1$, then, by our hypothesis we have $x_jx_k >_{\text{lex}} x_jx_{j+1}$, which is impossible.

One may easy check that the sets $A_1 = \{x_2x_n\}, A_2 = \{x_2x_{n-1}, x_3x_n\}, A_3 = \{x_2x_{n-2}, x_3x_{n-1}, x_4x_n\}, \ldots, A_{n-2} = \{x_2x_3, x_3x_4, \ldots, x_jx_{j+1}\}$ verify the conditions from Lemma 2.1, thus $\operatorname{ara}(I) \leq n-2$. Since we also have $\operatorname{proj\,dim}_S(S/I) = n-2$, we get that $\operatorname{ara}(I) = \operatorname{proj\,dim}_S(S/I)$.

To finish the proof, we only need to consider the case $j = 2$, that is, $u = x_1x_i$ and $v = x_2x_k$ for some i and k such that $v >_{\text{lex}} x_{i-1}x_n$. Note that, in particular, we have $i - 1 \geq 2$, that is, $i \geq 3$.

If $i > k$, then, as in the proof of Lemma 1.6, we obtain that the Taylor resolution of I is minimal. This implies that $\operatorname{proj\,dim}_S(S/I) = \mu(I)$, where $\mu(I)$ denotes the number of the minimal monomial generators of I. Therefore, $\operatorname{ara}(I) = \mu(I) = \operatorname{proj\,dim}_S(S/I)$.

If $k \geq i$, we show that $\operatorname{ara}(I) = \operatorname{proj\,dim}_S(S/I) = n-2$ by using again Lemma 2.1. In this case we put the generators of I in a 2-row tableau.

$$\begin{array}{ccccccccc}
 & & x_1x_i & \ldots & x_1x_k & x_1x_{k+1} & \ldots & x_1x_n \\
x_2x_3 & \ldots & x_2x_i & \ldots & x_2x_k & & &
\end{array}$$

If $i > 3$, we add to the second row the monomials $x_1x_2x_{k+1}, \ldots, x_1x_2x_n$.

We get the tableau

$$\begin{array}{ccccccccc}
 & & x_1x_i & \ldots & x_1x_k & x_1x_{k+1} & \ldots & x_1x_n \\
x_2x_3 & \ldots & x_2x_i & \ldots & x_2x_k & \underline{x_1x_2x_{k+1}} & \ldots & \underline{x_1x_2x_n}
\end{array}$$

and set

$$A_1 = \{x_1x_2x_n\}, A_2 = \{x_1x_n, x_1x_2x_{n-1}\}, A_3 = \{x_1x_{n-1}, x_1x_2x_{n-2}\}, \ldots$$

$$\ldots, A_{n-k} = \{x_1x_{k+2}, x_1x_2x_{k+1}\}, A_{n-k+1} = \{x_1x_{k+1}, x_2x_k\}, \ldots, A_{n-2} = \{x_2x_3\}.$$

If $i = 3$, then we add the monomials $x_1x_2x_{k+1}, \ldots, x_1x_2x_{n-1}$ to the initial tableau and get

$$\begin{array}{cccccccc} x_1x_3 & x_1x_4 & \ldots & x_1x_k & x_1x_{k+1} & \ldots & x_1x_{n-1} & x_1x_n \\ x_2x_3 & x_2x_4 & \ldots & x_2x_k & \underline{x_1x_2x_{k+1}} & \ldots & \underline{x_1x_2x_{n-1}} & \end{array}$$

We set

$$A_1 = \{x_1x_3\}, A_2 = \{x_1x_4, x_2x_3\}, A_3 = \{x_1x_5, x_2x_4\}, \ldots, A_{k-2} = \{x_1x_k, x_2x_{k-1}\},$$
$$A_{k-1} = \{x_1x_{k+1}, x_2x_k\}, A_k = \{x_1x_{k+2}, x_1x_2x_{k+1}\}, \ldots, A_{n-2} = \{x_1x_n, x_1x_2x_{n-1}\}.$$

In both cases, by using Lemma 2.1, we get $\operatorname{proj}\dim_S(S/I) = n-2 \leq \operatorname{ara}(I) \leq n-2$, hence $\operatorname{ara}(I) = n-2 = \operatorname{proj}\dim_S(S/I)$. □

We recall that an ideal $I \subset S$ is called a *set-theoretic complete intersection* if $\operatorname{ara}(I) = \operatorname{ht}(I)$. For squarefree monomial ideals we $\operatorname{ara}(I) \geq \operatorname{proj}\dim_S(S/I)$, by using again (1). If $\operatorname{ht}(I) = \operatorname{ara}(I)$, we get

$$\operatorname{ht}(I) \geq \operatorname{proj}\dim_S(S/I) = n - \operatorname{depth}(S/I) \geq n - \dim(S/I) = \operatorname{ht}(I).$$

Therefore, we derive the following implication for squarefree monomial ideals:

set-theoretic complete intersection $\Rightarrow$ Cohen-Macaulay.

For lexsegment edge ideals the converse is also true, by Theorem 2.2.

Corollary 2.3. *Let I be a lexsegment edge ideal. Then the following statements are equivalent:*

(a) I is Cohen-Macaulay.

(b) I is a set-theoretic complete intersection.

3 Arithmetical rank of the Alexander dual of a lexsegment edge ideal

As before, let $u = x_1x_i, v = x_jx_k$ be two squarefree monomials of degree 2 such that $u \geq_{\text{lex}} v$ and $I = (L(u,v))$ the lexsegment edge ideal generated by the set $L(u,v)$.

Let I^* be the Alexander dual ideal of I. Then we have

$$\begin{aligned} I^* \quad = \quad & (x_1, x_i) \cap (x_1, x_{i+1}) \cdots \cap (x_1, x_n) \cap (x_2, x_3) \cap (x_2, x_4) \cap \cdots \cap (x_2, x_n) \\ & \cap (x_3, x_4) \cap (x_3, x_5) \cap \cdots \cap (x_3, x_n) \cap \cdots \cap (x_j, x_{j+1}) \cap \cdots \cap (x_j, x_k), \end{aligned}$$

which is an unmixed ideal of height two (see, e.g., [20, Proposition 1.1]). In this section we show the equality $\operatorname{ara}(I^*) = \operatorname{proj}\dim_S(S/I^*)$. Since we have $\operatorname{proj}\dim_S(S/I^*) = \operatorname{reg} I$ [20, Corollary 1.6], by Proposition 1.15 we have the following:

Proposition 3.1. *Let $I = (L(u,v))$ be a lexsegment edge ideal where $u = x_1x_i, v = x_jx_k, j \geq 2$. Then*

$$\operatorname{proj}\dim_S(S/I^*) = \begin{cases} 3, & \text{if } i \geq j+2 \text{ and } x_n \nmid v \\ 2, & \text{otherwise.} \end{cases}$$

Now we determine the arithmetical rank of the Alexander dual of a lexsegment edge ideal.

Theorem 3.2. *Let $I = (L(u,v))$ be a lexsegment edge ideal. Then*

$$\operatorname{ara}(I^*) = \operatorname{proj}\dim_S(S/I^*).$$

Proof: We may assume that $u = x_1x_i, v = x_jx_k$. If $j = 1$, then $I^* = (x_1, x_i) \cap (x_1, x_{i+1}) \cap \cdots \cap (x_1, x_k) = (x_1, x_ix_{i+1} \dots x_k)$ is a (set-theoretic) complete intersection. Hence we may assume that $j \geq 2$.

Now we assume that $i \leq j+1$ or $k = n$. Then we have $\operatorname{proj}\dim_S(S/I^*) = \operatorname{ht} I^* = 2$, and S/I^* is Cohen-Macaulay. In this case I^* is a set-theoretic complete intersection by Kimura [11]. Hence $\operatorname{ara}(I^*) = \operatorname{proj}\dim_S(S/I^*) = 2$.

Next we assume that $i \geq j+2$ and $k \neq n$. Let J^* be the Alexander dual ideal of $J = (L(x_1x_i, x_{j-1}x_n))$. Then we have $\operatorname{ara}(J^*) = \operatorname{proj}\dim_S(S/J^*) = 2$. Hence there exist $f_1, f_2 \in S$ such that $\sqrt{(f_1, f_2)} = J^*$. Then we have

$$\begin{aligned} I^* &= J^* \cap (x_j, x_{j+1}) \cap (x_j, x_{j+2}) \cap \cdots \cap (x_j, x_k) \\ &= \sqrt{(f_1, f_2)} \cap (x_j, x_{j+1}x_{j+2} \dots x_k) \\ &= \sqrt{(f_1 f_2)(x_j, x_{j+1}x_{j+2} \dots x_k)} \\ &= \sqrt{(x_jf_1, x_jf_2, x_{j+1}x_{j+2} \dots x_kf_1, x_{j+1}x_{j+2} \dots x_kf_2)} \\ &= \sqrt{(x_jf_1, x_jf_2 + x_{j+1}x_{j+2} \dots x_kf_1, x_{j+1}x_{j+2} \dots x_kf_2)}. \end{aligned}$$

For the last equality we need only to justify the inclusion from the left part to the right part. This follows immediately if we notice that x_jf_2 and $x_{j+1}x_{j+2} \dots x_kf_1$ are solutions of the equation

$$t^2 - (x_jf_2 + x_{j+1}x_{j+2} \dots x_kf_1)t + x_jx_{j+1}x_{j+2} \dots x_kf_1f_2 = 0.$$

We have $3 = \operatorname{proj}\dim_S(S/I^*) \leq \operatorname{ara}(I^*) \leq 3$. Hence $\operatorname{ara}(I^*) = \operatorname{proj}\dim_S(S/I^*) = 3$, as desired. ∎

References

[1] Aramova, A., Herzog, J., Hibi, T., *Squarefree lexsegment ideals*, Math. Z. **228** (1998), 353–378.

[2] Barile, M., *On the arithmetical rank of the edge ideals of forests*, Comm. Algebra **36** (2008), 4678-4703.

[3] Barile,M., Terai, N., *Arithmetical ranks of Stanley-Reisner ideals of simplicial complexes with a cone*, to appear in Comm. Algebra, arXiv:0809.2194.

[4] Barile,M., Terai, N., *The Stanley-Reisner ideals of polygons as set-theoretic complete intersections*, Preprint, arXiv:0909.1900.

[5] Bonanzinga, V., Ene, V., Olteanu, A., Sorrenti, L., *An overview on the minimal free resolutions of lexsegment ideals*, in Combinatorial Aspects of Commutative Algebra, V. Ene, E. Miller, Eds, Contemporary Mathematics, AMS, **502** (2009), 5-24.

[6] Bonanzinga, V., Sorrenti, L., *Squarefree lexsegment ideals with linear resolution*, Bollettino UMI, Serie IX, Vol I, N.2 (2008), 275-291.

[7] Bonanzinga, V., Sorrenti, L., *Cohen-Macaulay squarefree lexsegment ideals generated in degree 2*, in Combinatorial Aspects of Commutative Algebra, V. Ene, E. Miller, Eds, Contemporary Mathematics, AMS, **502** (2009), 25-33.

[8] Gräbe, H.-G., *Uber den arithmetischen Rang quadratfreier Potenzproduktideale*, Math. Nachr. **120** (1985), 217-227.

[9] Herzog, J., *A generalization of the Taylor complex construction*, Comm. Algebra **35** (2007), 1747-1756.

[10] Kalai, G., Meshulam, R., *Intersections of Leray complexes and regularity of monomial ideals*, Journal of Combinatorial Theory Ser. A.**113** (2006), 1586-1592.

[11] Kimura, K., *Arithmetical rank of Cohen-Macaulay squarefree monomial ideals of height two*, to appear in J. Commutative Alg.

[12] Kimura, K., Terai, N., Yoshida, K., *Arithmetical rank of squarefree monomial ideals of small arithmetic degree*, J. Algebraic Combin. **29** (2009), 389-404.

[13] Kimura, K., Terai, N., Yoshida, K., *Arithmetical rank of squarefree monomial ideals of deviation two*, in Combinatorial Aspects of Commutative Algebra, V. Ene, E. Miller, Eds, Contemporary Mathematics, AMS, **502** (2009), 73-112.

[14] Kummini, M., *Regularity, depth and arithmetical rank of bipartite edge ideals*, to appear in J. Algebraic Combin., arXiv:0902.0473.

[15] LYUBEZNIK, G., *On the local cohomology modules $H^i_{\mathfrak{a}}(R)$ for ideals $\mathfrak{a}$ generated by monomials in an R-sequence*, in Complete Intersections, Acireale, 1983, S. Greco and R. Strano Eds., Lecture Notes in Mathematics, Springer-Verlag, **1092** (1984), 214–220.

[16] MORALES, M., *Simplicial ideals, 2-linear ideals and arithmetical rank*, math.AC/0702668.

[17] SCHENZEL, P., VOGEL, W., *On set-theoretic complete intersections*, J. Algebra **48** (1977), 401-408.

[18] SCHMITT, TH., VOGEL, W., *Note on set-theoretic intersections of subvarieties of projective space*, Math. Ann. **245** (1979), 247-253.

[19] TERAI, N., *Eisenbud-Goto inequality for Stanley-Reisner rings*, in Geometric and combinatorial aspects of commutative algebra (Messina, 1999), J. Herzog, G. Restuccia, Eds, Lecture Notes in Pure and Appl. Math., Dekker, **217** (2001), 379-391 .

[20] TERAI, N., *Alexander duality in Stanley–Reisner rings*, in Affine Algebraic Geometry, in honor of Professor Masayoshi Miyanishi on the Occasion of His Retirement from Osaka University, T. Hibi, Ed, Osaka University Press (2007), 449-462.

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