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An analytical technique to solve the BBM-BBM system

by
Pengzhan Huang

Abstract

In this paper, the homotopy analysis method is applied to obtain the solutions of BBM-BBM system. This analytic technique is valid for dealing with the nonlinearity and provides a convenient way of controlling the convergence region and rate of the series solution. The results obtained by the present method are compared with exact solutions, and it is seen that they are in excellent agreement.

Key Words: BBM-BBM system, Homotopy analysis method, Convergence analysis.
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1 Introduction

To describe the two-dimensional propagation of surface waves in a uniform horizontal channel filled with an irrotational, incompressible, inviscid fluid, the BBM-BBM system

$$N_t + W_x + (NW)_x - \frac{1}{6}N_{xxt} = 0,$$
$$W_t + N_x + WW_x - \frac{1}{6}W_{xxt} = 0, \tag{1}$$

was previously formulated by Bona and Chen in [9]. Dougalis et al. [12] analyzed three initial-boundary-value problems for the BBM-BBM system on a smooth plane domain. Bona and Chen [9] considered the BBM-BBM system with nonhomogeneous Dirichlet boundary conditions and they discretized this system using a fully discrete finite difference scheme of fourth-order accuracy in space and time. Chatzipantelidis [11] discretized the same system in the case of homogeneous Dirichlet boundary conditions using the standard Galerkin method in space and high order explicit multistep schemes in time.

In 1992, Liao [17] first employed the basic ideas of the homotopy [13] in topology to propose an analytic technique for nonlinear problems, namely the homotopy analysis method (HAM). Thereafter, the HAM has been developed [6-12] and has been widely applied in many subjects [13-28]. Different from perturbation techniques, the HAM is independent of any small/large physical parameters, and it contains the auxiliary parameter $\hbar$ (this parameter has been renamed to the convergence-control parameter [22] recently), which provides us with a simple way to adjust and control the convergence region of solution series.

The motivation of this paper is to employ the HAM to solve the BBM-BBM system. The rest of this paper is organized as follows. In secion 2, the notations and basic definitions of HAM have been introduced. In section 3, we extend the application of the HAM to construct approximate solutions for the BBM-BBM system. Moreover, the convergence analysis and computed experiments are presented. In the last section, we draw some conclusions.

2 Basic idea of homotopy analysis method

Consider the following nonlinear differential equation

$$\mathcal{N}(u(x,t)) = 0, \tag{2}$$

where $\mathcal{N}$ is a nonlinear operator, x and t denote the independent variables, $u(x,t)$ is an unknown function, respectively. For simplicity, we ignore all boundary and initial conditions, which can be treated without any difficulties. By means of generalizing the traditional homotopy method, Liao [20] constructs the zeroth-order deformation equation

$$(1-q)\mathcal{L}(\phi(x,t;q) - u_0(x,t)) = q\hbar H(x,t)\mathcal{N}(\phi(x,t;q)), \tag{3}$$

where $q \in [0,1]$ is the embedding (homotopy) parameter, $\hbar$ is the convergence-control parameter, $H(x,t)$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$, and $\phi(x,t;q)$ is an unknown function. It is obvious that when the embedding parameter $q = 0$ and $q = 1$, Eq. (3) becomes

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t), \tag{4}$$

respectively. Therefore, as q increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q, one has

$$\phi(x,t;q) = u_0(x,t) + \sum_{n=1}^{+\infty} u_n(x,t)q^n, \tag{5}$$

where

$$u_n(x,t) = \frac{1}{n!}\frac{\partial^n \phi(x,t;q)}{\partial q^n}\bigg|_{q=0}. \tag{6}$$

The convergence of the series (5) depends upon the convergence-control parameter $\hbar$. Supposing that $\hbar$ is chosen so properly that the series (5) is convergent at $q = 1$, we have, by means of (4), the solution series

$$u(x,t) = \phi(x,t;1) = u_0(x,t) + \sum_{n=1}^{+\infty} u_n(x,t), \tag{7}$$

which must be one of the solutions of the original equation, as proved by Liao [20].

Define the vector

$$\mathbf{u}_n = (u_0(x,t), u_1(x,t), \ldots, u_n(x,t)).$$

Differentiating the zeroth-order deformation equation (3) n times with respect to q and then setting $q = 0$ and finally dividing them by $n!$, we obtain the nth-order deformation equation

$$\mathcal{L}(u_n(x,t) - \delta_n u_{n-1}(x,t)) = \hbar H(x,t) R_n(\mathbf{u}_{n-1}), \tag{8}$$

where

$$R_n(\mathbf{u}_{n-1}) = \frac{1}{(n-1)!} \frac{\partial^{n-1} \mathcal{N}(\phi(x,t;q))}{\partial q^{n-1}}\bigg|_{q=0}, \tag{9}$$

and

$$\delta_n = \begin{cases} 0, & n = 1, \\ 1, & n \geq 2. \end{cases} \tag{10}$$

3 Solution of BBM-BBM system by homotopy analysis method

In this paper, we will solve (1) with the initial conditions

$$W(x,0) = 3k \operatorname{sech}^2 \frac{3}{\sqrt{10}}(x - x_0),$$

$$N(x,0) = \frac{15}{4}\left(-2 + \cosh\left(3\sqrt{\frac{2}{5}}(x - x_0)\right)\right) \operatorname{sech}^4\left(\frac{3(x - x_0)}{\sqrt{10}}\right). \tag{11}$$

The Eqs. (1) with the initial conditions (11) have the exact travelling-wave solutions [9]

$$W(x,t) = 3k \operatorname{sech}^2 \frac{3}{\sqrt{10}}(x - kt - x_0),$$

$$N(x,t) = \frac{15}{4}\left(-2 + \cosh\left(3\sqrt{\frac{2}{5}}(x - kt - x_0)\right)\right) \operatorname{sech}^4\left(\frac{3(x - kt - x_0)}{\sqrt{10}}\right), \tag{12}$$

where k and x_0 are constants.

In order to solve (1) by HAM, it is straightforward to choose the initial approximations

$$W_0(x,t) = W(x,0),$$
$$N_0(x,t) = N(x,0), \tag{13}$$

and the auxiliary linear operators,

$$\mathcal{L}_N(\phi(x,t;q)) = \frac{\partial \phi(x,t;q)}{\partial t},$$
$$\mathcal{L}_W(\psi(x,t;q)) = \frac{\partial \psi(x,t;q)}{\partial t}, \tag{14}$$

with the property

$$\mathcal{L}_N(c_1) = 0, \quad \mathcal{L}_W(c_2) = 0, \tag{15}$$

where c_1 and c_2 are the integration constants. For simplicity, using (1), we define the nonlinear operators as follows,

$$\begin{aligned}\mathcal{N}_N(\phi(x,t;q),\psi(x,t;q)) &= \frac{\partial \phi(x,t;q)}{\partial t} + \frac{\partial \psi(x,t;q)}{\partial x} + \phi(x,t;q)\frac{\partial \psi(x,t;q)}{\partial x} \\ &\quad + \psi(x,t;q)\frac{\partial \phi(x,t;q)}{\partial x} - \frac{1}{6}\frac{\partial^3 \phi(x,t;q)}{\partial x^2 \partial t},\end{aligned}$$

$$\begin{aligned}\mathcal{N}_W(\phi(x,t;q),\psi(x,t;q)) &= \frac{\partial \psi(x,t;q)}{\partial t} + \frac{\partial \phi(x,t;q)}{\partial x} + \psi(x,t;q)\frac{\partial \psi(x,t;q)}{\partial x} \\ &\quad - \frac{1}{6}\frac{\partial^3 \psi(x,t;q)}{\partial x^2 \partial t}.\end{aligned} \tag{16}$$

Let $\hbar_N$ and $\hbar_W$ denote the convergence-control parameters, $H_N(x,t)$ and $H_W(x,t)$ denote the auxiliary functions, respectively.

Using the above definition, we construct the zero-order deformation equations

$$(1-q)\mathcal{L}_N(\phi(x,t;q) - N_0(x,t)) = q\hbar_N H_N(x,t)\mathcal{N}_N(\phi(x,t;q),\psi(x,t;q)),$$
$$(1-q)\mathcal{L}_W(\psi(x,t;q) - W_0(x,t)) = q\hbar_W H_W(x,t)\mathcal{N}_W(\phi(x,t;q),\psi(x,t;q)). \tag{17}$$

Obviously, for $q = 0$ and $q = 1$,

$$\phi(x,t;0) = N_0(x,t), \qquad \phi(x,t;1) = N(x,t),$$
$$\psi(x,t;0) = W_0(x,t), \qquad \psi(x,t;1) = W(x,t). \tag{18}$$

For simplicity, we suppose $\hbar_N = \hbar_W = \hbar$ and $H_N(x,t) = H_W(x,t) = 1$.

Differentiating equations (17) n times with respect to q and dividing them by $n!$, we have the nth-order deformation equations

$$\mathcal{L}_N(N_n(x,t) - \delta_n N_{n-1}(x,t)) = \hbar R_{N,n}(\mathbf{N}_{n-1}, \mathbf{W}_{n-1}),$$

$$\mathcal{L}_W(W_n(x,t) - \delta_n W_{n-1}(x,t)) = \hbar R_{W,n}(\mathbf{N}_{n-1}, \mathbf{W}_{n-1}), \tag{19}$$

subject to initial conditions

$$\begin{aligned} N_n(x,0) &= 0, \\ W_n(x,0) &= 0, \end{aligned} \tag{20}$$

where

$$\begin{aligned} R_{N,n}(\mathbf{N}_{n-1}, \mathbf{W}_{n-1}) &= \frac{\partial N_{n-1}(x,t)}{\partial t} + \frac{\partial W_{n-1}(x,t)}{\partial x} - \frac{1}{6}\frac{\partial^3 N_{n-1}(x,t)}{\partial x^2 \partial t} \\ &+ \sum_{j=0}^{n-1}\left(\frac{\partial N_j(x,t)}{\partial x} W_{n-1-j}(x,t) + N_j(x,t)\frac{\partial W_{n-1-j}(x,t)}{\partial x}\right), \end{aligned}$$

$$\begin{aligned} R_{W,n}(\mathbf{N}_{n-1}, \mathbf{W}_{n-1}) &= \frac{\partial W_{n-1}(x,t)}{\partial t} + \frac{\partial N_{n-1}(x,t)}{\partial x} - \frac{1}{6}\frac{\partial^3 W_{n-1}(x,t)}{\partial x^2 \partial t} \\ &+ \sum_{j=0}^{n-1}\left(\frac{\partial W_j(x,t)}{\partial x} W_{n-1-j}(x,t)\right), \end{aligned} \tag{21}$$

$\mathbf{N}_n = (N_0(x,t), N_1(x,t), \ldots, N_n(x,t))$, $\mathbf{W}_n = (W_0(x,t), W_1(x,t), \ldots, W_n(x,t))$, and δ_n is defined by (10). The solutions of the nth-order deformation equations (19) for $n \geq 1$ become

$$\begin{aligned} N_n(x,t) &= \delta_n N_{n-1}(x,t) + \hbar \mathcal{L}_N^{-1}(R_{N,n}(\mathbf{N}_{n-1}, \mathbf{W}_{n-1})), \\ W_n(x,t) &= \delta_n W_{n-1}(x,t) + \hbar \mathcal{L}_W^{-1}(R_{W,n}(\mathbf{N}_{n-1}, \mathbf{W}_{n-1})). \end{aligned} \tag{22}$$

According to Eqs. (13)-(22) and taking $k = 5/2$ and $x_0 = 20$ in (11), we obtain

$$\begin{aligned} N_1(x,t) &= \hbar t \,\mathrm{sech}^2(0.6x-12)\Big(-9\tanh(0.6x-12) + 16.875\sinh(0.6x-12) \\ &\quad \mathrm{sech}^4(0.6x-12) + \big(18 - 9\cosh(0.6x-12)\big) \\ &\quad \mathrm{sech}^4(0.6x-12)\tanh(0.6x-12)\Big), \end{aligned}$$

$$\begin{aligned} W_1(x,t) &= \hbar t \,\mathrm{sech}^4(0.6x-12)\Big(2.25\sinh(0.6x-12) - \tanh(0.6x-12) \\ &\quad \big(49.5 + 9\cosh(0.6x-12)\big)\Big). \end{aligned} \tag{23}$$

$N_n(x)$ and $W_n(x)$ $(n = 2, 3, \ldots)$ can be calculated similarly. Then the series solutions obtained by the HAM can be written in the form

$$N(x,t) = N_0(x,t) + N_1(x,t) + N_2(x,t) + \cdots, \tag{24}$$

$$W(x,t) = W_0(x,t) + W_1(x,t) + W_2(x,t) + \cdots. \tag{25}$$

Theorem 3.1. *If the series*

$$N(x,t) = N_0(x,t) + \sum_{n=1}^{+\infty} N_n(x,t)$$

and

$$W(x,t) = W_0(x,t) + \sum_{n=1}^{+\infty} W_n(x,t)$$

converge, where $N_n(x,t)$ and $W_n(x,t)$ are governed by the Eqs. (22) *under the definitions* (21)*, then they must be the exact solutions of Eqs.* (1) *with initial conditions* (11)*.*

Proof: If the series (24) and (25) are convergent, then we can write

$$s_1 = \sum_{n=0}^{+\infty} N_n, \quad s_2 = \sum_{n=0}^{+\infty} W_n.$$

Thus,

$$\lim_{m\to+\infty} N_m = 0 \text{ and } \lim_{m\to+\infty} W_m = 0$$

are hold. Due to (14) and (19), we have

$$\begin{aligned} &\hbar \sum_{n=1}^{+\infty} R_{N,n}(\mathbf{N}_{n-1}, \mathbf{W}_{n-1}) = \lim_{m\to+\infty} \sum_{n=1}^{m} \mathcal{L}_N \Big(N_n(x,t) - \delta_n N_{n-1}(x,t) \Big) \\ = \ & \mathcal{L}_N \left(\lim_{m\to+\infty} \sum_{n=1}^{m} \Big(N_n(x,t) - \delta_n N_{n-1}(x,t) \Big) \right) \\ = \ & \mathcal{L}_N \left(\lim_{m\to+\infty} N_m(x,t) \right) = 0, \end{aligned}$$

and

$$\begin{aligned} &\hbar \sum_{n=1}^{+\infty} R_{W,n}(\mathbf{N}_{n-1}, \mathbf{W}_{n-1}) = \lim_{m\to+\infty} \sum_{n=1}^{m} \mathcal{L}_W \Big(W_n(x,t) - \delta_n W_{n-1}(x,t) \Big) \\ = \ & \mathcal{L}_W \left(\lim_{m\to+\infty} \sum_{n=1}^{m} \Big(W_n(x,t) - \delta_n W_{n-1}(x,t) \Big) \right) \\ = \ & \mathcal{L}_W \left(\lim_{m\to+\infty} W_m(x,t) \right) = 0. \end{aligned}$$

Since $\hbar \neq 0$, we arrive at

$$\sum_{n=1}^{+\infty} R_{N,n}(\mathbf{N}_{n-1}, \mathbf{W}_{n-1}) = 0,$$

and

$$\sum_{n=1}^{+\infty} R_{W,n}(\mathbf{N}_{n-1}, \mathbf{W}_{n-1}) = 0.$$

Substituting (21) in the above expressions, we have

$$\begin{aligned}
& \sum_{n=1}^{+\infty} R_{N,n}(\mathbf{N}_{n-1}, \mathbf{W}_{n-1}) \\
= & \sum_{n=1}^{+\infty} \left((N_{n-1})_t + (W_{n-1})_x + \sum_{j=0}^{n-1} \left(N_j W_{n-1-j} \right)_x - \frac{1}{6}(N_{n-1})_{xxt} \right) \\
= & (s_1)_t + (s_2)_x + (s_1 s_2)_x - \frac{1}{6}(s_1)_{xxt} \\
= & 0,
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=1}^{+\infty} R_{W,n}(\mathbf{N}_{n-1}, \mathbf{W}_{n-1}) \\
= & \sum_{n=1}^{+\infty} \left((W_{n-1})_t + (N_{n-1})_x + \sum_{j=0}^{n-1} W_j (W_{n-1-j})_x - \frac{1}{6}(W_{n-1})_{xxt} \right) \\
= & (s_2)_t + (s_1)_x + s_2 (s_2)_x - \frac{1}{6}(s_2)_{xxt} \\
= & 0.
\end{aligned} \tag{26}$$

Moreover, due to initial conditions (13) and (20), we get

$$s_1(x,0) = \sum_{i=0}^{+\infty} N_i(x,0) = N_0(x,0) = N(x,0),$$

and

$$s_2(x,0) = \sum_{i=0}^{+\infty} W_i(x,0) = W_0(x,0) = W(x,0).$$

Therefore, $s_1(x,t)$ and $s_2(x,t)$ satisfy Eqs. (1) and (11), and they are the exact solutions of Eqs. (1) with initial conditions (11). □

As pointed out by Liao [20], the convergence and rate of approximation for the HAM solution strongly depends on the values of the convergence-control parameter $\hbar$. The validity of the method is based on such an assumption that the series converges at $q = 1$. It is the convergence-control parameter $\hbar$ which ensures that this assumption can be satisfied. Generally, by means of the $\hbar$-curve, it is straightforward to choose a proper value of $\hbar$ to ensure that the solution series is convergent.

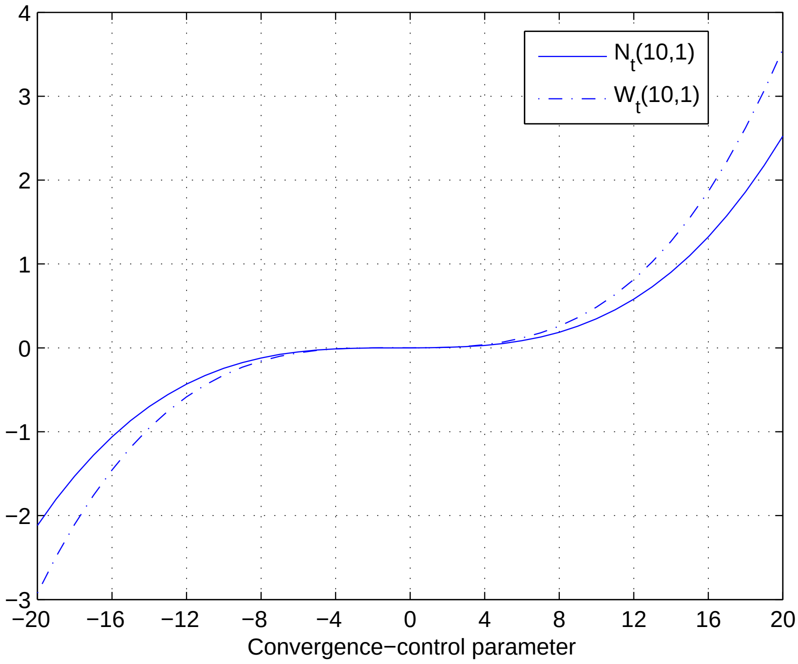

Fig. 1. The $\hbar$-curves of $N_t(10,1)$ and $W_t(10,1)$ obtained by the 3th-order approximation of the HAM with $k=5/2$ and $x_0=20$.

To find the range of admissible values of $\hbar$, $\hbar$-curves of $N_t(10,1)$ and $W_t(10,1)$ obtain by the 3th-order approximation of the HAM with $k=5/2$ and $x_0=20$ for the BBM-BBM system are plotted in Fig. 1. It is observed that the series solutions (24) and (25) converge to the exact solutions of (1), whenever $-4 \leq \hbar \leq 2$.

Table 1
Absolute errors for $N(x,t)$ obtained by the 3th-order approximate solution of the HAM for $\hbar=-1$ with $k=5/2$ and $x_0=20$

	t				
x	0.1	1.0	5.0	10.0	20.0
0.1	1.9920E−10	2.3561E−9	5.5911E−8	3.8764E−7	2.9818E−6
0.5	3.2192E−10	3.8076E−9	9.0356E−8	6.2645E−7	4.8188E−6
1.0	5.8657E−10	6.9378E−9	1.6464E−7	1.1415E−6	8.7805E−6
5.0	7.1254E−8	8.4279E−7	2.0001E−5	1.3868E−4	1.0668E−3

In Tables 1 and 2, the absolute errors for $N(x,t)$ and $W(x,t)$ obtained by the 3th-order approximation of the HAM for $\hbar=-1$ with $k=5/2$ and $x_0=20$ are

Table 2
Absolute errors for $W(x,t)$ obtained by the 3th-order approximate solution of the HAM for $\hbar = -1$ with $k = 5/2$ and $x_0 = 20$

	t				
x	0.1	1.0	5.0	10.0	20.0
0.1	3.4456E−10	2.5740E−9	3.5293E−8	1.3734E−7	5.4549E−7
0.5	5.5684E−10	4.1598E−9	5.7036E−8	2.2195E−7	8.8153E−7
1.0	1.0146E−9	7.5796E−9	1.0393E−7	4.0440E−7	1.6062E−6
5.0	1.2327E−7	9.2078E−7	1.2621E−5	4.9087E−5	1.9476E−4

shown, respectively. From these tables, we already noted that the solutions of the HAM are in excellent agreement with the exact solutions.

4 Conclusions

In this paper, the HAM is applied to obtain the solutions of the BBM-BBM system. The HAM provides us with a flexible way to control the convergence of approximation series. Moreover, the convergence analysis and computed experiments are presented. The results verify the validity and potential of the HAM for the studies of nonlinear system. Therefore, it is suggested to use the HAM to get the solutions of nonlinear problems in science and engineering effectively.

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College of Mathematics and System Sciences,
Xinjiang University,
Urumqi 830046, P.R. China
E-mail: `hpzh007@yahoo.cn`