Bull. Math. Soc. Sci. Math. Roumanie
Tome 54(102) No. 3, 2011, 245–260

Multiplicity of algebras of minors of extended Hankel matrices

by
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Abstract

In this paper we give a formula to determine the multiplicity of the algebras A_t of minors of an extended Hankel matrix. The multiplicity of A_t coincides with the number of facets of a simplicial complex Δ_t. The facets of Δ_t are related to skew tableaux and we may express the multiplicity of A_t as a sum of the number $f^{\lambda/\mu}$ of standard skew tableaux of shape λ/μ for certain λ and μ.

Key Words: Multiplicity, determinantal ideal, extended Hankel matrix.
2010 Mathematics Subject Classification: Primary 13C40, Secondary 14M12.

1 Introduction

Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field k and let $k[X]$ be the polynomial ring over k whose indeterminates are the entries of X. Denote by M_t the set of the t-minors of X. Furthermore denote by I_t the ideal of $k[X]$ generated by M_t, by $R_t = k[X]/I_t$ and by A_t the k-subalgebra of $k[X]$ generated by M_t.

The variety associated to R_t is the subset of $\mathrm{Hom}_k(k^m, k^n)$ of maps of rank $< t$ while the variety associated to A_t is the variety of exterior powers of linear maps, that is, the closure image of the map $\mathrm{Hom}_k(k^m, k^n) \to \mathrm{Hom}_k(\wedge^t k^m, \wedge^t k^n)$ sending f to $\wedge^t f$.

Formulas for the multiplicity $e(R_t)$ of R_t (i.e. the degree of the associated variety) are classical and given by Giambelli [9], Abhyankar [1] and Galligo [8]. In particular in [12] Herzog and Trung used Gröbner bases deformations and a formula of Gessel and Viennot to prove that:

$$e(R_t) = \det\left(\binom{m+n-i-j}{m-i}\right)_{1\le i,j\le t-1}.$$

On the other hand the multiplicity $e(A_t)$ of A_t is not known in general despite to the fact that the homological and arithmetical property of A_t are well understood, see the paper of Bruns and Conca [2]. However if $t = m \le n$, then the algebra A_t is the coordinate ring of the Grassmann variety $\mathrm{Grass}(m-1, n-1)$ of the $(m-1)$-planes in a projective space of dimension $n-1$, and its multiplicity was computed by Schubert [15] in 1886; see the book of Harris [11, page 247]:

$$e(A_m) = \frac{1!2!\cdots(m-1)!(mr)!}{r!(r+1)!\cdots(m+r-1)!},$$

where $r = n - m$.

The goal of the paper is to present a combinatorial formula for the multiplicity for the algebra of minors associated with extended Hankel matrices. The ideals of minors of these matrices define the secant varieties of the so-called balanced rational normal scrolls while the variety associated with the algebra of minors can be seen as a subvariety of the variety of exterior powers of linear maps.

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring, and fix a positive integer c. The extended Hankel matrices are defined as follows:

$$X_{n,c,t} = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-(t-1)c} \\ x_{1+c} & x_{2+c} & \cdots & \cdots & \cdots \\ x_{1+2c} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1+(t-1)c} & \cdots & \cdots & \cdots & x_n \end{pmatrix}$$

for $t = 1, \ldots, m = \lfloor \frac{n+c}{c+1} \rfloor$. To simplify the notation we set $X_t = X_{n,c,t}$. As above we denote by M_t the set of t-minors of X_t and I_t, R_t and A_t respectively the ideal generated of R by M_t, the quotient ring R/I_t and the k-subalgebra of R generated bt M_t.

In [13] we have studied the algebra A_t in details. In particular we proved that the set M_t is a Sagbi basis of A_t. This allows us to deform A_t to a toric algebra $\mathcal{A}_t$ whose generators are the c-chains with length t (see below for the definition). Moreover, a square-free Gröbner basis of the binomial ideal defining $\mathcal{A}_t$ is described in [13, Theorem 5.1.4]. So we are led to consider the associated simplicial complex Δ_t. It turns out that the multiplicity of A_t is exactly the number of facets of Δ_t. We construct a bijection between the set of facets of Δ_t and a certain set of standard skew tableaux, see Theorem 3.7 and Theorem 3.17. Given a partition μ of m and a partition λ of n such that $\mu \subseteq \lambda$ we denote by f^λ the number of standard tableaux of shape λ with entries in $\{1, \ldots, n\}$ and by $f^{\lambda/\mu}$ the number of standard skew tableaux of shape λ/μ with entries in $\{1, \ldots, n-m\}$.

It turns out that:

(i)
$$e(A_m) = f^\lambda,$$

where $\lambda = (\underbrace{v, \ldots, v}_{m})$ and $v = n - (m-1)(c+1) - 1$.

(ii) If $2 \le t < m$, then

$$e(A_t) = \sum_{a \in \Lambda_{t-1}} f^{\lambda(a)/\mu(a)},$$

where Λ_{t-1} is the set of the c-chains of $\{2+c, \ldots, n-(c+1)\}$ of length $t-1$ and $\lambda(a)$, $\mu(a)$ are defined as in 3.14.

We are grateful to Prof. Aldo Conca for the discussions during the writing of the paper. Some results in this paper have been conjectured and confirmed by using the computer algebra package CoCoA [4].

2 The multiplicity of algebra of t-minors of extended Hankel matrices

Definition 2.1. In $\mathbb{N}$ we introduce the following partial order:

$$i \le_c j \qquad \text{if and only if} \qquad i = j \text{ or } i + c < j.$$

We write $i <_c j$ if $i \le_c j$ and $i \ne j$. We say that a sequence of integers $a_1, a_2, \ldots, a_s$ is a c-chain if $a_1 <_c a_2 <_c \cdots <_c a_s$. We set:

$$\Omega_t = \{\text{the set of } c\text{-chains of length } t\}.$$

Let $a = a_1, \ldots, a_t$ and $b = b_1, \ldots, b_t$ be two c-chains such that $(a_1, \ldots, a_t) \le (b_1, \ldots, b_t)$ with respect to the lexicographic order. If $a_i \le b_i$ for all $i = 1, \ldots, t$ and $b_i \le a_{i+1}$ for all $i = 2, \ldots, t-1$ then (a, b) is called *a sorted pair*. More generally, let $a^{(1)}, \ldots, a^{(k)}$ be c-chains length t. The table $A = (a_j^{(i)})$ is called sorted if $(a^{(i)}, a^{(j)})$ is sorted pair for all $1 \le i < j \le k$.

Lemma 2.2. *Let $a = a_1, \ldots, a_t$ and $b = b_1, \ldots, b_t$ be two c-chains. There exists a sorted pair $((c_1, \ldots, c_t), (d_1, \ldots, d_t))$ such that*

$$\text{multiset}\{a_1, \ldots, a_t, b_1, \ldots, b_t\} = \text{multiset}\{c_1, \ldots, c_t, d_1, \ldots, d_t\}.$$

Proof: We arrange the multiset $\{a_1, \ldots, a_t, b_1, \ldots, b_t\}$ as the multiset
$$\{\alpha_1, \alpha_2, \ldots, \alpha_{2t}\}$$
such that $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_{2t}$ and fill them into the table of size $2 \times t$ follow the *zigzag way*. This means:

$$(c_1, c_2, \ldots, c_t) = (\alpha_1, \alpha_3, \ldots, \alpha_{2t-1}), (d_1, d_2, \ldots, d_t) = (\alpha_2, \alpha_4, \ldots, \alpha_{2t}).$$

We need prove that $(c_1, \ldots, c_t), (d_1, \ldots, d_t)$ are c-chains. Because $a = a_1, \ldots, a_t$ and $b = b_1, \ldots, b_t$ are two c-chains, then

$$|\,\text{multiset}\{a_1, \ldots, a_t, b_1, \ldots, b_t\} \cap [a_i, a_i + c]| \le 2$$

and

$$|\operatorname{multiset}\{a_1,\dots,a_t,b_1,\dots,b_t\}\cap[b_i,b_i+c]|\le 2$$

for all $i=1,\dots,t$. Assume that $(c_1,\dots,c_t)$ is not a c-chain. There exists an index k such that $c_k\le c_{k+1}\le c_k+c$. This implies that

$$c_k,d_k,c_{k+1}\in\operatorname{multiset}\{a_1,\dots,a_t,b_1,\dots,b_t\}\cap[c_k,c_k+c].$$

This is a contradiction. □

The pair (c,d) in 2.2 is unique. It is called *the sorted pair reduction from* (a,b).

Let $R=k[x_1,x_2,\dots,x_n]$ be the polynomial ring over field k respects to the degree-lexicographic term order. We say that a monomial $x_{a_1}\cdots x_{a_s}$ is a c-chain of R if its indices form a c-chain. Denote by $\mathcal{A}_t$ the k-subalgebra of R generated by the elements $x_{a_1}x_{a_2}\cdots x_{a_t}$ with $x_{a_1}x_{a_2}\cdots x_{a_t}$ is a c-chain of R.

Denote by M_t the set of t-minors of X_t and A_t the k-subalgebra of R generated by the elements of M_t. In [13], we have shown that the set of t-minors form a Sagbi basis of A_t. By [3, Corollary 2.5], we have

$$e(A_t)=e(\operatorname{in}A_t)=e(\mathcal{A}_t).$$

We take a family of indeterminates $Y=(Y_a)_{a\in\Omega_t}$ and consider the presentation of $\mathcal{A}_t$ as:

$$\Phi:k[Y]\to\mathcal{A}_t$$

defined by sending Y_a to $x_a=x_{a_1}x_{a_2}\cdots x_{a_t}$, where $a=(a_1,\dots,a_t)$. Therefore, $\mathcal{A}_t\cong k[Y]/\ker\Phi$. So,

$$e(\mathcal{A}_t)=e(k[Y]/\ker\Phi)=e(k[Y]/\operatorname{in}(\ker\Phi)).$$

In [13, Theorem 5.1.4], a Gröbner basis of $\ker(\Phi)$ has been described. It follows that :

$$\operatorname{in}(\ker(\Phi))=(Y_aY_b:\text{(a,b) is a non-sorted pair}).$$

Therefore we can associate with A_t a simplicial complex Δ_t on the set Ω_t. The faces of Δ_t are the subsets $\{a^{(1)},a^{(2)},\dots,a^{(k)}\}$ of Ω_t such that the table $(a^{(i)})_{i=1}^k$ is sorted. The facets of Δ_t correspond to *maximal sorted tables*, that is sorted tables which are maximal under inclusion. Summing up, the multiplicity of A_t equals the number of maximal sorted tables.

We now describe some properties of the maximal sorted tables. We have the following lemma:

Lemma 2.3. *Let $A=(a_j^{(i)})$ be a sorted table with size $h\times t$.*
(a) If $t=m$, then A is a maximal sorted table if and only if the following conditions satisfy:

- $a^{(1)}=(1,2+c,\dots,1+(m-1)(c+1))$ *and* $a^{(h)}=(n-(m-1)(c+1),n-(m-2)(c+1),\dots,n)$,

- $\sum_{i=1}^{t} a_i^{(j+1)} - \sum_{i=1}^{t} a_i^{(j)} = 1$ *for all* $j = 1, \ldots, h-1$.

(b) If $2 \leq t < m$*, then* A *is a maximal sorted table if and only if the following conditions satisfy:*

- $a_1^{(1)} = 1$ *and* $a_t^{(h)} = n$,
- $a_i^{(h)} = a_{i+1}^{(1)}$ *for all* $i = 1, \ldots, t-1$,
- $\sum_{i=1}^{t} a_i^{(j+1)} - \sum_{i=1}^{t} a_i^{(j)} = 1$ *for all* $j = 1, \ldots, h-1$.

Proof: The maximality of the sorted table implies that $\sum_{i=1}^{t} a_i^{(j+1)} - \sum_{i=1}^{t} a_i^{(j)} = 1$ for all $j = 1, \ldots, h-1$.
(a) Because $a^{(i)} = (a_1^i, \ldots, a_t^i)$ is a c-chain of $\{1, \ldots, n\}$, then $1 + (j-1)(c+1) \leq a_j^{(i)} \leq n - (t-j)(c+1)$ for all $j = 1, \ldots, t$. Moreover $t = m$, there does not exist any c-chain of $\{1, \ldots, n\}$ of length $t+1$. Thus $a_j^{(h)} \leq n - (t-j)(c+1) \leq 1 + j(c+1)$ for all $j = 1, \ldots, t-1$. If $a^{(1)} \neq (1, 2+c, \ldots, 1+(m-1)(c+1))$ or $a^{(h)} \neq (n-(m-1)(c+1), n-(m-2)(c+1), \ldots, n)$, then we can extended the sorted table by adding the row $(1, 2+c, \ldots, 1+(m-1)(c+1))$ in the top of the table or the row $(n-(m-1)(c+1), n-(m-2)(c+1), \ldots, n)$ in the bottom of the table.
(b) If $2 \leq t < m$, then $(a_2^{(1)}, a_3^{(1)}, \ldots, a_t^{(1)})$ is a c-chain of $\{2+c, \ldots, n-(c+1)\}$. The desired assertion follows now from the definition of a sorted table. $\square$

Example 2.4. (a) Let $n = 9$, $t = 3$ and $c = 2$. Then

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 1 & 4 & 8 \\ 1 & 5 & 8 \\ 1 & 5 & 9 \\ 2 & 5 & 9 \\ 2 & 6 & 9 \\ 3 & 6 & 9 \end{pmatrix}$$

is a maximal sorted table.
(b) Let $n = 9$, $t = 3$ and $c = 1$. Then

$$A = \begin{pmatrix} 1 & 3 & 6 \\ 1 & 3 & 7 \\ 1 & 4 & 7 \\ 2 & 4 & 7 \\ 2 & 4 & 8 \\ 2 & 5 & 8 \\ 2 & 6 & 8 \\ 3 & 6 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

is a maximal sorted table.

Let $a = (a_1, \ldots, a_t), b = (b_1, \ldots, b_t)$ be two c-chains length t with $a_i \le b_i$ for all $i = 1, \ldots, t$. A maximal sorted tables with a is the first row and b is the last row can be interpreted as a path in $\mathbb{Z}^t$ from a to b. If we denote by $Q_c\binom{a}{b}$ the number of paths in the subregion of $\mathbb{Z}^t$ described by:

$$\begin{cases} a_i \le x_i \le b_i & \text{for all } i = 1, \ldots, t \\ c + 1 \le x_{i+1} - x_i & \text{for all } i = 1 \ldots, t - 1 \end{cases}.$$

we have that $Q_c\binom{a}{b}$ is the number of maximal sorted tables such that a is the first row and b is the last row.

We define the following function:

$$\chi_c(a_1, a_2, \ldots, a_t) = \begin{cases} 1 & \text{if } a_1, \ldots, a_t \text{ is a } c\text{ -chain} \\ 0 & \text{if } a_1, \ldots, a_t \text{ is not a } c\text{ -chain} \end{cases}.$$

The number $Q_c\binom{a}{b}$ can be determined recursively by:

$$Q_c\binom{a_1,\ldots,a_t}{b_1,\ldots,b_t} = \sum_{i=1}^{t} \chi(b_1, \ldots, b_{i-1}, b_i - 1, b_{i+1}, \ldots, b_t) Q_c\binom{a_1,\ldots,a_t}{b_1,\ldots,b_i-1,\ldots,b_t},$$

where $Q_c\binom{a}{a} = 1$ and $Q_c\binom{g_1,\ldots,g_t}{h_1,\ldots,h_t} = 0$ if $g_i > h_i$ for some i.

Set

$$P_c(a_1, \ldots, a_{t-1}) = Q_c\binom{1,a_1,\ldots,a_{t-1}}{a_1,\ldots,a_{t-1},n}.$$

Corollary 2.5. *(a) If $t = m$, then*

$$e(A_t) = Q_c\binom{a}{b},$$

where $a = (1, c+2, \ldots, 1 + (t-1)(c+1))$ and $b = (n - (m-1)(c+1), n - (m-2)(c+1), \ldots, n)$.
(b) If $2 \le t < m$, then

$$e(A_t) = \sum_{a \in \Lambda_{t-1}} P_c(a),$$

where Λ_{t-1} is the set of c-chains of $\{2 + c, \ldots, n - (c+1)\}$ of length $t - 1$.

Formulas for computing the number of paths in subregions of $\mathbb{Z}^t$ for $t > 2$ are very difficult to obtain. But in $\mathbb{Z}^2$ those formulas can be found and can be used to compute the multiplicity of A_2, see [3, Corollary 4.5]. On the other hand, 2.5 implies the following:

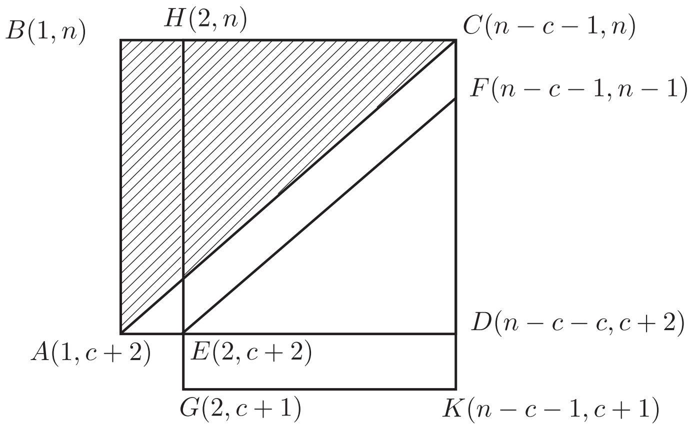
Corollary 2.6. *(a) If* $\lfloor \frac{n+c}{c+1} \rfloor = 2$*, then*

$$e(A_2) = Q_c\binom{1,c+2}{n-c-1,n}.$$

(b) If $\lfloor \frac{n+c}{c+1} \rfloor > 2$*, then*

$$e(A_2) = \sum_{a=c+2}^{n-c-1} P_c(a)$$

In [3] the formulas for $e(A_2)$ are obtained inductively. We close this section by showing that we can get those formula by a simple counting. Indeed we have:

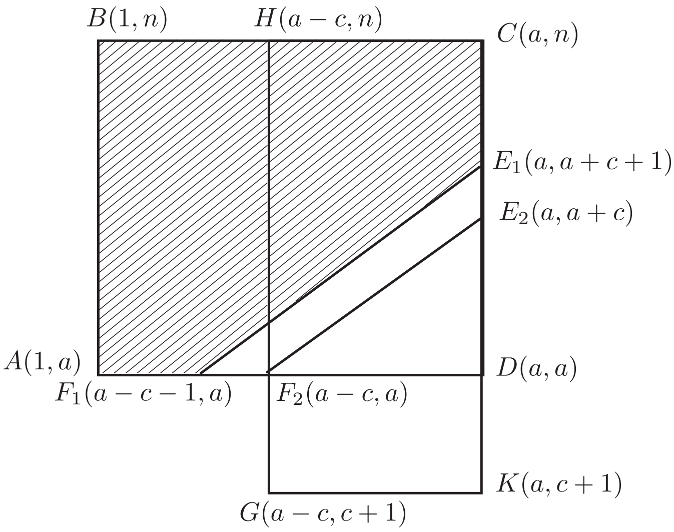


One has:

$$\begin{aligned} Q_c\binom{1,c+2}{n-c-1,n} &= |\{\text{paths from A to C in } \Delta ABC\}| \\ &= |\{\text{paths from A to C in } \Box ABCD\}| \\ &\quad -|\{\text{paths from A to C through a point in } \Delta DEF\}| \\ &= |\{\text{paths from A to C in } \Box ABCD\}| \\ &\quad -|\{\text{paths from G to C in } \Box GHCK\}| \\ &= \binom{2n-2c-4}{n-c-2} - \binom{2n-2c-4}{n-c-3}. \end{aligned}$$

So,

$$e(A_2) = \binom{2n-2c-4}{n-c-2} - \binom{2n-2c-4}{n-c-3}.$$



$$\begin{aligned}
P_c(a) &= |\{\text{paths from A to C in } ABCE_1F_1\}| \\
&= |\{\text{paths from A to C in } \square ABCD\}| \\
&\quad -|\{\text{paths from A to C through a point in } \Delta DE_2F_2\}| \\
&= |\{\text{paths from A to C in } \square ABCD\}| \\
&\quad -|\{\text{paths from G to C in } \square GHCK\}| \\
&= \binom{n-1}{a-1} - \binom{n-1}{c}.
\end{aligned}$$

So,

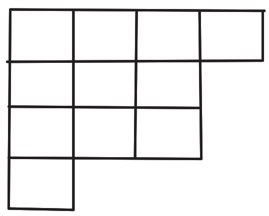
$$e(A_2) = \sum_{a=c+2}^{n-c-1} \binom{n-1}{a-1} - (n-2c-2)\binom{n-1}{c}.$$

3 Standard tableau, standard skew tableau and the formula of the multiplicity

We give in this section a combinatorial formula for $Q_c\binom{a}{b}$ and $P_c(a)$. We will deal with standard tableaux and standard skew tableaux.

Definition 3.1. Suppose $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. The *shape of* λ (or *Ferrers diagram*) is an array of n boxes into l left-justified rows with row i having λ_i boxes. We label every boxes with the set of numbers $\{1, \ldots, n\}$. It is called *a* λ*-tableau.* A λ-tableau is called *standard tableau* if its rows and its columns are increasing sequences.

Example 3.2. Let $\lambda = (4, 3, 3, 1)$.



λ-shape

2	5	8	3
7	10	1	
11	4	6	
9			

λ-tableau

1	3	6	10
2	4	7	
5	9	11	
8			

standard λ-tableau

The number of standard λ-tableaux is denoted by f^λ. This is the dimension of *Specht module* S^λ in group representation theory, see the book of Sagan [14]. The number f^λ is determined by the Hook formula, see, for instance, the paper of Frame, Robinson and Thrall [7].

Definition 3.3. If $v = (i, j)$ is a box in the shape λ, then the *Hook length* is

$$h_v = |\{(i, j')|j' \geq j\} \cup \{(i', j)|i' \geq i\}|$$

Theorem 3.4 (Hook formula)**.** *If* $\lambda \vdash n$, *then*

$$f^\lambda = \frac{n!}{\prod_{v \in \lambda} h_v}$$

Example 3.5. Let $\lambda = (4, 3, 3, 1)$. We have

7	5	4	1
5	3	2	
4	2	1	
1			

hook-length table

and

$$f^\lambda = \frac{11!}{7.5.5.4.4.3.2.2.1.1} = 1188.$$

On the other hand, the number f^λ can be expressed by following determinantal formula, see the papers of Frobenius [5] or Young [16]. Set $1/r! = 0$ if $r < 0$.

Theorem 3.6 (Determinantal formula)**.** *If* $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$, *then*

$$f^\lambda = n! \left| \frac{1}{(\lambda_i - i + j)!} \right|,$$

where the determinant is l *by* l.

We have the following theorem:

Theorem 3.7. *Let* $a = (1, c+2, \ldots, 1+(t-1)(c+1))$ *and* $b = (n-(m-1)(c+1), n-(m-2)(c+1), \ldots, n)$. *Then*

$$Q_c\binom{a}{b} = f^\lambda,$$

where $\lambda = (\underbrace{r, \ldots, r}_{t})$ *and* $r = n-(t-1)(c+1)-1$.

Proof: We will build a bijection between the set of maximal sorted tables of the form $\begin{pmatrix} a \\ \vdots \\ b \end{pmatrix}$ and the set of standard λ-tableaux. Set $s = t(n-1-(t-1)(c+1))$.

Let $A = (a_j^{(i)})$ be a maximal sorted table with a is the first row and b is the last row. By 2.3, table A has size $(s+1)\times m$ and $a_k^{(s+1)} - a_k^{(1)} = n-1-(t-1)(c+1)$ for all $k = 1, \ldots, m$. Moreover, $\sum_{i=1}^m a_i^{(j+1)} - \sum_{i=1}^m a_i^{(j)} = 1$ for all $j = 1, \ldots, s$, this implies that for each $j \in \{1, \ldots, s\}$ there exists an unique index $t_j \in \{1, \ldots, m\}$ such that $a_{t_j}^{(j+1)} = a_{t_j}^{(j)} + 1$. This is an injection between the set of maximal sorted tables and a set of multisets $\{t_1, \ldots, t_s\}$ with $t_j \in \{1, \ldots, m\}$. It is easy to see that $|\{j \mid t_j = k\}| = a_k^{(s+1)} - a_k^{(1)}$ for every multiset in this case. We will label the boxes of λ-shape step by step from 1 to s. If $t_j = k$, then we label the first empty box from the left in the $(t+1-k)$th row with the number j. We will prove that the λ-tableau is a standard λ-tableau. Obviously, the rows are increasing. Moreover, for each $j \in \{1, \ldots, s\}$, the set $\{i \mid i \le j, t_i = k\}$ is the content of the $(t+1-k)$th row in the jth step. Because the first row of the table A is tight (i.e., $a_{i+1}^{(1)} - a_i^{(1)} = c+1$ for all $i = 1, \ldots, t-1$), this implies that:

$$|\{i \mid i \le j, t_i = t\}| \ge |\{i \mid i \le j, t_i = t-1\}| \ge \cdots \ge |\{i \mid i \le j, t_i = 1\}|$$

for all $j = 1, \ldots, s$.

If there exists a place (u, v) in the λ-shape such that $\lambda_{u,v} = j_1 > \lambda_{u+1,v} = j_2$, then we have:

$$|\{i \mid i \le j_2, t_i = t+1-u\}| < |\{i \mid i \le j_2, t_i = t-u\}|$$

in the j_2th step. This is a contradiction. Therefore the λ-tableau is a standard λ-tableau.

Conversely, let B be a standard λ-tableau. We will rebuild the maximal sorted table A of the inverse correspondence. We define the ith row of A by induction on $i \in \{1, \ldots, s+1\}$ from the first row a. If the number i belongs to kth row of B, then we add 1 to the k-component of the ith row to have the $(i+1)$th row. It is easy to check that A is a maximal sorted table of the form $\begin{pmatrix} a \\ \vdots \\ b \end{pmatrix}$. □

This implies the following corollaries:

Corollary 3.8.

$$e(A_m) = f^\lambda,$$

where $\lambda = (\underbrace{r, \ldots, r}_{m})$ *and* $r = n - (m-1)(c+1) - 1$.

Corollary 3.9. *If* $\lfloor \frac{n+c}{c+1} \rfloor = 2$, *then*

$$e(A_2) = f^{(n-c-2, n-c-2)}.$$

If $t = m$, then A_m is isomorphic to the coordinate ring Grass$(m-1, m+r-1)$ of the Grassmann variety of $(m-1)$-planes in a projective space of dimension $m+r-1$ with r as in 3.8. We recall the formula of the degree of the Grassmann variety is given by Schubert in 1886, it is:

$$d(m,r) := \text{degree}(\text{Grass}(m-1, m+r-1)) = \frac{1!2!\cdots(m-1)!(mr)!}{r!(r+1)!\cdots(r+m-1)!}$$

Corollary 3.10.

$$e(A_m) = \frac{1!2!\cdots(m-1)!(mr)!}{r!(r+1)!\cdots(r+m-1)!}$$

Proof: By 3.8, $e(A_m) = f^\lambda$ with $\lambda = (\underbrace{r, \ldots, r}_{m})$. We have the hook length table:

$m+r-1$	$m+r-2$	$\cdots$	m
$m+r-2$	$m+r-3$	$\cdots$	
$\vdots$	$\vdots$		$\vdots$
r	$r-1$	2	1

It is easy to see that

$$e(A_m) = f^\lambda = \frac{(mr)!}{\prod_{v \in \lambda} h_\lambda(v)}$$

$$= \frac{(mr)!}{\frac{r!}{0!}\frac{(r+1)!}{1!}\cdots\frac{(r+m-1)!}{(m-1)!}} = \frac{1!2!\cdots(m-1)!(mr)!}{r!(r+1)!\cdots(r+m-1)!}$$

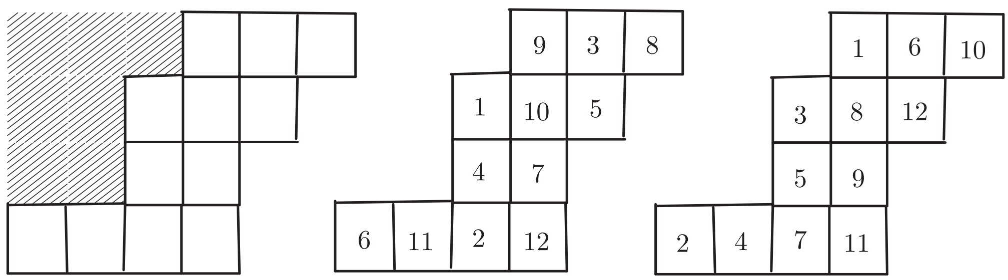
□

Definition 3.11. Let $\mu \subseteq \lambda$ be two shapes. The corresponding *skew shape* is the set of boxes

$$\lambda/\mu = \{c \mid c \in \lambda \text{ and } c \notin \mu\}.$$

If λ is a partition of n and μ is a partition of m, then we label every boxes of λ/μ with the set of numbers $\{1, \dots, n-m\}$. It is called *a skew λ/μ-tableau*. Moreover, it is called *a standard skew λ/μ-tableau* if its rows and its columns are increasing sequences. Denote by $f^{\lambda/\mu}$ the number of standard skew λ/μ-tableaux.

Example 3.12. Let $\lambda = (6, 5, 4, 4)$ and $\mu = (3, 2, 2)$.



λ/μ-shape　　skew λ/μ-tableau　　standard skew λ/μ-tableau

In [6, Main Theorem], Feit gives the determinantal formula for the number of standard skew tableaux. We have the following:

Theorem 3.13. *If $\mu = (\mu_1, \dots, \mu_s) \subseteq \lambda = (\lambda_1, \dots, \lambda_s)$ and $n = \sum_{i=1}^{s}(\lambda_i - \mu_i)$, then*

$$f^{\lambda/\mu} = n!\left|\frac{1}{(\lambda_i - \mu_j - i + j)!}\right|,$$

where the determinant is s by s.

Now we will give the combinatorial formula to determinate the number $P_c(a)$ in 2.5 with $t < m$ and $a \in \Lambda_{t-1}$.

Definition 3.14. Let $a = (a_1, \dots, a_{t-1}) \in \Lambda_{t-1}$. We define the sequences $\lambda(a) = (\lambda_1, \lambda_2, \dots, \lambda_t)$ and $\mu(a) = (\mu_1, \mu_2, \dots, \mu_{t-1}, \mu_t)$ as follows:

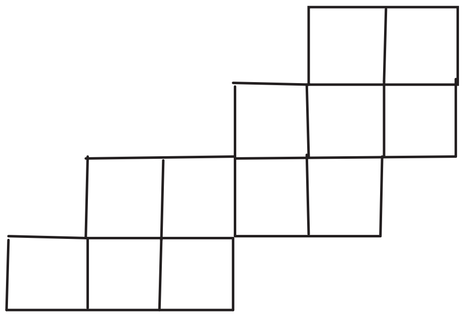
$$\lambda_1 = n - 1 - (t-1)(c+1), \lambda_i = a_{t+1-i} - 1 - (t-i)(c+1) \text{ for all } i = 2, \dots, t$$

and

$$\mu_i = \lambda_{i+1} - (c+1) \text{ for all } i = 1, \dots, t-1, \mu_t = 0.$$

Remark 3.15. *Let $a = (a_1, \dots, a_{t-1}) \in \Lambda_{t-1}$ and λ_i, μ_i are defined as above for all $i = 1, \dots, t$. Then $\lambda_i > \mu_i \geq 0$ for all $i = 1, \dots, t$ and $\lambda(a)$, $\mu(a)$ are decreasing sequences . In particular, $\mu(a) \subseteq \lambda(a)$ and $\sum_{i=1}^{t}(\lambda_i - \mu_i) = n - 1$.*

Example 3.16. Let $n = 13$, $c = 1$, $a = (4, 8, 11)$. Then $\lambda(a) = (6, 6, 5, 3)$ and $\mu(a) = (4, 3, 1, 0)$.



$\lambda(a)/\mu(a)$

Theorem 3.17. *Let* $a = (a_1, \ldots, a_{t-1}) \in \Lambda_{t-1}$. *Then one has:*

$$P_c(a) = f^{\lambda(a)/\mu(a)}.$$

Proof: We will construct a bijection between the set of maximal sorted tables of the form $\begin{pmatrix} 1 & a_1 & \cdots & a_{t-2} & a_{t-1} \\ & & \vdots & & \\ a_1 & a_2 & \cdots & a_{t-1} & n \end{pmatrix}$ and the set of standard skew $\lambda(a)/\mu(a)$-tableaux.

Let $A = (a_j^{(i)})$ be a maximal sorted table of above form. By 2.3, the table A has size $n \times t$ and $\sum_{i=1}^m a_i^{(j+1)} - \sum_{i=1}^m a_i^{(j)} = 1$ for all $j = 1, \ldots, s$. Therefore there exists an unique index $t_j \in \{1, \ldots, t\}$ such that $a_{t_j}^{(j+1)} = a_{t_j}^{(j)} + 1$ for each $j \in \{1, \ldots, n-1\}$. It is easy to see that $|\{j \;|t_j = k\}| = a_k^{(n)} - a_k^{(1)} = a_k - a_{k-1}$ with $a_0 = 1, a_t = n$. If $t_j = k$, we say that using k-step in the jth step. We will label the boxes of $\lambda(a)/\mu(a)$-shape step by step from 1 to $n-1$. If $t_j = k$, then we label the first empty box from the left in the $(t+1-k)$th row with the number j. We will prove that the skew $\lambda(a)/\mu(a)$-tableau is a standard skew $\lambda(a)/\mu(a)$-tableau.

For each $j \in \{1, \ldots, n-1\}$, the set $\{i \;|i \leq j, t_i = k\}$ is the content of the $(t+1-k)$th row in the jth step. In particular, $|\{i \;|i \leq j, t_i = k\}| = |\{k$-steps which are used from the first step to the jth step$\}|$. For each $j = 1, \ldots, s$ we have:

$$|\{i \;|i \leq j, t_i = k+1\}| + a_k - a_{k-1} - (c+1) \geq |\{i \;|i \leq j, t_i = k\}|$$

for all $k = 1, \ldots, t-1$.

If there exists a place (u, v) in the $\lambda(a)/\mu(a)$-shape such that $\lambda_{u,v} = j_1 > \lambda_{u+1,v} = j_2$, then we have

$$|\{i \;|i \leq j_2, t_i = k+1\}| + a_k - a_{k-1} - (c+1) < |\{i \;|i \leq j_2, t_i = k\}|$$

in the j_2th step with $k = t-u$. This is a contradiction. Therefore the $\lambda(a)/\mu(a)$-tableau is a standard skew $\lambda(a)/\mu(a)$-tableau.

Conversely, let B be a standard skew $\lambda(a)/\mu(a)$-tableau. We will rebuild the maximal sorted table A of the inverse correspondence. We define the ith row of A by induction on $i \in \{1, \ldots, s+1\}$ from the first row $(1, a_1, \ldots, a_{t-1})$. If the number i belongs to kth row of B, then we add 1 to the k-component of the ith row to have the $(i+1)$th row. It is easy to see that A is a maximal sorted table of the form as above. □

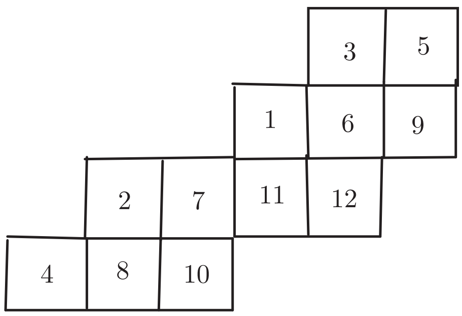
Example 3.18. Let $n = 13$, $c = 1$, $a = (4, 8, 11)$ as in 3.16. Given a maximal sorted table

$$A = \begin{pmatrix} 1 & 4 & 8 & 11 \\ 1 & 4 & 9 & 11 \\ 1 & 5 & 9 & 11 \\ 1 & 5 & 9 & 12 \\ 2 & 5 & 9 & 12 \\ 2 & 5 & 9 & 13 \\ 2 & 5 & 10 & 13 \\ 2 & 6 & 10 & 13 \\ 3 & 6 & 10 & 13 \\ 3 & 6 & 11 & 13 \\ 4 & 6 & 11 & 13 \\ 4 & 7 & 11 & 13 \\ 4 & 8 & 11 & 13 \end{pmatrix}$$

corresponds with a multiset

$$\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}\} = \{3, 2, 4, 1, 4, 3, 2, 1, 3, 1, 2, 2\}.$$

If $t_i = j$, then the number i stays in the $(5-j)$th of the $\lambda(a)/\mu(a)$-shape. So, the maximal sorted table A corresponds with the standard skew $\lambda(a)/\mu(a)$-tableau



standard skew $\lambda(a)/\mu(a)$-tableau

Corollary 3.19. *If $2 \le t < m$, then*

$$e(A_t) = \sum_{a \in \Lambda_{t-1}} f^{\lambda(a)/\mu(a)},$$

where Λ_{t-1} is the set of c-chains of $\{2+c, \ldots, n-(c+1)\}$ of length $t-1$.

Example 3.20. Let $n = 9, c = 1$ and $t = 3$. We have:

$$\begin{aligned} e(A_3) &= P_1(3,5) + P_1(3,6) + P_1(3,7) + P(4,6) + P(4,7) + P(5,7) \\ &= f^{(4,2,2)} + f^{(4,3,2)/(1,0,0)} + f^{(4,4,2)/(2,0,0)} + f^{(4,3,3)/(1,1,0)} \\ &\quad + f^{(4,4,3)/(2,1,0)} + f^{(4,4,4)/(2,2,0)} \\ &= 56 + 168 + 140 + 98 + 168 + 56 = 686. \end{aligned}$$

One can write down a simple routing to compute the number $f^{\lambda/\mu}$. We have done it by using CoCoA program. For instance, for n=30, c=3 and t=5 we get

$$e(A_t) = 6431457245628453210.$$

Acknowledgements: The author would like to thank the National Foundation for Science and Technology Development of Viet Nam (NAFOSTED) for helping the research.

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Received: 30.09.2010
Accepted: 11.04.2011.

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