A simple presentation of the handlebody group of genus 2 *

by

CLEMENT RADU POPESCU

Abstract

For genus \( g = 2 \) I simplify Wajnryb’s presentation of the handlebody group.

Key Words: Handlebody group, group presentation.

2010 Mathematics Subject Classification: Primary 20F05, 57M05; Secondary 20F38, 57M60.

1 Introduction

Let \( S_{g,b,k} \) be a closed surface of genus \( g \), \( h \) boundary components and \( k \) distinguished points. I’ll use the notation \( S_g \) for \( S_{g,0,0} \). The mapping class group, \( \mathcal{M}_{g,b,[k]} \) is the group of all isotopy classes of orientation preserving homeomorphisms which keep the boundary and the distinguished points pointwise fixed. \( \mathcal{M}_{g,b,k} \) is the group of all isotopy classes of orientation preserving homeomorphisms which keep the boundary pointwise fixed, and is also fixing the set of distinguished points possibly permuting them. We have \( \mathcal{M}_{g,b,[k]} \hookrightarrow \mathcal{M}_{g,b,k} \). I’ll use the notation \( \mathcal{M}_g \) for \( \mathcal{M}_{g,0,0} \). There are some important instances of these groups. For example the braid group \( B_n \) is shown in [1], theorem 1.10, to be isomorphic with \( \mathcal{M}_{0,1,n} \), and so the pure braid group \( P_n \) is isomorphic with \( \mathcal{M}_{0,1,[n]} \).

For a long time it was an open problem to obtain a presentation for \( \mathcal{M}_{g,1,0} \). In [7] MacCool proved, using purely algebraic methods, that \( \mathcal{M}_{g,1,0} \) is finitely presented for any genus \( g \). Hatcher and Thurston in [4] made a crucial breakthrough in the subject developing an algorithm for obtaining a finite presentation for \( \mathcal{M}_{g,1,0} \). Using this algorithm, Harer in [3] obtained a finite, explicit presentation. Finally it was Wajnryb, who in [10] gave a simple presentation for \( \mathcal{M}_{g,0,0} \) and \( \mathcal{M}_{g,1,0} \).

Using similar techniques as in [4], he found in [11] a presentation of the handlebody group. The handlebody group is the subgroup of \( \mathcal{M}_g \) formed by all the

*This is part of my Ph.D. Thesis, which I prepared at COLUMBIA UNIVERSITY under the supervision of Prof. Joan Birman.
isotopy classes of orientation preserving homeomorphisms of $S_g$, which extend to the entire handlebody $H_g$. I'll denote it by $H_g$. The presentation obtained by Wajnryb in [11] is long and complicated.

In this note I will give a simple presentation for $H_2$ starting from Wajnryb’s presentation. The numbers of generators and relations are considerably reduced (referee’s observation). For topological reasons, such a simplification for higher genera doesn’t work.

A similar presentation was obtained by Hirose in [5] using a different method than Wajnryb’s. In my thesis [8], I showed that the two presentations of $H_2$ are equivalent. At the end of the paper I’ll give some details about this equivalence.

In Figure 1 it is shown a system of curves on the surface $S_g$. The isotopy classes of the curves $\alpha_i$’s will be called meridians, and those of $\beta_i$’s will be called longitudes.

**Figure 1: Surface of genus $g$**

**Definition 1.** A (positive) Dehn-twist with respect to a simple closed curve $\gamma$, denoted by $T_\gamma$, is a homeomorphism of the oriented surface $S_g$, which is supported in a regular neighborhood of $\gamma$ and is obtained as follows: one cuts open the surface along $\gamma$ and rotates one end of it with $360^\circ$ to the right and then glue back the surface. This is done in such a way that on the boundary and the complement of the regular neighborhood, $T_\gamma$ is the identity map.

I’ll call $\gamma$ the support of the Dehn-twist.

The effect of a positive $T_\gamma$ on any segment which intercepts the curve $\gamma$ transversally in one point is as follows: cut the segment at the interception point and rotate to the right once around $\gamma$. The following is a well known result (see [11]):

**Lemma 2.** If $h$ is a homeomorphism of the surface $S_g$, and $T_\gamma$ is a Dehn-twist, then $T_{h(\gamma)} \equiv hT_\gamma h^{-1}$. 
Remark 3. I’ll use the following notation $h \ast g \equiv hgh^{-1}$. So in this notation Lemma 2 can be written $T_{h(\gamma)} \equiv h \ast T_{\gamma}$.

Roman letters will be used for a Dehn-twist with the support denoted by the corresponding Greek letter. For the curves in Figure 1 we have $a_i \equiv T_{\alpha_i}$, $b_i \equiv T_{\beta_i}$, $c_i \equiv T_{\gamma_i}$.

Definition 4. A meridian curve on the surface is one which represents a non-trivial homotopy class in $\pi_1(S_g, \ast)$ and bounds a properly embedded disc in the handlebody $H_g$.

Consider $N_{\alpha}$ the normal subgroup of $\pi_1(S_g, \ast)$ generated by the homotopy classes of the meridians $\alpha_1, \ldots, \alpha_g$ (see Figure 1). Let $\# : M_g \longrightarrow \text{Aut}(\pi_1(S_g, \ast))$ the homomorphism which takes a homeomorphism into the automorphism of the fundamental group of the surface, and the image of a homeomorphism will be denoted with a $\#$-subscript ($h \mapsto h_\#$). In [2] Griffiths shows that $h \in H_g$ if and only if $h_\#(N_{\alpha}) \subset N_{\alpha}$. A set of generators for $H_g$ was obtained by Suzuki in [9].

Wajnryb has described in [11] an algorithm for getting a presentation of the handlebody group. This is similar to Hatcher–Thurston’s algorithm in [4] and to Wanjnryb’s algorithm in [10]. For the handlebody $H_g$ of genus $g$ there exists an associated 2-dimensional complex $X$, called the cut-system complex of the handlebody (different from the one constructed for a surface $S_g$). The vertices are cut-systems (collection of $g$ disjoint meridian curves). Another cut-system is obtained if we replace one curve with another meridian curve disjoint from all the curves in the cut system. Such vertices are joined by an edge. Moreover to any triangle we associate a face, thus we obtain a 2-dimensional simplicial complex, called $X$.

$H_g$ acts transitively on the vertices of $X$, and this action can be extended to a simplicial action on $X$. Using this action Wajnryb obtains the presentation for $H_g$.

In Section 2, I will give some details about Wajnryb’s presentation for any $g$, and in Section 3 the reduction I obtained for the case $H_2$.

2 Wajnryb’s algorithm

For the convenience of the reader I will describe in this section in some detail Wajnryb’s algorithm. He proved in [11], Theorem 13, that the complex $X$, described above is connected and simply connected. The 0-skeleton $X^{(0)}$ has a preferred vertex represented by the cut-system formed by the collection of the $g$ meridians, one for each handle. It is denoted $\tilde{v}_0 \equiv < \alpha_1, \alpha_2, \ldots, \alpha_g >$. Wajnryb’s algorithm has a few steps.

Step 1: Find a presentation of the stabilizer of $\tilde{v}_0$ denoted by $\mathcal{K}$. The homeomorphisms in $\mathcal{K}$ either preserve $\tilde{v}_0$ pointwise or permutes the $\alpha_i$’s or changes their respective orientations. A presentation for $\mathcal{K}$ can be found using the following exact sequences.
In both sequences (1) and (2) \( \pm \Sigma_g \) is the discrete group of signed permutations. \( \pm \Sigma_g \) is the group of permutations of \( \{-g, 1-g, \ldots, 1, 2, \ldots g\} \) such that \( \sigma(-i) \equiv -\sigma(i) \). The homomorphism \( \pm \Sigma_g \rightarrow \Sigma_g \) is the forgetting sign homomorphism and the sequence (1) splits. \( K_0 \) is the subgroup of \( K \), fixing all the \( \alpha_i \)'s pointwise. To find a presentation of \( K_0 \), one needs first a presentation for \( M_{0,2g,0} \) which can be obtained from the following diagram.

Using this presentation of \( M_{0,2g,0} \) and the following exact sequence

\[
1 \rightarrow \mathbb{Z}^g \rightarrow M_{0,2g,0} \rightarrow K_0 \rightarrow 1 \quad (4)
\]

one gets a presentation of \( K_0 \).

**Step 2:** Other cut-systems can be obtained using "translations". Such a translation is given by the homeomorphism \( r_{i,j} \) which changes \( \alpha_j \) into \( \gamma_{i,j} \) keeping all the other \( \alpha_i \)'s \( i \neq j \) fixed (see [11]). For the benefit of the reader let me recall the definition of the curves \( \gamma_{i,j} \). Let \( i, j \) be integers, such that \( -g \leq i \leq j \leq g \). Then, \( \gamma_{i,j} \) is the curve on the disc with \( 2g \) discs removed (see Figure 2) which encloses all the holes from \( i \) to \( j \). If \( i \equiv j \), then the curve \( \gamma_{i,j} \) is \( \alpha_i \). Wajnryb, using connectedness of \( X \), proved that the generators of \( K \) together with the elements \( r_{i,j} \) generate \( \mathcal{H}_g \).

**Step 3:** There are finitely many edge orbits modulo the action of \( \mathcal{H}_g \). To each edge we associate its stabilizer \( K_{i,j} \). Relations are coming from the conjugations \( r_{i,j} hr_{i,j}^{-1} \) and \( r_{i,j}^{-1} hr_{i,j} \in K \) for any \( h \in K_{i,j} \). For more details see [11].

Using the above described algorithm he was able to find a presentation of the handlebody group which, unfortunately, is very complicated.

Let me give a brief description of the homeomorphisms used in Wajnryb’s presentation. In Figure 2 we see some particular meridian curves. The curves denoted by \( \gamma_1 \) are the same in both Figure 1 and 2. A positive Dehn-twist
is considered to be taken in a counterclockwise direction. Looking carefully at Definition 1 one sees that the Dehn-twist does not depend on the orientation of the curve.

A half Dehn-twist on $\delta_{-1,1}$ is the twist of the first knob and will be denoted by $o$. It changes the orientation of both the first meridian and longitude. From its definition we see that $o^2 \equiv d_{-1,1}$. It is easily seen that $o(\delta_{1,2}) \equiv \gamma_1$. Using Remark 3 we get that $o \ast d_{1,2} \equiv c_1$. Other important homeomorphisms on the surface are $t_1$ which exchange the meridians $\alpha_i$ and $\alpha_{i+1}$, fixing the others. Another homeomorphism which exchanges $\alpha_i$ and $\alpha_{i+1}$, also exchanging $\beta_i$ and $\beta_{i+1}$, is $k_i \equiv a_i a_{i+1} t_i d_{i,i+1}^{-1}$. I'll mention also $z$, which is a rotation of the surface about the z-axis (the z-axis is considered to pierce the surface in its center of symmetry as drawn in Figure 1, with positive direction from bellow to above), changing the i-th hole into the $g-i+1$-st hole, considering that $i > 0$. From Figure 2 it is clear that a curve $\delta_{i,j}$ is the one which encloses in one of the two regions determined on the disc, the holes $\partial_i$ and $\partial_j$. The Dehn-twists with these supports can be considered to be the generators of the $P_{2g} \simeq M_{0,1,2g}$. There is one homeomorphism, $z_j$ which belongs to the stabilizer of an edge of type $(i,j)$ in the 1-skeleton of $X$. In fact $z_j$ has the form $z_j \equiv k_{j-1} k_{j-1} \cdots k_{g+j-1} z$ and it is not the conjugation of $z$ by the product of $k_i$'s.

In the case $g \equiv 1$ it is proved by Wajnryb in [11], Theorem 14, and detailed in [8], Theorem 2.2, that $H_1 \simeq \mathbb{Z} \oplus \mathbb{Z}_2$.

In the next statement I will use the same notations as in [11], and I will write explicitly Wajnryb’s presentation for the case of genus $g \equiv 2$. I will also introduce in the presentation some generators together with their defining relations.

Figure 2: A disc with $2g$ holes
Theorem 5 (Wajuryb). A presentation of $\mathcal{H}_2$ is given by:

Generators: $a_1, a_2, d_{-2}, d_{-21}, d_{-22}, d_{-11}, d_{-12}, d_{12}, o, o_2, t, r, z, e$.

Defining relations:

(D1) $o^{-1}t^{-1}a^{-1} * d_{12} \equiv d_{-21}$

(D2) $t^{-1}a^{-1} * d_{12} \equiv d_{-21}$

(D3) $o^{-1} * d_{12} \equiv d_{-12}$

(D4) $t^{-1}d_{12} * d_{-11} \equiv d_{-22}$

(D5) $o_2 \equiv t d_{12}^{-1} * o$

(D6) $z \equiv a_1^{-1}a_2^{-1} o t o d_{12}$

(D7) $e \equiv o z o^{-1} z$

(P1) $a_1 \equiv a_2 ; a_i \equiv d_{kl}$

(P2.1) $d_{-21}^{-1}d_{-21}d_{-21} \equiv d_{-21}d_{-11}d_{-21}d_{-11}^{-1}$

(P2.2) $d_{-21}^{-1}d_{-11}d_{-21} \equiv d_{-21}d_{-11}d_{-21}$

(P2.3) $d_{-21}^{-1}d_{-22}d_{-21} \equiv d_{-22}d_{12}d_{-22}d_{12}^{-1}d_{-22}^{-1}$

(P2.4) $d_{-21}^{-1}d_{-22}d_{-21} \equiv d_{-22}d_{12}d_{-22}d_{12}^{-1}d_{-22}^{-1}$

(P2.5) $d_{-21}^{-1}d_{-22}d_{-11} \equiv d_{-22}$

(P2.6) $d_{-21}^{-1}d_{-12}d_{-21} \equiv d_{-22}d_{-12}d_{-12}^{-1}$

(P2.7) $d_{-21}^{-1}d_{-12}d_{-21} \equiv d_{-22}d_{12}d_{-22}d_{12}^{-1}d_{-22}^{-1}$

(P2.8) $d_{-11}^{-1}d_{-12}d_{-11} \equiv d_{-12}d_{12}d_{-12}d_{12}^{-1}d_{-12}^{-1}$

(P2.9) $d_{-11}^{-1}d_{12}d_{-12}^{-1} \equiv d_{12}$

(P2.10) $d_{-21}^{-1}d_{12}d_{-21} \equiv d_{-22}d_{12}d_{-22}^{-1}$

(P2.11) $d_{-11}^{-1}d_{12}d_{-11} \equiv d_{-12}d_{12}d_{-12}$

(P3) $d_{-21}^{-1}d_{-21}d_{-12}d_{-11}d_{-12}d_{12} \equiv a_1^4 a_2^4$

(P4.1) $d_{-11}^{-1}d_{12}d_{12} \equiv a_1^4 a_2^2$

(P4.2) $d_{-21}^{-1}d_{22}d_{12} \equiv a_1^4 a_2^4$

(P4.3) $d_{-11}^{-1}d_{12}d_{-12} \equiv a_1^4 a_2^2$

(P4.4) $d_{-11}^{-1}d_{22}d_{-12} \equiv a_1^4 a_2^4$

(P5) empty for genus 2

(P6) $o^2 \equiv d_{-11}; t^2 \equiv d_{12}^{-1}a_1^{-2}a_2^2$

(P7) $t * a_1 \equiv a_2; o \equiv a_i, \ i \equiv 1, 2$

(P8) $t \equiv d_{12}^{-1}; o t o t o t o; o \equiv d_{-22}$

(P9) $r^2 \equiv a_2^4 o_2 d_{12} a_2 d_{12}^2$

(P10 a) $r * a_2 \equiv d_{12}$; $r \equiv a_1$

(P10 b − d) empty for genus 2

(P10 e) $r \equiv e$

(P10 f) $r * d_{12} \equiv a_2$

(P10 g) $r * d_{-2} \equiv d_{-11}d_{12}a_1^{-2}a_2^2$

(P10 h − k) empty for genus 2

(P11) $r t r \equiv t r t$

(P12) empty for genus 2
3 A simple presentation for $\mathcal{H}_2$

In this section I will prove the following:

**Theorem 6.** There is a simple presentation of $\mathcal{H}_2$.

*Generators:* $a_1, a_2, d, o, t, r$.

*Relations:*

\[
d \Leftrightarrow oto; \quad odod \equiv a_1^2a_2^2; \quad o^2 \Leftrightarrow t^{-1}d; \quad z \equiv a_1^{-1}a_2^{-1}oto; \quad a_1 \Leftrightarrow a_2; \quad a_1 \Leftrightarrow d; \quad t^2 \equiv d^2a_1^{-2}a_2^{-2}; \quad o \Leftrightarrow a_1; \quad t \ast a_1 \equiv a_2; \quad t \Leftrightarrow d; \quad ototo \equiv d; \quad r \Leftrightarrow a_1; \quad r \Leftrightarrow zo; \quad z \equiv a_2d^2a_1^{-2}a_2^{-2}; \quad r \Leftrightarrow a_2; \quad r \Leftrightarrow ozo; \quad rtr \equiv trt.
\]

**Proof:** Use (P4.4) and (P4.1) in (P3) and also the commuting relations (P1) to get

(P3)$^\prime$ \quad $d_{-11} \equiv d_{-22}$

Replace $d_{-22}$ with $d_{-11}$ in relations (P4.1) - (P4.4), denoted (P4.1)$'$ - (P4.4)$'$.

Using the commuting relations (P1) and (P4.3)$'$ one gets

(P4.1)$''$ \quad $d_{12} \equiv d_{-2-1}$.

Rewrite again (P4.2)$'$ - (P4.4)$'$, and use (P4.1)$''$ to obtain (P4.2)$''$ - (P4.4)$''$.

Conjugate (P4.4)$''$ with $d_{12}^{-1}$ and modulo the commuting relations (P1) one gets again (P4.2)$''$. So (P4.4)$''$ is redundant. So far (P3) - (P4.4) look like:

(P3)$'$ \quad $d_{-11} \equiv d_{-22}$

(P4.1)$''$ \quad $d_{-2-1} \equiv d_{12}$

(P4.2)$''$ \quad $d_{-21} \equiv a_1^2a_2^2d_{12}^{-1}d_{-11}^{-1}$

(P4.3)$''$ \quad $d_{-12} \equiv a_1^2a_2^2d_{-11}^{-1}d_{12}^{-1}$

(P4.4)$''$ \quad redundant

Using the above relations in (P2.1) - (P2.11) (the pure braid relations), all these become trivial modulo the commuting relations (P1). This is the main reduction in the presentation, which does not take place for any higher genus. So, for $g \equiv 2$ the pure braid relations are redundant modulo (P1) and consequences of (P3) - (P4.4).

Using Tietze operations, we replace in the remaining relations the expressions for $d_{-2-1}, d_{-21}, d_{-22}, d_{-12}$ and remove these generators together with (P3)$''$ - (P4.4)$''$. I will remove also generators $d_{-11}, a_2, e$ together with first part of (P6), (D5) and (D7) replacing first their expressions everywhere else.

At this stage the presentation looks like this:

*Generators:* $a_1, a_2, d_{12}, o, t, r, z$

*Relations:*

(D1) \quad $o^{-1}t^{-1}o^{-1} \ast d_{12} \equiv d_{12}$

(D2) \quad $t^{-1}o^{-1} \ast d_{12} \equiv a_1^2a_2^2d_{12}^{-1}o^{-2}$
\[(D3) \quad o^{-1} * d_{12} \equiv a_1^2 a_2^2 o^{-2} d_{12}^{-1} \]

\[(D4') \quad t^{-1} d_{12} * o^2 \equiv o^2 \]

\[(D6') \quad z \equiv a_1^{-1} a_2^{-1} o do d_{12} \]

\[(P1) \quad a_1 \equiv a_2 ; \quad a_i \equiv d_{12} ; \quad a_i \equiv o^2 \]

\[(P6') \quad t^2 \equiv d_{12}^2 a_1^{-2} a_2^{-2} \]

\[(P7) \quad t * a_1 \equiv a_2 ; \quad o \equiv a_i \]

\[(P8) \quad t \equiv d_{12} ; \quad oto \equiv toto ; \quad o \equiv o^2 \]

\[(P9') \quad r^2 \equiv a_2^{-4} (td_{12}^{-1} o)d_{12} (td_{12}^{-1} o)d_{12}^{-1} \]

\[(P10a) \quad r * a_2 \equiv d_{12} ; \quad r \equiv a_1 \]

\[(P10c') \quad r \equiv o^{-1} o^{-1} z \]

\[(P10d') \quad r * d_{12} \equiv a_2 \]

\[(P10g') \quad r * d_{12} \equiv a_2^2 a_2^{-2} d_{12}^{-1} d_{12} a_1^{-2} a_2^{-1} \]

\[(P11) \quad rtr \equiv trt \]

Let \(d_{12} \equiv d \) (it is the only \(d_{ij} \) left). In the above \((D1)'\) is equivalent with \((D1)\):

\[(P9') \quad r^2 \equiv o \quad \text{using} \quad (P1) \quad \text{to get} \quad \text{(D2)'} \quad \text{odod} \equiv a_1^2 a_2^2. \]

Conjugating the above with \(d\) and using \((P1)\) we get a redundant relation:

\[(D2)'' \quad \text{dodo} \equiv a_1^2 a_2^2 \equiv \text{odod}. \]

Rewrite \((D3)\) using \((D2)''\) as follows:

\[o^{-1} do \equiv a_1^2 a_2^2 o^{-2} d_{12}^{-1} \equiv o^{-1} dodo \equiv a_1^2 a_2^2 o^{-1} (\text{using} \quad (D2)'' \quad \text{odod} \equiv a_1^2 a_2^2). \]

Relation \((D4)')\) is equivalent with \(t^{-1} d_{12} \equiv o^2.\)

In \((P1)\) the last part, \(o^2 \equiv a_i, \) is redundant from the second part of \((P7)\) which is \(o \equiv a_i.\)

In \((P8)\) we can get rid of \(o \equiv o^2.\)

Lastly in \((P10g)'\), because of the commuting relations \((P1)\) and \((P7)\) we’ll get \((P10f)'\). So \((P10g)'\) is redundant.

Rewrite \((P9)'\). Use for this \((P8), \quad (D2)'' \quad \text{and} \quad (P1)\) to get \(r^2 \equiv d_{12}^{-2} a_1^{-2} a_2^{-2}.\)

Because \((P10f)'\) and the above one easily gets that the first part of \((P10a)\) is redundant.

This is the presentation in the statement of Theorem 6. \(\square\)

Another simple presentation of the handlebody group of genus 2 was obtained by Hirose in [5]. He used a result obtained by R. Kramer in [6], which gives a decomposition of \(\mathcal{H}_2\) as an amalgamated free product as follows:
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\[ \mathcal{H}_2 \cong K \ast K_{\alpha_1 \cup \alpha_2 \gamma_1} K_{\alpha_1 \cup \alpha_2 \cup \gamma_1}. \]

Here \( K_{X_1, \ldots, X_k} \) is the group of isotopy classes of homeomorphisms which have the property that \( h(X_i) \equiv X_i \) for \( 1 \leq i \leq k \). \( X_i \) represents a subset of the surface. Hirose calculates presentations of all three groups occurring in the above amalgamated free product and a presentation for \( \mathcal{H}_2 \).

In my thesis [8], I proved the following result:

**Theorem 7.** The simplified presentation of \( \mathcal{H}_2 \) in Theorem 6 and Hirose’s presentation of \( \mathcal{H}_2 \) are equivalent.

**Sketch of the proof.** First I proved that the presentations of the stabilizer \( K \) are equivalent. Hirose’s presentation of \( K \) has five generators and nine relations.

The two stabilizers are generated by 3 meridian Dehn twists, the switch of the first knob, and a homeomorphism which interchanges the meridians \( \alpha_1 \) and \( \alpha_2 \). In the case of simplified Wajnryb presentation (Theorem 6) this homeomorphism is \( t \), and in Hirose’s presentation is \( z \equiv a_1^{-1}a_2^{-1}otod \).

To obtain the presentation for the handlebody groups, we need in both cases one more generator, which is a “translation”. In Wajnryb’s case the translation is \( r \) which transforms \( a_2 \) in \( d \), and in Hirose’s case is \( d_{31} \) which transforms \( a_1 \) into \( c_1 \). We also know that \( c_1 \equiv o^{-1} * d \).

The equivalence is obtained using Tietze transformations. \( \square \)

**References**


Received: 21.09.2009
Revised: 28.01.2010
Accepted: 08.06.2010.

Inst. of Math. "Simion Stoilow",
P.O. Box 1-764, RO-014700,
Bucharest, Romania
E-mail: Radu.Popescu@imar.ro