Bull. Math. Soc. Sci. Math. Roumanie
Tome 54(102) No. 3, 2011, 281–293

Topology of plane curves and discriminants

by
Zahid Raza

Abstract

We explore the relation between the topology of the fibers of a polynomial in two complex variables and the degree of the associated discriminant. This gives, in particular, lower and upper bounds for this degree, and the polynomials realizing these bounds, or even close values, can be described geometrically, see Theorems 3.1, 3.2, 3.3 and 3.4.

Key Words: Plane curves, polar curves, discriminant.
2010 Mathematics Subject Classification: Primary 14H50, Secondary 14H20, 32S15.

1 Introduction

The following general result by R. Thom, A. Varchenko and J.-L. Verdier is well-known, see [14], [17], [18], [16].

Theorem 1.1. *Let* $f : \mathbb{C}^n \to \mathbb{C}$ *be a complex polynomial map, with* $n \geq 2$*. Then there is a minimal finite set* $B(f)$ *in* $\mathbb{C}$ *such that*

$$f : f^{-1}(\mathbb{C} \setminus B(f)) \to \mathbb{C} \setminus B(f)$$

is a locally trivial fibration.

This bifurcation set $B(f)$ contains two types of points : the critical values $C(f)$ of f and the critical values at infinity $I(f)$ of f. In the case $n \geq 3$ the description of the set $I(f)$ is rather involved, see [15].

In this paper we concentrate on the simpler case $n = 2$. First we recall the usual definition of $I(f)$ in terms of jumps of the Milnor number of the singularities of f at infinity. Then we recall a result by Krasiňski relating the sum of the Milnor number of the singularities of f at infinity to the degree $\deg \Delta(x)$ of the discriminant $\Delta(x)$ of f with respect to y. In fact we make this result more precise in Proposition 2.6 and relate it to the set $I(f)$ in Corollary 2.7. It is surprising that this explicit and constructive description of the set of critical values at infinity $I(f)$, which apparently goes back to Krasiǹski and Hà, is present neither in Durfee's excellent survey [8], nor in the recent monograph by Tibăr [16].

In Proposition 2.9 we relate the degree of $\Delta(x)$ to the topology of the affine curve $\mathcal{C}' : f = 0$ supposed to be reduced and irreducible. For such a situation we obtain the following bounds

$$m := d - 1 \leq \deg \Delta(x) \leq M := d^2 - d$$

where $d = \deg_y f = \deg f$. In the third section we describe the irreducible curves $\mathcal{C}'$ for which $\deg \Delta(x) \in \{m, M-2, M-1, M\}$. We also investigate the relation between the possible values of $\deg \Delta(x)$ and the geometry of the curve $\mathcal{C}'$ when $d = 3$.

In the final section we discuss our results in the second section the light of more sophisticated results which are valid in arbitrary dimension obtained in [5] and [15].

2 Teissier's result and Krasiǹski's formula

First we recall what is a critical value at infinity in the case $n = 2$.

Definition 2.1. *Let $f(x,y) \in \mathbb{C}[x,y]$ be a reduced polynomial and let $\mathcal{C}$ be the projective closure of the affine curve $\mathcal{C}' : f(x,y) = 0$. Let $\mathcal{C}_\infty = \mathcal{C} \cap \mathcal{L}_\infty$ be the set of points at infinity of $\mathcal{C}$. We consider the projective curves $\mathcal{C}_t$ (possibly with multiple factors) given by the equations*

$$\mathcal{C}_t : F(x,y,z) - tz^d = 0$$

where $d = \deg f$ and $F(x,y,z) = z^d f(x/z, y/z)$. Clearly $\mathcal{C}_\infty = \mathcal{C}_t \cap \mathcal{L}_\infty$ for any $t \in \mathbb{C}$. Let $\mu_p^t = \mu_p(\mathcal{C}_t)$ be the Milnor number of $\mathcal{C}_t$ at $p \in \mathcal{C}_\infty$. If a multiple component of $\mathcal{C}_t$ passes through p, then we set $\mu_p^t = +\infty$. Let

$$\mu_p^{min} = \min\{\mu_p^t : t \in \mathbb{C}\}. \tag{2.1}$$

With this notation one has

$$I(f) = \{t \in \mathbb{C} : \text{there is a } p \in \mathcal{C}_\infty \text{ such that } \mu_p^t > \mu_p^{min}\} \tag{2.2}$$

see for instance [10] or [6], pp. 20-22. The elements of $I(f)$ are called critical values at infinity of the polynomial function $f : \mathbb{C}^2 \longmapsto \mathbb{C}$. Different equivalent definitions of critical values at infinity are discussed in [8]. For every $t \in \mathbb{C}$ we put

$$\lambda^t(f) = \sum_{p \in \mathcal{C}_\infty} \left(\mu_p^t - \mu_p^{min}\right).$$

If $\mathcal{C}_t$ is non reduced then $\lambda^t(f) = +\infty$. Note that if $\mathcal{C}$ and $\mathcal{L}_\infty$ meet with multiplicity 1 at every intersection point, then $I(f) = \emptyset$. One says in such a case that $\mathcal{C}$ is transversal to the line at infinity $\mathcal{L}_\infty$, see also Theorem 3.1 below.

Lemma 2.2. *(Teissier's Formula)*
Let $\mathcal{C}'$ be the curve defined by the reduced polynomial $f(x,y)=0$ such that the line $x=a$ intersects the curve $\mathcal{C}'$ in a finite number of points. Then for any point $p=(a,b)\in\mathcal{C}'$ one has

$$(f,\frac{\partial f}{\partial y})_p=\mu_p(f)+(f,x-a)_p-1\,. \tag{2.3}$$

This is a two dimensional case of a formula due to Teissier, see [13], Chap.2, Proposition 1.2.

Definition 2.3. *(Polar curve)*
Let $\mathcal{C}\subset\mathbb{P}^2$ be a reduced projective curve given by $F=0$ and $q=[q_0:q_1:q_2]\in\mathbb{P}^2$ such that $q_0\frac{\partial F}{\partial x}+q_1\frac{\partial F}{\partial Y}+q_2\frac{\partial F}{\partial z}\neq 0$ in $\mathbb{C}[x,y,z]$.
The polar curve $\Gamma_q(\mathcal{C})$ of $\mathcal{C}$ with respect to q is the projective plane curve defined by the equation $q_0\frac{\partial F}{\partial x}+q_1\frac{\partial F}{\partial Y}+q_2\frac{\partial F}{\partial z}=0$.

Lemma 2.4. *(Projective Version of Teissier's Formula)*
For any point $p\neq q$ on the projective curve $\mathcal{C}$, one has

$$(\mathcal{C},\Gamma_q(\mathcal{C}))_p=\mu_p(\mathcal{C})+(\mathcal{C},\bar{pq})_p-1$$

where $\bar{pq}$ is the line passing through the points p and q.

Proof: We choose coordinates on $\mathbb{P}^2$ so that $p=[0:0:1]$, $q=[0:1:0]$. Then the above formula reduces to Teissier's local formula at the origin

$$(f,\frac{\partial f}{\partial y})_0=\mu_0(f)+(f,x)_0-1.$$

□

Let $f(x,y)\in\mathbb{C}[x,y]$ be a reduced polynomial such that $\deg_y f=\deg f=d>1$. Let $\Delta(x,T)=disc_y(f(x,y)-T)=\mathcal{R}_y(f(x,y)-T,\frac{\partial f}{\partial y})$ be the y discriminant regarded as a polynomial in $\mathbb{C}[x,T]$. Let us write

$$\Delta(x,T)=\Delta_0(T)x^N+\cdots+\Delta_N(T), where\ \ \Delta_0(T)\neq 0\in\mathbb{C}[T].$$

Let $\Delta(x)=disc_y f(x,y)$ be the y-discriminant of the polynomial f.

Proposition 2.5. *(Krasiǹski's Formula)*
Let f be a polynomial of two variables x,y which is x-regular, i.e. the equality $\deg_y f=\deg f$ holds. Then, for any value of $t\in\mathbb{C}$, we have

$$\sum_{p\in\mathcal{C}_\infty}\mu_p(\mathcal{C}_t)=c-\deg_x(\Delta(x,t))$$

where c is a constant independent of t.

This formula is due to Krasiǹski, for details see [12]. We improve this formula by calculating the value of this constant c.

Proposition 2.6. *Let* $f(x,y) \in \mathbb{C}[x,y]$ *be a reduced polynomial such that* $\deg_y f = \deg f = d > 1$ *and let* $\mathcal{C}$ *be the projective closure of the affine curve* $f(x,y) = 0$. *Let* $\mathcal{C}_\infty$ *be the set of points at infinity of* $\mathcal{C}$ *and let* $k = \#\mathcal{C}_\infty$. *Then*

$$\deg \Delta(x) = d(d-2) + k - \sum_{p \in \mathcal{C}_\infty} \mu_p(\mathcal{C}). \tag{2.4}$$

Proof: If $F(x,y,z)$ is the homogenous polynomial corresponding to $f(x,y)$, then $\mathcal{C}$ is defined by $F(x,y,z) = 0$. Then $\frac{\partial F}{\partial y}$ corresponds to $\frac{\partial f}{\partial y}$ because we have assumed that $\deg_y f = \deg f$. Let $q = (0:1:0)$, note that $q \notin \mathcal{C}$ and consider the corresponding polar curve $\Gamma_q(\mathcal{C}) : \frac{\partial F}{\partial y} = 0$. Then by well known properties of the discriminant, we have

$$\begin{aligned}
\deg \Delta(x) &= \deg \mathcal{R}_y(f, \frac{\partial f}{\partial y}) = \sum_{p \in \mathbb{C}^2} (f, \frac{\partial f}{\partial y})_p = \sum_{p \in \mathcal{C} \setminus \mathcal{L}_\infty} (\mathcal{C}, \Gamma_q(\mathcal{C}))_p \\
&= \sum_{p \in \mathcal{C}} (\mathcal{C}, \Gamma_q(\mathcal{C}))_p - \sum_{p \in \mathcal{L}_\infty} (\mathcal{C}, \Gamma_q(\mathcal{C}))_p.
\end{aligned}$$

By Bézout's Theorem for $\mathcal{C}, \Gamma_q(\mathcal{C})$ and Lemma 2.4 for $\mathcal{C}, \Gamma_q(\mathcal{C})$ we get

$$\begin{aligned}
\deg \Delta(x) &= d(d-1) - \sum_{p \in \mathcal{C}_\infty} \mu_p(\mathcal{C}) - \sum_{p \in \mathcal{C}_\infty} ((\mathcal{C}, \bar{pq})_p - 1) \\
&= d(d-1) - \sum_{p \in \mathcal{C}_\infty} \mu_p(\mathcal{C}) - d + \sum_{p \in \mathcal{C}_\infty} 1 \\
&= d(d-2) + k - \sum_{p \in \mathcal{C}_\infty} \mu_p(\mathcal{C}).
\end{aligned}$$

□

Corollary 2.7. *With the above notation*

$$\lambda^t(f) = N - \deg_x \Delta(x,t) \text{ and } I(f) = \{t \in \mathbb{C} : \Delta_0(t) = 0\}.$$

Proof: By Proposition 2.6 applied to $f - t$ we get

$$\deg_x \Delta(x,t) = d(d-2) + k - \sum_{p \in \mathcal{C}_\infty} \mu_p^t$$

and

$$\deg_x \Delta(x,T) = d(d-2) + k - \sum_{p \in \mathcal{C}_\infty} \mu_p^{min}.$$

Subtracting the above two equations we get the result. □

Remark 2.8. *In the above Propositions we need the condition* $\deg f_y = \deg f = d > 1$*. Indeed, if* $\deg f_y < \deg f$*, we cannot apply the affine form of the Bézout's theorem to get the equality*

$$\deg \mathcal{R}_y(f, \frac{\partial f}{\partial y}) = \sum_{p\in\mathbb{C}^2} (f, \frac{\partial f}{\partial y})_p$$

in the proof of Proposition 2.6, because the point $[0:1:0]$ *at infinity lies on both curves defined by the equations* F *and* $\frac{\partial F}{\partial y}$*. As an explicit example, consider* $f(x,y) = x^2y - x$*. Then* $\Delta(x,t) = 1$*, so we can derive no information about the critical values at infinity. But we know that* f *has a critical value at infinity, namely* $t = 0$*. See Example 2.11. below for more on this polynomial.*

Proposition 2.9. *Let* $f(x,y) \in \mathbb{C}[x,y]$ *be a reduced polynomial such that* $\deg_y f = \deg f = d > 1$*. Assume that the fiber* $\mathcal{C}'_t = f^{-1}(t)$ *is reduced and irreducible, then*

$$\deg_x \Delta(x,t) = d - 1 + b_1(\mathcal{C}'_t) + \mu(\mathcal{C}'_t)$$

where $\mu(\mathcal{C}'_t)$ *is the total Milnor number of* $\mathcal{C}'_t$*.*

Proof: With the above notation, we obviously have

$$\chi(\mathcal{C}'_t) = \chi(\mathcal{C}_t) - k.$$

Using next Corollary (5.4.4) in [6], p. 162, we get

$$\chi(\mathcal{C}'_t) = \chi(\mathcal{C}_{d,smooth}) + \sum_{i=1}^{k+s} \mu_{p_i}(\mathcal{C}_t) - k$$

where $\{p_{k+1}, ..., p_{k+s}\}$ are the singularities of the affine curve $\mathcal{C}'_t$ and $\mathcal{C}_{d,smooth}$ is a smooth plane curve of degree d. By Proposition 2.6, we get

$$\chi(\mathcal{C}'_t) = \chi(\mathcal{C}_{d,smooth}) + d(d-2) + k - deg_x\Delta(x,t) - k + \sum_{i=k+1}^{k+s} \mu_{p_i}(\mathcal{C}_t).$$

But

$$\chi(\mathcal{C}_{d,smooth}) = 2 - 2g = 3d - d^2$$

and, by definition, the total Milnor number $\mu(\mathcal{C}'_t)$ is precisely the sum $\sum_{i=k+1}^{k+s} \mu_{p_i}(\mathcal{C}_t)$. It follows that

$$\chi(\mathcal{C}'_t) = d - \deg_x \Delta(x,t) + \mu(\mathcal{C}'_t).$$

On the other hand $\mathcal{C}'_t$ is a connected affine curve, so it has the homotopy type of a CW-complex of dimension one, i.e. a bouquet of $b_1(\mathcal{C}'_t)$ circles S^1. It follows that

$$\chi(\mathcal{C}'_t) = 1 - b_1(\mathcal{C}'_t)$$

which yields the result. □

Remark 2.10. *It was later communicated to the author that the results 2.6, 2.7, 2.9 also obtained by the Polish mathematicians Janusz Gwozdziewicz and Arkadiusz Ploski in [9].*

Example 2.11. *(Broughton Polynomial)*
Let us consider the polynomial

$$f = x^2y - x$$

introduced in [4]. First we have to make f *an* x*-regular polynomial: for this we substitute* $x = x + y$ *and* $y = y$ *in the polynomial* f *and get*

$$f = x^2y + 2xy^2 + y^3 - x - y.$$

Then the discriminant is given by

$$\Delta(x,t) = -4tx^3 + x^2 - 18tx - 27t^2 + 4.$$

From this we see that $t = 0$ *is the only critical value at infinity and* $\lambda^0(f) = 1$.

Example 2.12. *(Briançon Polynomials)*
These polynomials were introduced in [3] and further studied and generalized in [2]. Set $s = xy+1$, $p = sx-1$, $g = (p^4+p^3+p^2x)x^{-2}$ *and finally* $f = g+a_1ps+a_0s$, *where we have two cases depending on the values of* a_0 *and* a_1.
Case(I) $a_0 = -\frac{1}{3},\ a_1 = \frac{5}{3}$

First we have to make f x*- regular : for this we substitute* $x = x+y$ *and* $y = y$ *in the polynomial* f. *We get the first Briançon polynomial*

$$f_B = y^{10} + 6y^9x + (15x^2 + 4)y^8 + (20x^3 + 20x - 3)y^7 + (15x^4 + 40x^2 - 12x+ 6)y^6+(6x^5+40x^3-18x^2+24x-\tfrac{19}{3})y^5+(x^6+20x^4-12x^3+36x^2-19x+7)y^4+ (4x^5 - 3x^4 + 24x^3 - 19x^2 + 18x - \tfrac{11}{3})y^3 + (6x^4 - \tfrac{19}{3}x^3 + 15x^2 - \tfrac{22}{3}x + 3)y^2+ (4x^3 - \tfrac{11}{3}x^2 + 4x - \tfrac{4}{3})y + x^2 - \tfrac{1}{3}x - 1 - t.$$

The corresponding discriminant is given by

$$\Delta(x,t) = (147456t^3 + 524288t^2 + \tfrac{4194304}{9}t)x^{13}+ (2654208t^5+23113728t^4+72963840t^3+\tfrac{900005888}{9}t^2+\tfrac{1367932928}{27}t+\tfrac{2097152}{9})x^{12}+ (11943936t^7 + 92565504t^6 + 273411072t^5 + 362007040t^4 + \tfrac{1430708224}{9}t^3- \tfrac{5250188224}{81}t^2 - \tfrac{24647948288}{729}t + \tfrac{680574976}{27})x^{11} + (-11943936t^8 - 111144960t^7 -429161472t^6 - 924229888t^5 - \tfrac{12616195712}{9}t^4 - \tfrac{151270623088}{81}t^3- \tfrac{462308130121}{243}t^2 - \tfrac{2033537129248}{2187}t - \tfrac{29936945920}{729})x^{10} + \cdots$$

where the dots represent the terms of lower degree of x. *Hence, we get* $\Delta_0(t) = 147456t^3+524288t^2+\frac{4194304}{9}t = 0$ *and solving this equation, we get the bifurcation values* $t = 0\,,\ -\frac{16}{9}$. *Hence the set of critical values at infinity is* $I(f) = \{0, -\frac{16}{9}\}$

and $\lambda^0(f) = 1$ *(since* $\Delta_1(0) \neq 0$*) and* $\lambda^{-\frac{16}{9}}(f) = 3$ *(since* $\Delta_1(-\frac{16}{9}) = \Delta_2(-\frac{16}{9}) = 0$ *and* $\Delta_3(-\frac{16}{9}) \neq 0$*).*

Cases(II) $a_0 = \frac{1}{9},\ a_1 = \frac{7}{9}$
First we have to make f *in* x *regular: for this we substitute* $x = x + y$ *and* $y = y$ *in the polynomial* f*. We get the second Briançon polynomial*
$f_C = y^{10} + 6xy^9 + (15x^2 + 4)y^8 + (20x^3 + 20x - 3)y^7 + (15x^4 + 40x^2 - 12x + 6)y^6 + (6x^5 + 40x^3 - 18x^2 + 24x - \frac{65}{9})y^5 + (x^6 + 20x^4 - 12x^3 + 36x^2 - \frac{65}{3}x + 7)y^4 + (4x^5 - 3x^4 + 24x^3 - \frac{65}{3}x^2 + 18x - \frac{49}{9})y^3 + (6x^4 - \frac{65}{9}x^3 + 15x^2 - \frac{98}{9}x + \frac{13}{3})y^2 + (4x^3 - \frac{49}{9}x^2 + \frac{16}{3}x - \frac{20}{9})y + x^2 - \frac{11}{9}x + \frac{1}{3} - t.$
Then the discriminant is given by

$$\begin{aligned}\Delta(x,t) = {} & (\tfrac{16384}{9}t^3 + \tfrac{2097152}{729}t^2 + \tfrac{67108864}{59049}t)x^{13} + (294912t^5 + \tfrac{1814528}{9}t^4 - \tfrac{56963840}{243}t^3 \\ & - \tfrac{9882959872}{59049}t^2 - \tfrac{6625951744}{4782969}t)x^{12} + (11943936t^7 + 34172928t^6 + \tfrac{109577216}{3}t^5 \\ & + \tfrac{12665979392}{729}t^4 + \tfrac{22929418240}{6561}t^3 + \tfrac{421145362496}{1594323}t^2 - \tfrac{8127423733760}{387420489}t + \tfrac{262144}{19683})x^{11} \\ & + (-11943936t^8 - 41545728t^7 - \tfrac{554957824}{9}t^6 - \tfrac{38802641152}{729}t^5 - \tfrac{1357119867008}{59049}t^4 \\ & + \tfrac{2783829240464}{4782969}t^3 + \tfrac{116400715008485}{43046721}t^2 + \tfrac{56796331333120}{1162261467}t - \tfrac{21901312}{1594323})x^{10} + \cdots\end{aligned}$$

where dots represents the terms of lower degree of x*. We get,* $\Delta_0(t) = \frac{16384}{9}t^3 + \frac{2097152}{729}t^2 + \frac{67108864}{59049}t = 0$ *and solving this equation, we get the bifurcation values* $t = 0\,,\ -\frac{64}{81}$.
Hence the set of critical values at infinity is $I(f) = \{0, -\frac{64}{81}\}$ *and* $\lambda^0(f) = 2$ *(since,* $\Delta_0(0) = \Delta_1(0) = 0$ *and* $\Delta_2(0) \neq 0$*)and* $\lambda^{-\frac{64}{81}}(f) = 2$ *(since* $\Delta_0(-\frac{64}{81}) = \Delta_1(-\frac{64}{81}) = 0$ *and* $\Delta_2(-\frac{64}{81}) \neq 0$*).*

3 On the degree of the discriminant

In this section we assume that the polynomial $f(x,y)$ is a primitive polynomial, that is there does not exist a polynomial $h(x,y) \in \mathbb{C}[x,y]$ such that $f(x,y) = g(h(x,y))$, where $g \in \mathbb{C}[t]$ and $\deg(g) \geq 2$. Equivalently, a primitive polynomial is a polynomial whose generic fiber F, i.e. the fiber of the fibration in the Introduction, is connected, see [7]. We also assume that the 0-level curve $\mathcal{C}' : f = 0$ is irreducible.

Theorem 3.1. *The discriminant of a plane affine curve* $\mathcal{C}' : f = 0$ *whose equation* f *is* $x - regular$ *satisfies the inequality*

$$\deg \Delta(x) \leq M := d^2 - d.$$

Equality holds if and only if the projective closure $\mathcal{C}$ *of* $\mathcal{C}'$ *is smooth along the line at infinity* L_∞ *and the intersection* $\mathcal{C} \cap L_\infty$ *is transverse or, equivalently, the top degree homogeneous component of* f *has no multiple factor. In this situation one has*

$$\deg_x \Delta(x,t) = M$$

for all $t \in \mathbb{C}$*. In particular* $I(f) = \emptyset$ *in this case.*

Proof. Let us suppose that the equality holds i.e. $\deg \Delta(x) = d^2 - d$, then by Proposition 2.6, we get $d(d-1) = d(d-2) + k - \sum_{p \in \mathcal{C}_\infty} \mu_p(\mathcal{C}) \Rightarrow k = d$. Hence, the projective closure is smooth along the line at infinity $\mathcal{L}_\infty$ and the intersection $\mathcal{C} \cap \mathcal{L}_\infty$ is transverse. Conversely, suppose that the curve is smooth and transversal along the line at infinity, then again by Proposition 2.6, we have the equality $\deg \Delta(x) = d^2 - d$. The $x-$regular polynomial f of degree d can be written as $f = F^d + F^{d-1} + \cdots + F^0$, where F^i is homogeneous component of degree i and its partial derivative is given as $\frac{\partial f}{\partial y} = dF^{d-1} + (d-1)F^{d-2} + \cdots + C$. Then y-discriminant has at most degree $M = d(d-1)$. Since the resultant of two forms equal to zero if and only if they have common component. Hence, $\deg \Delta(x) = d^2 - d$ if and only if the top degree homogeneous component of f has no multiple factors, hence the result follows.

Theorem 3.2. *The discriminant of an irreducible plane affine curve $\mathcal{C}' : f = 0$ whose equation f is $x - regular$ satisfies the inequality*

$$\deg \Delta(x) \geq m := d - 1.$$

Equality holds if and only if the curve $\mathcal{C}'$ is smooth and contractible. In this situation one has

$$\deg_x \Delta(x, t) = m$$

for all $t \in \mathbb{C}$. In particular $I(f) = \emptyset$ in this case and f is obtained from a linear form by composing with a biregular automorphism of $\mathbb{C}^2$.

Proof: Let $f(x,y) \in \mathbb{C}[x,y]$ be a reduced polynomial such that $\deg_y f = \deg f = d > 1$. Assume that the fiber $\mathcal{C}'_t = f^{-1}(t)$ is reduced and irreducible, then by Proposition 2.9, we have

$$\deg \Delta(x) \geq d - 1.$$

If the curve is smooth and contractible, then again by Proposition 2.9, we get the equality $\deg \Delta(x) = d - 1$. Conversely, suppose that $\deg \Delta(x) = d - 1$, then Proposition 2.9 implies that $b_1(\mathcal{C}'_t) + \mu(\mathcal{C}'_t) = 0$, which is only possibly when both equal to zero, hence the curve $\mathcal{C}'$ is smooth and contractible. The last claim follows by the celebrated Abhyankar-Moh Theorem, see [1]. □

Beyond these two results referring to the maximal possible value M and the minimal possible value m for the degree of $\Delta(x)$, one can say the following.

Theorem 3.3. *The discriminant of a plane affine curve $\mathcal{C}' : f = 0$ whose equation f is $x - regular$ satisfies the equality*

$$\deg \Delta(x) = M - 1$$

if and only if the projective closure $\mathcal{C}$ of $\mathcal{C}'$ is smooth along the line at infinity L_∞ and the intersection $\mathcal{C} \cap L_\infty$ is transverse except at exactly one point p_1 where L_∞ is a simple tangent to $\mathcal{C}$. In this situation one has

$$\deg_x \Delta(x, t) = M - 1$$

for all $t \in \mathbb{C}$*. In particular* $I(f) = \emptyset$ *in this case.*

Proof: Note that in this case $k = d - 1$ and this explains the first claims.

To explain the last claim, suppose that the singularity of $\mathcal{C}$ at a point at infinity (say at $p = (1 : 0 : 0)$, such that we may use y, z as local coordinates at p) is defined locally by a function germ $g(y, z) = 0$. Then, the same singular point, but this time on the deformed curve $\mathcal{C}_t$ is defined by

$$g_t(y, z) = g(y, z) + tz^d. \tag{3.1}$$

In our case $d > 1$ and hence, if $\mathcal{C}$ is smooth at p, then the same is true for $\mathcal{C}_t$, and all these curves have the same tangent at p. □

Theorem 3.4. *The discriminant of a plane affine curve* $\mathcal{C}' : f = 0$ *whose equation* f *is* $x-regular$ *satisfies the equality*

$$\deg \Delta(x) = M - 2$$

if and only if one of the two cases (A) and (B) below holds.

(A) the projective closure $\mathcal{C}$ *of* $\mathcal{C}'$ *is smooth along the line at infinity* L_∞ *and either*

(i) $d \geq 3$ *and the intersection* $\mathcal{C} \cap L_\infty$ *is transverse except at exactly one point* p_1 *where* L_∞ *is a simple inflectional tangent to* $\mathcal{C}$*, or*

(ii) $d \geq 4$ *and the intersection* $\mathcal{C} \cap L_\infty$ *is transverse except at exactly two points* p_1 *and* p_2 *such that* L_∞ *is a simple tangent to* $\mathcal{C}$ *at each point* p_1 *and* p_2*.*

(B) $d \geq 3$ *and the projective closure* $\mathcal{C}$ *of* $\mathcal{C}'$ *is smooth along the line at infinity* L_∞ *except at one point* p_1 *which is a node on* $\mathcal{C}$*. The intersection* $\mathcal{C} \cap L_\infty$ *is transverse, where transversality at the point* p_1 *means that* L_∞ *is not in the tangent cone of* $\mathcal{C}$ *at* p_1*.*

In this situation one has

$$\deg_x \Delta(x, t) = M - 2$$

for all $t \in \mathbb{C}$ *and hence* $I(f) = \emptyset$*.*

Proof: Set $\mu_\infty(\mathcal{C}) = \sum_{p \in \mathcal{C}_\infty} \mu_p(\mathcal{C})$. Using Proposition 2.6, it follows that in this case one can have either $k = d - 2$ and $\mu_\infty(\mathcal{C}) = 0$ (and this leads to the case (A), or $k = d - 1$ and $\mu_\infty(\mathcal{C}) = 1$ (and this leads to the case (B)). Details are left to the reader. The last claim follows from equation (3.1), which shows that all singularities $(\mathcal{C}_t, p_1)$ are nodes with the same tangent cone. □

Remark 3.5. *If* f *is not* x*-regular, let* f_s *be the polynomial obtained from* f *by setting* $X = x + sy$ *and* $Y = sx + y$*. Then* f_s *is* x*-regular for* $s \in \mathbb{C}^*$ *generic, and obviously one has for such an* s

$$\deg_x disc_y(f_s(x, y) - t) \geq \deg_x disc_y(f(x, y) - t) = \deg_x \Delta(x, t).$$

For instance, in the absence of the x-regularity assumption on f, Theorem 2.9 becomes

$$\deg_x \Delta(x,t) \leq d - 1 + b_1(\mathcal{C}'_t) + \mu(\mathcal{C}'_t).$$

This remark and above Theorems yield the following.

Corollary 3.6. *Assume that f is a primitive polynomial of degree $d \geq 3$ such that for some $t \in \mathbb{C}$ one has $\deg_x \Delta(x,t) \geq M - 2$, with $M = d^2 - d$. Then $I(f) = \emptyset$.*

Remark 2.8 shows that such a result fails when $M - 2$ is replaced by $M - 3$.

A different approach is to fix the degree d of the polynomial f and discuss the degree of the discriminant $\deg_x \Delta(x,t)$ in function of the coefficients of f. Here is one example of this approach.

Example 3.7. *(Case $d = 3$)*
A polynomial of degree three which is $x-$regular and without constant term can be normalized to the following form

$$f(x,y) = y^3 + (ax^2 + c)y + Ax^3 + Bx^2 + Cx.$$

(First we kill in the usual way the term in y^2, then we kill the term in x in the coefficient of y.) Then the y-discriminant of $f(x,y) - t$ is

$$\Delta(x,t) = (-4a^3 - 27A^2)x^6 - 54ABx^5 + (-54AC - 12a^2c - 27B^2)x^4 +$$

$$+(54At - 54BC)x^3 + (54Bt - 27C^2 - 12ac^2)x^2 + 54Ctx - 4c^3 - 27t^2.$$

In this case $M = 9 - 3 = 6$ and $m = 3 - 1 = 2$. So the (interesting) coefficients of x^6, x^5, x^4, x^3, x^2 are respectively $\Delta_0 = 4a^3 + 27A^2$, $\Delta_1 = AB$, $\Delta_2 = 18AC + 4a^2c + 9B^2$, $\Delta_3 = At - BC$, $\Delta_4 = 18Bt - 9C^2 - 4ac^2$. The condition $\Delta_0 = 0$ says precisely that we are not in the situation of Theorem 3.1, i.e. the homogeneous part of degree 3 has a multiple root. The condition $\Delta_1 = 0$ (added to the condition $\Delta_0 = 0$) leads to two possible cases.

Case(i) $A = 0$
Then $a = 0$ and if $B \neq 0$, then we are exactly in the case (A) (i) of Theorem 3.4 (since in this case there is exactly one point at infinity). On the other hand, if $B = 0$ as well, then $\Delta_2 = \Delta_3 = 0$ and $\Delta_4 = -9C^2$. The polynomial becomes $f(x,y) = y^3 + cy + Cx - t$. This polynomial is equivalent to a linear form as in Theorem 3.2 if $C \neq 0$. If $C = 0$, then the polynomial f is non-primitive, case excluded from our discussion.

Case(ii) $A \neq 0$
Then $a \neq 0,\ B = 0$. Then the polynomial becomes $f(x,y) = y^3 + ax^2y + Ax^3 + Cx + cy$. We know that the corresponding top form $f_3 = y^3 + ax^2y + Ax^3$ has a double root i.e. $f_3 = (y - ux)^2(y + 2ux)$ for some $u \in \mathbb{C}^*$. Comparing the coefficients we get $A = 2u^3,\ a = -3u^2$.

Then we have $\Delta_2 = 18u^3C + 18u^4c$. *Hence* $\Delta_2 \neq 0$ *if and only if* $u \neq \frac{-C}{c}$ *and this corresponds to the case* (B) *of Theorem 3.4. Indeed, there are now two points at infinity and an irreducible cubic curve cannot have a bitangent! (If* $c = 0$ *and* $C \neq 0$ *we are in the same situation).*

Now $\Delta_2 = 0$ *may occur either when* $C = c = 0$ *and then* f *is reducible and non-reduced (case excluded by our assumptions), or when* $c \neq 0$ *and* $u = \frac{-C}{c}$*, which leads again to a reducible polynomial*

$$f(x, y) = (y - ux)[(y - ux)(y + 2ux) + c].$$

After a coordinate change $X = y - ux$, $Y = y + 2ux$*, this is essentially the Broughton polynomial considered in Example 2.11*

4 Polar invariants and global γ^*-invariants of affine curves

In this section we use the above notation and discuss briefly some of the results of the papers [5] and [15] in the light of our results in the second section above. In these papers the results are obtained for arbitrary dimensional hypersurfaces using non-proper Morse theory or generalized Lefschetz pencils. However, in the case of plane curves our results are as complete as theirs and obtained by far less complicated techniques.

In [5] the authors consider affine polar curves $\Gamma_a(\mathcal{C}')$ for $a = (a_1, a_2) \in \mathbb{C}^2$ defined to be the union of the irreducible components of the curve

$$a_1 \frac{\partial f}{\partial x} + a_2 \frac{\partial f}{\partial y} = 0$$

which are not contained in the singular locus of f. It follows from Example 8 and Theorem 3' in [5] that if $\mathcal{C}'_t$ is smooth and irreducible, then the intersection multiplicity $(\mathcal{C}_t, \Gamma_a(\mathcal{C}'))$ is independent of a as long as $(a_1 : a_2 : 0) \notin \mathcal{C}_\infty$. Note that for an x-regular polynomial f, $a = (0 : 1)$ is such a good choice.

With this particular choice for a (assuming f to be x-regular) we clearly have

$$(\mathcal{C}_t, \Gamma_a(\mathcal{C}')) = \sum_{p \in \mathbb{C}^2} (f - t, \frac{\partial f}{\partial y})_p = \deg_x \Delta(x, t).$$

Hence the polar invariant of the affine curve $\mathcal{C}'_t$ introduced in Definition 2 in [5] is nothing else but the x-degree of the discriminant $\Delta(x, t)$. For a smooth curve $\mathcal{C}'_t$ this equality follows also from Proposition 5, (i) in [5] as well. Moreover, Theorem 3 in [5] implies that, for a smooth curve $\mathcal{C}'_t$, one has

$$\deg_x \Delta(x, t) = d - 1 + b_1(\mathcal{C}'_t).$$

This is just a special case of Proposition 2.9 above.

In [15] the author introduces the global γ^*-invariants for a hypersurface having only isolated singularities, see Definition 3.3. In the case of plane curves, one has

$$\gamma^1(\mathcal{C}'_t) = \deg_x \Delta(x, t) \text{ and } \gamma^0(\mathcal{C}'_t) = d.$$

With these equalities in mind, we see that Theorem 3.6, (iii) in [15] yields exactly Proposition 2.9. Moreover, the description of the critical values at infinity given in Corollary 2.7 is equivalent to Theorem 4.6 in [15].

5 Acknowledgement

I am indebted to the anonymous referee for his valuable comments and suggestions on my previous version of paper. Author also acknowledge the financial support from HEC and Abdus Salam School of Mathematical Sciences GCU, Lahore, Pakistan.

References

[1] S. S. Abhyankar, T. T. Moh: *Embeddings of the line in the plane*, J. Reine Angew. Math. 276(1975), 148–166.

[2] E. Artal Bartolo, P. Cassou-Noguès, I. Luengo Velasco: *On polynomials whose fibers are irreducible with no critical points*, Math. Ann. 299 (1994), 477-490.

[3] J. Briançon: *Tentative ratée au Jacobien par la voie du polygone de Newton*, preprint 1985, Université de Nice.

[4] S. A. Broughton: *Milnor numbers and the topology of polynomial hypersurfaces*, Invent.Math. 92(1988), 217–241.

[5] P. Cassou-Noguès, A. Dimca: *Topology of complex polynomials via polar curves*, Kodai Math. J. 22(1999),131–139.

[6] A. Dimca: Singularities and Topology of Hypersurfaces, Universitext, Springer, 1992.

[7] A. Dimca, L. Paunescu: *On the connectivity of complex affine hypersurfaces II*, Topology 39(2000) 1035-1043.

[8] A.H. Durfee: Five definitions of Critical Points at Infinity, Singularities: the Brieskorn Anniversary Volume, Progress in Mathematics 162 (1998), 345–360.

[9] J. Gwozdziewicz, A. Ploski: *Formulae for the singularities at infinity of plane curves,* Univ, Iagel. Acta Math. 39(2001), 109-123.

[10] H.V. Hà, D.T. Lê: *Sur la topologie des polynômes complexes*, Acta Math. Vietnamica 9(1984), 21–32.

[11] H.V. Hà, T.S. Phạm: *Remark on the equisingularity of families of affine plane curves*, Ann. Polon. Math. 68 (1998), no. 3, 275–280.

[12] T. Krasiǹski: *The level sets of polynomials in two variables and the Jacobian Conjecture*, Acta Univ. Łòdż. UŁ, Łòdż (1991)(in Polish).

[13] B. Teissier: *Cycles évanescents, section planes et conditions de Whitney*, Astérisque No.7-8 (1973).

[14] R. Thom: *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc. 75(1969), 249–312.

[15] M. Tibăr: *Asymptotic equisingularity and topology of complex hypersurfaces*, Int. Math. Research Notices 18(1998), 979–990.

[16] M. Tibăr: *Polynomials and vanishing cycles*, Cambridge Tracts in Math. 170, CUP, 2007.

[17] A. N. Varchenko: *Theorems on topological equisingularity of families of algebraic varieties and families of polynomial maps*, Izvestyia Akad. Nauk 36(1972), 957–1019.

[18] J.-L. Verdier: *Stratification de Whitney et Théorème de Bertini-Sard*, Invent. Math. 36(1976), 295–312.

Received: 05.02.2010
Revised: 08.01.2011
Accepted: 04.03.2011.

National University of Computer
and Emerging Sciences Lahore Campus,
B-Block, Faisal Town, Lahore,
Pakistan
E-mail: `zahid.raza@nu.edu.pk`