Bull. Math. Soc. Sci. Math. Roumanie
Tome 54(102) No. 3, 2011, 203–211

PF and PP-properties in Hurwitz series ring

by
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Abstract

In this note we continue the study of Hurwitz series ring HA introduced by Keigher. We focus on the PF and PP-properties in HA and its sub-ring hA of Hurwitz polynomials.

Key Words: Hurwitz series and polynomials; PP-ring; PF-ring.
2010 Mathematics Subject Classification: Primary 13F25, Secondary 13B25; 13C12.

Introduction

Let A be a commutative ring with unit and HA the set of formal expressions of the type $f = \sum_{i:0}^{\infty} a_i X^i$ where $a_i \in A$. When $g = \sum_{i:0}^{\infty} b_i X^i$ then $f + g = \sum_{i:0}^{\infty} (a_i + b_i) X^i$ and $f * g = \sum_{n:0}^{\infty} c_n X^n$ with $c_n = \sum_{i:0}^{n} C_n^i a_i b_{n-i}$. With these two operations HA is a commutative ring with identity. It was introduced and studied by Keigher in [3]. It plays an important role in many areas, especially in differential algebra. Its behavior depends deeply on the characteristic of A. In the positive characteristic case, it is closely connected to the structure of A. In the first section, we give some general properties of the ring HA. The main result says that if $f = \sum_{i:0}^{\infty} a_i X^i$ and $g = \sum_{i:0}^{\infty} b_i X^i \in HA$, then $f * g = 0$ if and only if $a_i b_j = 0$ for all $i, j \in \mathbb{N}$. In the second section, we prove that the Hurwitz polynomials ring hA is a PF-ring if and only if A is a PF-ring and a torsion free $\mathbb{Z}-$module. The analogous for HA is true when A is Noetherian. In the third section, we prove that hA is PP if and only if A is a PP-ring and a torsion free $\mathbb{Z}-$module. For HA, we define an order $\leq$ on $Bool(A)$, the set of all idempotents of A, by $a \leq b$ if $ab = a^2$. Then HA is a PP-ring if and only if A is a PP-ring and a torsion free $\mathbb{Z}-$module and any finite or countable set of $Bool(A)$ admits a least upper bound.

1 Generalities.

Proposition 1.1. *The Hurwitz series* $f = \sum_{j:0}^{\infty} a_j X^j \in (X, X^2, \ldots, X^m)$ *if and only if* $a_0 = 0$ *and for every* $j > m$, a_j *is a linear expansion of* $C_j^1, C_j^2, \ldots, C_j^m$ *with coefficients in* A.

Proof: $'' \Longrightarrow''$ Let $f = \sum_{k:1}^{m} X^k * f_k$, with $f_k = \sum_{i:0}^{\infty} a_{k,i} X^i \in HA$. Since $X^k * f_k = \sum_{i:k}^{\infty} a_{k,i-k} C_i^k X^i$ then $a_0 = 0$ and for each $j > m$, $a_j = \sum_{k:1}^{m} a_{k,j-k} C_j^k$.
$'' \Longleftarrow''$ For $j > m$, let $a_j = \sum_{k:1}^{m} b_{j,k} C_j^k$, so $a_j X^j = \sum_{k:1}^{m} b_{j,k} C_j^k X^j = \sum_{k:1}^{m} b_{j,k} X^k * X^{j-k}$. Then $f = \sum_{j:1}^{\infty} a_j X^j = \sum_{j:1}^{m} a_j X^j + \sum_{j:m+1}^{\infty} \sum_{k:1}^{m} b_{j,k} X^k * X^{j-k} = \sum_{k:1}^{m} X^k * \big(a_k + \sum_{j:m+1}^{\infty} b_{j,k} X^{j-k}) \in (X, X^2, \ldots, X^m)$. □

Corollary 1.2. *The series* $f = \sum_{j:0}^{\infty} a_j X^j \in (X, X^2, \ldots, X^m)$ *if and only if for each* $j \in \mathbb{N}$, $a_j X^j \in (X, X^2, \ldots, X^m)$.

Remark 1.3. *The analogous of Proposition 1.1 and Corollary 1.2 are true in Hurwitz polynomials ring.*

Proposition 1.4. *Suppose that the ring* A *is reduced and the* $\mathbb{Z}-$*module* A *is torsion free. Let* $f = \sum_{i:0}^{\infty} a_i X^i$ *and* $g = \sum_{j:0}^{\infty} b_j X^j \in HA$. *Then* $f * g = 0$ *if and only if* $a_i b_j = 0$ *for all* $i, j \in \mathbb{N}$. *In particular,* $(0 : f) = H(0 : a_i, i \in \mathbb{N})$.

Proof: First step: If $(a_k X^k + a_{k+1} X^{k+1} + \ldots) * g = 0$ then $a_k g = 0$. Indeed, the coefficient of X^k in this product is $a_k b_0 = 0$. Suppose by induction that $a_k b_i = 0$ for any $i < n$. The coefficient of X^{k+n} in the product is $\sum_{i:0}^{k+n} C_{k+n}^i a_i b_{k+n-i} = \sum_{i:k}^{k+n} C_{k+n}^i a_i b_{k+n-i} = \sum_{j:0}^{n} C_{k+n}^{k+n-j} a_{k+n-j} b_j = 0$. Multiplying by a_k we obtain $C_{k+n}^k a_k^2 b_n = 0$. Since A is torsion free then $a_k^2 b_n = 0$ so $(a_k b_n)^2 = 0$. Since A is reduced $a_k b_n = 0$.

Second step: Let $f = \sum_{i:0}^{\infty} a_i X^i$ and $g = \sum_{i:0}^{\infty} b_i X^i$ such that $f * g = 0$. By the first step, $a_0 g = 0$. Suppose by induction that $a_i g = 0$ for $0 \leq i \leq n$. Then

$(a_{n+1}X^{n+1}+\dots)*g=(f-\sum_{i:0}^{n}a_iX^i)*g=f*g-\sum_{i:0}^{n}X^i*(a_ig)=0$. By the first step, $a_{n+1}g=0$. So for all $i,j\in\mathbb{N}$, $a_ib_j=0$. □

Examples.

All the hypotheses in Proposition 1.4 are necessary.

1. Let A be any ring, $0\neq a\in A$ and $m\geq 2$ an integer such that $ma=0$. Then $aX*X^{m-1}=maX^m=0$. But $a.1=a\neq 0$. For the non-reduced ring $A=\mathbb{Z}/4\mathbb{Z}$, we have $X*\bar{2}X=\bar{4}X^2=0$. But here $\mathbb{Z}/4\mathbb{Z}$ is not a torsion free $\mathbb{Z}-$module.

2. Let K be a commutative field of zero characteristic and $A=K[U,V]/(U^2,V^2)$ where U and V are indeterminates over K. The ring A is not reduced. Put $u=\bar{U}$ and $v=\bar{V}$. Then A is a $K-$vector space with basis $\{1,u,v,uv\}$ and a torsion free $\mathbb{Z}-$module. In HA, we have $(u-vX)*(u+vX)=u^2-v^2X^{(2)}=0$. But $uv\neq 0$.

Corollary 1.5. *Suppose that the ring A is reduced and the $\mathbb{Z}-$module A is torsion free. Let $f_1,\dots,f_n\in HA$ be such that $f_1*\dots*f_n=0$. Then $a_1\dots a_n=0$ for each coefficient a_i of f_i.*

Proof: Let a_i any coefficient of f_i. Since $f_1*(f_2*\dots*f_n)=0$ by Proposition 1.4, $a_1b=0$ for any coefficient b of $f_2*\dots*f_n$. Thus $a_1(f_2*\dots*f_n)=0$ so $(a_1f_2)*(f_3*\dots*f_n)=0$. Since a_1a_2 is a coefficient of a_1f_1 we have $(a_1a_2)c=0$ for any coefficient c of $f_3*\dots*f_n$. Hence $a_1a_2(f_3*\dots*f_n)=0$. Continuing, we see that $a_1\dots a_n=0$. □

Notation.

For any ring A, $Nil(A)$ is its nilradical, i.e; the set of all the nilpotent elements of A and $T(A)=\{x\in A;\ \exists 0\neq m\in\mathbb{Z}, mx=0\}$ is the $\mathbb{Z}-$ sub-module of A consisting of elements with torsion.

Proposition 1.6. *The following conditions are equivalent for a ring A:*
1. $T(A)\subseteq Nil(A)$.
2. The $\mathbb{Z}-$ module $A/Nil(A)$ is torsion free.
3. If f and $g\in HA$ (resp hA) are such that $f*g\in H(Nil(A))$ (resp $h(NilA)$) then for each coefficient a of f and each coefficient b of g, $ab\in Nil(A)$.

Proof: $''(1)\implies(2)''$ Let $\bar{a}\in A/Nil(A)$ and $n\in\mathbb{N}^*$ such that $n\bar{a}=0$. Then $na\in Nil(A)$. There exists $k\in\mathbb{N}^*$ such that $0=(na)^k=n^ka^k$. Hence $a^k\in T(A)\subseteq Nil(A)$. So $a\in Nil(A)$ and $\bar{a}=0$.
$''(2)\implies(3)''$ $A/Nil(A)$ is a reduced ring and torsion free $\mathbb{Z}-$module. In $H(A/Nil(A))$, we have $\bar{f}*\bar{g}=0$. By Proposition 1.4, for each coefficient a of f and each coefficient b of g, we have $\bar{a}\bar{b}=0$ so $ab\in Nil(A)$.
$''(3)\implies(1)''$ Suppose that $T(A)\not\subseteq Nil(A)$ and let $a\in T(A)\setminus Nil(A)$. There exists $m\in\mathbb{N}^*$ such that $ma=0$. For the Hurwitz polynomials $f=X$ and

$g = aX^{m-1}$, we have $f * g = maX^m = 0 \in H(Nil(A))$. But $a1 = a$ is not nilpotent. □

2 Hurwitz series over a PF-ring

We recall the defintion of a PF-ring and we prove that it is reduced; i.e., without nontrivial nilpotent element. While we give necessary and sufficient conditions for Hurwitz polynomials ring to have the PF-property, the case of Hurwitz series is stated under the Noetherian hypothesis.

Definition 2.1. *A ring A is said to be PF if for each $a \in A$ and $x \in (0 : a)$ there exists $y \in (0 : a)$ such that x=xy.*

The following Lemma must be well known.

Lemma 2.2. *A PF-ring A is reduced.*

Proof: Let $0 \neq a \in Nil(A)$ and $n \geq 1$ the smallest integer such that $a^n = 0$. Then $n \geq 2$ and $a \in (0 : a^{n-1})$. Since A is a PF-ring, there exists $b \in (0 : a^{n-1})$ such that $ab = a$. So $0 = ba^{n-1} = (ba)a^{n-2} = aa^{n-2} = a^{n-1}$, which contradicts the minimality of n. □

Lemma 2.3. *If hA or HA is a PF-ring then A is a PF-ring and a torsion free $\mathbb{Z}-$module.*

Proof: Let $a \in A$ and $b \in (0 : a)_A$. Since hA (resp. HA) is PF-ring, there is $g = c_0 + c_1X + \ldots$ such that $ag = 0$ and $bg = b$. So $c_0a = 0$ and $bc_0 = b$. Then A is a PF-ring. Let $a \in A$ and $m \geq 1$ an integer such that $ma = 0$. Then $aX * X^{m-1} = maX^m = 0$, so $aX \in (0 : X^{m-1})$. Since hA (resp. HA) is a PF-ring there is $f = a_0 + a_1X + \ldots \in (0 : X^{m-1})$ such that $aX = aX * f = aa_0X + 2aa_1X^2 + \ldots$. So $a = aa_0$. But $0 = X^{m-1} * f = a_0X^{m-1} + \ldots$ so $a_0 = 0$ and $a = 0$. We conclude that the $\mathbb{Z}-$module A is torsion free. □

Corollary 2.4. *If charact$(A) = m > 0$ then hA and HA are not PF-rings.*

Proof: For each $a \in A$, $ma = 0$. □

Theorem 2.5. *The ring hA (resp. HA) is PF if and only if for each $f, g \in hA$ (resp. HA) such that $f * g = 0$ there is $c \in A$ such that $cf = 0$ and $cg = g$.*

Proof: Let $f = a_0 + a_1X + \ldots$ and $g = b_0 + b_1X + \ldots \in hA$ (resp. HA) such that $f*g = 0$. Since hA (resp. HA) is a PF-ring there exists $h = c_0 + c_1X + \ldots \in (0 : f)$ such that $g = h * g$ so $g * (h - 1) = 0$ and $f * h = 0$. By Lemma 2.3, A is a reduced ring and a torsion free $\mathbb{Z}-$module. By Proposition 1.4, $a_ic_j = 0$ for all $i, j \in \mathbb{N}$ and in particular $c_0f = 0$. Also $b_i(c_0 - 1) = 0$ for all $i \in \mathbb{N}$, so $b_ic_0 = b_i$ and $c_0g = g$. The converse is clear. □

Theorem 2.6. *Hurwitz polynomials ring over A is PF if and only if A is a PF-ring and a torsion free $\mathbb{Z}-$module.*

Proof: $'' \Longrightarrow''$ By Lemma 2.3. $'' \Longleftarrow''$ Let $f = a_0 + \ldots + a_mX^m$ and $g = g_0 + \ldots + g_nX^n \in hA$ such that $f * g = 0$. Since A is reduced then by Proposition 1.4, $a_ig_j = 0$ for all i, j. Put $J = (0 : g_0)_A \cap \ldots \cap (0 : g_n)_A$. Then $a_0, \ldots, a_m \in J$. If $x \in J$ since A is PF, for each $j = 0, \ldots, n$ there is $y_j \in (0 : g_j)_A$ such that $x = xy_j$. Let $y = y_0 \ldots y_n \in J$ then $xy = x$. Applying this for $a_0, a_1, \ldots, a_m \in J$, we find $b_0, b_1, \ldots, b_m \in J$ such that $a_0b_0 = a_0, a_1b_1 = a_1, \ldots, a_mb_m = a_m$. Define inductively the elements $c_0, \ldots, c_m$ of J by $c_0 = b_0$ and $c_i = c_{i-1}+b_i-c_{i-1}b_i$. Note that $a_0c_0 = a_0b_0 = a_0$. Suppose by induction that $(a_0+a_1X+\ldots+a_iX^i)c_i = a_0+a_1X+\ldots+a_iX^i$, for each $i \leq k$. Then $(a_0+a_1X+\ldots+a_kX^k+a_{k+1}X^{k+1})c_{k+1} = (a_0 + a_1X + \ldots + a_kX^k + a_{k+1}X^{k+1})(c_k + b_{k+1} - c_kb_{k+1}) = (a_0 + a_1X + \ldots + a_kX^k)c_k+(a_0+a_1X+\ldots+a_kX^k)(b_{k+1}-c_kb_{k+1})+a_{k+1}X^{k+1}(c_k+b_{k+1}-c_kb_{k+1}) = a_0+a_1X+\ldots+a_kX^k+a_{k+1}c_kX^{k+1}+a_{k+1}b_{k+1}X^{k+1}-c_ka_{k+1}b_{k+1}X^{k+1} = a_0+a_1X+\ldots+a_kX^k+a_{k+1}X^{k+1}$. So for $t = 0, 1, \ldots, m$, $(a_0+a_1X+\ldots+a_tX^t)c_t = a_0 + a_1X + \ldots + a_tX^t$. Take $c = c_m \in A$, then $cf = f$ and $c \in J = (0 : g_0)_A \cap \ldots \cap (0 : g_n)_A$ so $c \in (0 : g)_{hA}$. □

Corollary 2.7. *Suppose A Noetherian. Then HA is a PF-ring if and only if A is a PF-ring and a torsion free $\mathbb{Z}-$module.*

Proof: $'' \Longrightarrow''$ By Lemma 2.3. $'' \Longleftarrow''$ Let $f = a_0 + a_1X + \ldots$ and $g = g_0 + g_1X + \ldots \in HA$ such that $f \in (0 : g)$. Then $f * g = 0$ and since A is reduced then by Proposition 1.4, $a_ig_j = 0$ for all i, j. Let $C_f = (a_0, a_1, \ldots)$ and $C_g = (g_0, g_1, \ldots)$ be the contents of f and g respectively. Since A is Noetherian there are integers $m, n \in \mathbb{N}$ such that $C_f = (a_0, a_1, \ldots, a_m)$ and $C_g = (g_0, g_1, \ldots, g_n)$. Put $\tilde{f} = a_0 + a_1X + \ldots + a_mX^m$ and $\tilde{g} = g_0 + g_1X + \ldots + g_nX^n \in hA$. Then $\tilde{f} * \tilde{g} = 0$ so $\tilde{f} \in (0 : \tilde{g})_{hA}$. By Theorem 2.6, hA is PF. By Theorem 2.5, there is $c \in A$ such that $c\tilde{f} = \tilde{f}$ and $c \in (0 : \tilde{g})$. This means that $ca_0 = a_0, \ldots, ca_m = a_m$ and $cg_0 = \ldots = cg_n = 0$. So $ca = a$ for each $a \in C_f$ and $cb = 0$ for each $b \in C_g$. Then $cf = f$ and $c \in (0 : g)_{HA}$. Then HA is PF. □

3 Hurwitz series over a PP-ring

We give necessary and sufficient conditions for Hurwitz polynomials and series rings to have the PP-property.

Definition 3.1. *A ring A is said to be a PP-ring if the annihilator $(0 : a)$ of each element $a \in A$ is generated by an idempotent element of A.*

The following three lemmas are less or more known. We include their proofs for sake of the completeness.

Lemma 3.2. *Any PP-ring A is a PF-ring, so reduced.*

Proof: Let $a \in A$ then $(0 : a) = eA$ with $e^2 = e$. For each $x \in (0 : a)$, $x = eb$ with $b \in A$. So $x = eb = e^2 b = ex$ with $e \in eA = (0 : a)$. □

Lemma 3.3. *Let $e_1, \ldots, e_n$ be idempotent elements in a ring A. Then the intersection $e_1 A \cap \ldots \cap e_n A = eA$ where $e = e_1 e_2 \ldots e_n$ is an idempotent.*

Proof: By induction it suffices to prove the property for $n = 2$. Put $e = e_1 e_2 \in e_1 A \cap e_2 A$ then $eA \subseteq e_1 A \cap e_2 A$. Conversely, let $x \in e_1 A \cap e_2 A$ then $x = e_1 a$ and $x = e_2 b$ with $a, b \in A$. Then $e_2 x = e_1 e_2 a = ea$ and $e_2 x = e_2^2 b = e_2 b = x$ so $x = ea \in eA$ and $e_1 A \cap e_2 A \subseteq eA$. □

Lemma 3.4. *The following assertions are equivalent for a ring A*
1. A is a PP-ring.
2. For each $0 \neq a \in A$, there exists an idempotent $e \in A$ and a regular element $b \in A$ such that $a = be$.

Proof: $''(1) \Longrightarrow (2)''$ Let $0 \neq a \in A$ then $(0 : a) = Ae$ with $e^2 = e$ and $ae = 0$. We have $(1 - e)(e + a) = e + a - e^2 - ea = a$. Since $1 - e$ is idempotent, we just need to prove that $e + a$ is regular. Let $x \in A$ such that $x(e + a) = 0$ then $xe + xa = 0$. By multiplying by e we have $xe + xae = 0$ so $xe = 0$. The equality $xe + xa = 0$ becomes $xa = 0$ so $x \in (0 : a) = Ae$. Put $x = eb$ with $b \in A$. The equality $x(e + a) = 0$ becomes $0 = eb(e + a) = b(e^2 + ea) = be = x$ so $e + a$ is regular.
$''(2) \Longrightarrow (1)''$ Let $0 \neq a \in A$ then $a = be$ with $e^2 = e$ and b a regular element. Let $x \in (0 : a)$ then $0 = ax = bex$ so $ex = 0$ because b is regular. So $x = (1 - e)x \in (1 - e)A$ then $(0 : a) \subseteq (1 - e)A$. Conversely, $(1 - e)a = (1 - e)be = (e - e^2)b = 0$ so $1 - e \in (0 : a)$ then $(0 : a) = (1 - e)A$. □

This section is based on the result [1, Proposition 2.3] which states that $Bool(HA) = Bool(hA) = Bool(A)$ for any commutative ring A.

Lemma 3.5. *If hA or HA is a PP-ring then A is a PP-ring and a torsion free $\mathbb{Z}-$module.*

Proof: Let $r \in A$. Since hA (resp. HA) is a PP-ring, $(0 : r)_{hA} = ehA$ (resp. $(0 : r)_{HA} = eHA$) with $e \in Bool(HA) = Bool(hA) = Bool(A)$. Because $er = 0$ then $e \in (0 : r)_A$ so $eA \subseteq (0 : r)_A$. Conversely, let $l \in (0 : r)_A \subseteq (0 : r)_{hA} = ehA$ (resp. $(0 : r)_A \subseteq (0 : r)_{HA} = eHA$) then $l = ef$. So $l = ea_0 \in eA$ where a_0 is the constant term of f, then $(0 : r)_A \subseteq eA$. We conclude that $(0 : r)_A = eA$ and A is a PP-ring.
Let $a \in A$ and $m \geq 2$ be an integer such that $ma = 0$. By Lemma 3.4, there is $e \in Bool(A)$ and $f = a_0 + a_1 X + \ldots \in hA$ (resp. HA) a regular element such that $X = ef$. Then $ea_0 = ea_2 = \ldots = 0$ and $ea_1 = 1$. So $e \in U(A)$ but the only invertible idempotent is 1. Then $e = 1$, $a_1 = 1$ and $a_0 = a_2 = \ldots = 0$ so $X = f$ is a regular element. But $X * aX^{m-1} = maX^m = 0$ so $aX^{m-1} = 0$ and $a = 0$. This proves that A is a torsion free $\mathbb{Z}-$module.
An alternative proof: Since hA (resp. HA) is a PP-ring then it is a PF-ring. By Lemma 2.3, A is a torsion free $\mathbb{Z}-$module. □

Corollary 3.6. *If charact(A)* > 0 *then* hA *and* HA *are not PP-rings.*

Theorem 3.7. *The Hurwitz polynomials ring over* A *is PP if and only if* A *is a PP-ring and a torsion free* $\mathbb{Z}-$*module.*

Proof: $'' \Longrightarrow''$ By Lemma 3.5. $'' \Longleftarrow''$ Let $f = \sum_{i:0}^{n} a_i X^i \in hA$. Put $(0 : a_i) = e_i A$ with $e_i \in Bool(A)$ and $e = e_0 \ldots e_n \in Bool(A)$. By Lemma 3.3, $e_0 A \cap \ldots \cap e_n A = eA$. Because A is a PP-ring so it is reduced by Lemma 3.2. By Proposition 1.4, we have $(0 : f) = h(e_0 A \cap \ldots \cap e_n A) = ehA$. □

Lemma 3.8. *Let* A *be a reduced ring. The relation defined on* A *by* $a \leq b$ *if and only if* $ab = a^2$ *is an order.*

Proof: The reflexivity is clear. Let $a, b \in A$ such that $a \leq b$ and $b \leq a$, then $ab = a^2$ and $ab = b^2$. So $(a-b)^2 = a^2 + b^2 - 2ab = 0$. Since A is reduced $a - b = 0$ and $a = b$. So $\leq$ is anti-symmetric. Let $a, b, c \in A$ such that $a \leq b$ and $b \leq c$ then $ab = a^2$ and $bc = b^2$. Then $(ac - ab)^2 = a^2(c^2 + b^2 - 2cb) = ab(c^2 + b^2 - 2b^2) = ab(c^2 - b^2) = ab(c - b)(c + b) = a(bc - b^2)(c + b) = 0$. Since A is reduced then $ac - ab = 0$ so $ac = ab = a^2$ then $a \leq c$, which means that $\leq$ is transitive. □

Lemma 3.9. *Let* A *be a PP-ring and* $S \subseteq Bool(A)$*. Then* S *admits a least upper bound in* $Bool(A)$ *if and only if it admits a least upper bound in* $(A, \leq)$ *which is an idempotent.*

Proof: $'' \Longleftarrow''$ Clear. $'' \Longrightarrow''$ Let e be the least upper bound of S in $Bool(A)$ and $y \in A$ be any upper bound of S. Then $xy = x$ for each $x \in S$, so $x(1 - y) = 0$. By Lemma 3.4, $1 - y = fa$ with $f^2 = f$ and $a \in A$ a regular element. Since $0 = x(1 - y) = xfa$ then $xf = 0$ and $x(1 - f) = x$ so $x \leq 1 - f$ for each $x \in S$. By definition of e we have $e \leq 1 - f$ so $e(1 - f) = e$ and $ef = 0$. Then $e(1 - y) = efa = 0$ so $ey = e$. This means that $e \leq y$ and e is the least upper bound of S in $(A, \leq)$.

The PP-property in Hurwitz series ring is raised in [4] under strong hypotheses. In the following Theorem we characterize it completely. □

Theorem 3.10. *The following assertions are equivalent for a ring* A

1. HA *is a PP-ring.*

2. A *is a PP-ring and a torsion free* $\mathbb{Z}-$*module and any finite or countable set of* $Bool(A)$ *admits a least upper bound in* $(Bool(A), \leq)$*.*

3. A *is a PP-ring and a torsion free* $\mathbb{Z}-$*module and any finite or countable increasing sequence in* $Bool(A)$ *admits a least upper bound in* $(A, \leq)$ *that belongs to* $Bool(A)$*.*

Proof: $''(1) \Longrightarrow (2)''$ By Lemma 3.5, A is a PP-ring and a torsion free $\mathbb{Z}-$module. Let $C = \{e_i;\ i \in I\}$, $I \subseteq \mathbb{N}$, be a finite or countable set of nonzero idempotent elements of A. Put $0 \neq g = \sum_{i \in I} e_i X^i \in HA$. Since HA is a PP-ring, by

Lemma 3.4, $g = t\theta$ with $t \in Bool(HA) = Bool(A)$ and $\theta \in HA$ a regular element. We will prove that t is the least upper bound of C in $Bool(A)$. We have $(1-t)g = g - tg = g - t^2\theta = g - t\theta = g - g = 0$ so for each $i \in I$, $(1-t)e_i = 0$, then $te_i = e_i$ or $e_i \le t$. Then t is an upper bound of C. Let $t' \in Bool(A)$ be such that $e_i \le t'$ for each $i \in I$. Then $e_i t' = e_i$ so $(1-t')e_i = 0$ then $(1-t')g = 0$ or $(1-t')t\theta = 0$ and then $(1-t')t = 0$ because θ is a regular element. Then $t't = t$ so $t \le t'$. We conclude that t is the least upper bound for C.

$''(2) \implies (3)''$ By Lemma 3.9. $''(3) \implies (1)''$ Let $f = \sum_{i:0}^{\infty} a_i X^i \in HA$. Since A is reduced, $(0:f) = H\big(0:(a_0, a_1, \ldots)\big)$. But $0:(a_0, a_1, \ldots) = \bigcap_{i:0}^{\infty}(0:a_i) = \bigcap_{i:0}^{\infty} e_i A$ with $e_i \in Bool(A)$ because A is a PP-ring. Put $d_n = e_0 e_1 \ldots e_n \in Bool(A)$ for any $n \in \mathbb{N}$. Since $d_i \in e_i A$ then $\bigcap_{i:0}^{\infty} d_i A \subseteq \bigcap_{i:0}^{\infty} e_i A$. Conversely, let $x \in \bigcap_{i:0}^{\infty} e_i A$, then $x = e_0 a = d_0 a$ with $a \in A$ so $x \in d_0 A$. Suppose by induction that $x \in d_n A$ so $x = d_n b$ with $b \in A$. But $x \in e_{n+1} A$ so $x = e_{n+1} c$ with $c \in A$. Then $e_{n+1} x = e_{n+1} d_n b = d_{n+1} b$ and $e_{n+1} x = e_{n+1}^2 c = e_{n+1} c = x$. So $x = d_{n+1} b \in d_{n+1} A$. Then $\bigcap_{i:0}^{\infty} d_i A = \bigcap_{i:0}^{\infty} e_i A$. We have an increasing sequence of idempotents $1 - d_0 \le 1 - d_1 \le \ldots$ because $(1-d_n)(1-d_{n+1}) = 1 - d_{n+1} - d_n + d_n d_{n+1} = 1 - d_{n+1} - d_n + d_{n+1} = 1 - d_n$ so $1 - d_n \le 1 - d_{n+1}$. Let $d \in Bool(A)$ the least upper bound of $\{1 - d_n;\ n \in \mathbb{N}\}$ in $(A, \le)$. We will prove that $\bigcap_{i:0}^{\infty} d_i A = (1-d)A$. Since $1 - d_i \le d$ then $(1-d_i)d = 1 - d_i$ so $1 - d = d_i(1-d) \in d_i A$ then $(1-d)A \subseteq \bigcap_{i:0}^{\infty} d_i A$. Conversely, let $y \in \bigcap_{i:0}^{\infty} d_i A$. Then $y = d_i y_i$ with $y_i \in A$ so $(1-d_i)(1-y) = 1 - y - d_i + d_i y = 1 - y - d_i + d_i^2 y_i = 1 - y - d_i + d_i y_i = 1 - y - d_i + y = 1 - d_i$ so $1 - d_i \le 1 - y$. According to the definition of d we have $d \le 1 - y$ so $d = d(1-y) = d - dy$ then $dy = 0$. So $y(1-d) = y - yd = y$ then $y \in (1-d)A$. We conclude that $\bigcap_{i:0}^{\infty} e_i A = \bigcap_{i:0}^{\infty} d_i A = (1-d)A$. Then $(0:f) = H\big(\bigcap_{i:0}^{\infty} e_i A\big) = (1-d)HA$. □

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Received: 10.09.2019
Accepted: 03.06.2011.

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