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On the location of zeros of a polynomial

by
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Abstract

Observing that for the zeros of polynomial $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$, Cauchy's bound

$$|z| < 1 + A, \qquad A = \max_{0 \le j \le n-1} |a_j|$$

does not reflect the fact that for $A \to 0$, all zeros approach the origin $z = 0$, Boese and Luther suggested the proper bound

$$|z| < R',$$

$$R' = \begin{cases} \left\{A(1 - nA)/(1 - (nA)^{1/n})\right\}^{1/n}, A \le 1/n, \\ \min\left\{(1 + A)(1 - (A/((1 + A)^{n+1} - nA))), \right. \\ \left. \qquad 1 + 2((nA - 1)/(n + 1))\right\}, A \ge 1/n. \end{cases}$$

We have obtained a generalization of Boese and Luther's bound by considering the polynomial

$$z^n + a_p z^{n-p} + a_{p+1} z^{n-p-1} + \ldots + a_n, 1 \le p < n$$

and have also suggested certain related results.

Key Words: Cauchy's bound, angle-independent bound, angle-independent zero free bound.
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1 Introduction and statement of results

Concerning the zeros of a polynomial of degree n we have the following well known result due to Cauchy [2].

Theorem A. *All zeros of the polynomial*

$$f(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_0, \tag{1.1}$$

with

$$A = \max_{0 \le j \le n-1} |a_j|, \tag{1.2}$$

lie in the disc

$$|z| < 1 + A.$$

Boese and Luther [1] observed that Cauchy's bound does not reflect the fact that for

$$A \to 0,$$

all zeros approach

$$z = 0.$$

Accordingly they obtained the appropriate bound and proved

Theorem B. *All zeros of the polynomial* $f(z)$ *lie in the disc*

$$|z| < R', \tag{1.3}$$

where

$$R' = \begin{cases} \left\{A(1 - nA)/(1 - (nA)^{1/n})\right\}^{1/n}, \ A \le 1/n, \\ \min\left\{(1 + A)(1 - (A/((1 + A)^{n+1} - nA))),\right. \\ \left. 1 + 2((nA - 1)/(n + 1))\right\}, \ A \ge 1/n. \end{cases}$$

In this paper we have firstly obtained a generalization of Theorem B. More precisely we have proved

Theorem 1. *All the zeros of the polynomial*

$$q(z) = z^n + a_p z^{n-p} + a_{p+1} z^{n-p-1} + \ldots + a_n, 1 \le p < n, \tag{1.4}$$

with

$$D = \max_{p \le t \le n} |a_t| \tag{1.5}$$

and

$$R = \begin{cases} \min(\{1 + 2((D(n - p + 1) - 1)/(p + n))\}, \\ [D\left\{(((n - p + 1)D)^{(n-p+1)/n} - 1)/(((n - p + 1)D)^{1/n} - 1)\right\}]^{1/n}), & (n - p + 1)D < 1 \\ \min[\left\{D + ((n - p + 1)D)^{p-1}\right\}^{1/p} \\ \left\{1 - D(p(D + ((n - p + 1)D)^{p-1})^{(n+1)/p} - (n - p + 1)D)^{-1}\right\}, \\ \{1 + 2((D(n - p + 1) - 1)/(p + n))\}] & , (n - p + 1)D > 1, \\ 1 & , (n - p + 1)D = 1, \end{cases}$$

lie in the disc

$$|z| \le R.$$

***Remark* 1.** *In many cases Theorem 1 gives better bound than that obtained by Theorem B (e.g. for the polynomial*

$$q(z) = z^5 + a_4 z + a_5;\ |a_4| = 1, |a_5| = 4; p = 4,$$

all the zeros lie in

$$\begin{aligned} \textit{(i)}\ |z| &< 4.9985, \textit{ (by Theorem B)},\\ \textit{(ii)}\ |z| &< 2.6, \textit{ (by Theorem 1)}. \end{aligned}$$

By applying Theorem 1 on the ray θ = constant, we obtain

***Corollary* 1.** *Let*

$$\cos_+ t = \max(0, \cos t),\ \textit{for real } t \tag{1.6}$$

and

$$-a_k = A_k e^{i\alpha_k}, p \leq k \leq n. \tag{1.7}$$

Then the angle-independent bound $R(D)$, for all zeros of $q(z)$, of Theorem 1 can be replaced by $R(D(\theta))$, where

$$D(\theta) = \max_{p \leq k \leq n} \{A_k \cos_+(\alpha_k - k\theta)\}, \theta \in [0, 2\pi). \tag{1.8}$$

In the same paper [1] Boese and Luther obtained a zero free region around origin also, for an n^{th} degree polynomial and proved

Theorem C. *The polynomial*

$$f(z) := a_0 + a_1 z + \ldots + a_n z^n,\ a_0 a_n \neq 0, n \geq 2 \tag{1.9}$$

is zero-free in the open disk

$$|z| < R_0, \tag{1.10}$$

$$R_0 := \begin{cases} A/(1 + A - {\rho_0}^n), & 0 \leq A \leq n, \quad (1.11)\\ [1 + A - A(1 + A - (A/\rho_0))^{-1/n}]^{1/n}, & n \leq A, \quad (1.12) \end{cases}$$

$$\rho_0 := \begin{cases} A/(1 + A), & 0 \leq A \leq A_0, \quad (1.13)\\ \max\left\{A/(1 + A), 1 + \sqrt{D} - (3/(2n - 2))\right\}, & A_0 \leq A \leq n, \quad (1.14)\\ \left[A \max\left\{1/n, 1 - (n/A)^{1/n}\right\}\right]^{1/n}, & n \leq A, \quad (1.15) \end{cases}$$

$$\begin{aligned} A_0 &:= n(5/8 - (3/(4n - 4))),\\ A &:= |a_0|/\max\{|a_1|, |a_2|, \ldots, |a_n|\},\\ D &:= (9/(4(n - 1)^2)) - (6(1 - (A/n))/(n^2 - 1)). \end{aligned}$$

In this paper we have obtained a partial generalization of Theorem C also, by proving

Theorem 2. *The polynomial*

$$s(z) = a_n + a_{n-1}z + a_{n-2}z^{n-2} + \ldots + a_p z^{n-p} + a_0 z^n, a_0 a_n \neq 0; 1 \leq p < n, \quad (1.16)$$

with

$$E = |a_n|/\max(|a_{n-1}|, |a_{n-2}|, \ldots, |a_p|, |a_0|), \quad (1.17)$$

$$\rho = \begin{cases} \max(E/(n-p+1), E/(1+E)), & E < n-p+1 \quad (1.18) \\ (E\max(1/(n-p+1), 1/[1+((E/(n-p+1))^{-p/n}/ \\ \ (1-(E/(n-p+1))^{-1/n}))]))^{1/n}, & E > n-p+1 \quad (1.19) \end{cases}$$

and

$$R = \begin{cases} E/(1+E-\rho^{n-p}+\rho^{n-1}-\rho^n), & E < n-p+1, \\ \left\{ \begin{array}{l} 1+E-(1+E-\rho^{n-p}+\rho^{n-1}-(E/\rho))^{(n-p)/n}+ \\ (1+E-\rho^{n-p}+\rho^{n-1}-(E/\rho))^{(n-1)/n} \quad - \\ E(1+E-\rho^{n-p}+\rho^{n-1}-(E/\rho))^{-1/n} \end{array} \right\}^{1/n}, & E > n-p+1, \\ 1, & E = n-p+1, \end{cases}$$

is zero free in the disc

$$|z| < R.$$

***Remark* 2.** *Theorem 2, with the possibility*

$$E \geq n-p+1 \quad \& \quad p = 1$$

reduces to the corresponding possibility of Theroem C. But Theorem 2, with the possibility

$$E < n-p+1 \quad \& \quad p = 1$$

reduces partly to the corresponding possibility of Theorem C, (to be precise, only for $0 \leq A \leq A_0$*, (and to be more precise, Theorem 2 is a refinement of Theroem C, for* $0 \leq A \leq A_0$*)).*

***Remark* 3.** *To be more precise, for the possibility*

$$E \neq n-p+1,$$

the zero free disc is

$$|z| \leq R.$$

By applying Theorem 2 on the ray θ *= constant, we obtain*

***Corollary* 2.** *Let*

$$-\frac{a_k}{a_n} = A_k e^{i\alpha_k}, k = 0, p, p+1, \ldots, n-1; 1 \leq p < n\ a_0 a_n \neq 0, \tag{1.20}$$

$$\cos_+ t = \max(0, \cos t), \ \textit{for real } t \tag{1.21}$$

and

$$E(\theta) = 1/\max\Big\{A_0 \cos_+(\alpha_0 + n\theta), A_p \cos_+(\alpha_p + (n-p)\theta)\ , \\ A_{p+1}\cos_+(\alpha_{p+1} + (n-p-1)\theta), \ldots, A_{n-1}\cos_+(\alpha_{n-1} + \theta)\Big\}. \tag{1.22}$$

Then the angle-independent zero free bound $R(E)$ *for* $s(z)$, *of Theorem 2 can be replaced by* $R(E(\theta))$.

Finally, in this paper we have obtained a result, which again gives a zero free region around origin for an n^{th} degree polynomial, but better than those obtainable by many other known results. More precisely we have proved

Theorem 3. *The polynomial*

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n, \qquad a_0 a_n \neq 0, n \geq 2, \tag{1.23}$$

with

$$|a_1/a_0| = A_1, \tag{1.24}$$

$$\max_{2\leq j\leq n} |a_j/a_0| = \delta, \tag{1.25}$$

$$A_1/\left\{1 - (1/(1+A_1)^{n-1})\right\} = A_0, 0 < A_1, \tag{1.26}$$

$$G(A_1,\delta) = \begin{cases} [(A_1+1) - \sqrt{(A_1-1)^2 + 4\delta}]/(2(A_1-\delta)), & A_1 \neq \delta, \quad (1.27) \\ 1/(A_1+1), & A_1 = \delta, \quad (1.28) \end{cases}$$

$$\rho = \begin{cases} \max\{1/((n-1)\delta + A_1), G(A_1,\delta)\}, & (n-1)\delta + A_1 > 1\& \quad (1.29) \\ & \{A_1 \geq \delta \text{ or } \delta > \max(A_1, A_0) \text{ or } A_1 = 0\} \\ \max\Big\{((\delta - A_1)/(n\delta))^{1/(n-1)}, & \\ 1/((n-1)\delta + A_1), G(A_1,\delta)\}, & (n-1)\delta + A_1 > 1\ \&\ A_1 < \delta \leq A_0, \quad (1.30) \\ \max\Big[1/((n-1)\delta + A_1)^{1/n}, & \\ 1/(\delta\{1 - ((n-1)\delta + A_1)^{1/n}\}^{-1} + & \\ (A_1 - \delta)\{(n-1)\delta + A_1\}^{(n-1)/n})^{1/n}\Big] & ,(n-1)\delta + A_1 < 1 \quad (1.31) \end{cases}$$

and

$$R = \begin{cases} (1 + A_1 + (\delta - A_1)\rho - \delta\rho^n)^{-1}, & (n-1)\delta + A_1 > 1\ \& \\ & \{A_1 \ge \delta \text{ or } A_1 < \delta \le A_0\} \\ \rho, & (n-1)\delta + A_1 > 1\ \& \\ & \{\delta > \max(A_1, A_0) \text{ or } A_1 = 0\} \\ \{((1 + A_1)/\delta) - (1/(\delta\rho)) + (1 - (A_1/\delta))\rho\}^{1/n}, & (n-1)\delta + A_1 < 1\ \&\ A_1 \le \delta, \\ \rho, & (n-1)\delta + A_1 < 1\ \&\ A_1 > \delta \\ 1, & (n-1)\delta + A_1 = 1, \end{cases}$$

is zero free in the disc

$$|z| < R.$$

***Remark* 4.** *The polynomial*

$$p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3; |a_0| = 1, |a_1| = 6, |a_2| = 12, |a_3| = 8,$$

is zero free in

$$\begin{aligned} (i)\ |z| &< .077, (\textit{by Theorem C}), \\ (ii)\ |z| &< .077, (\textit{by [3, Exercise no. 2, p. 126]}), \\ (iii)\ |z| &< .125, (\textit{by Theorem 3}). \end{aligned}$$

***Remark* 5.** *To be more precise, for the possibility*

$$(n-1)\delta + A_1 > 1,$$

the zero free disc is

$$|z| \le R.$$

By applying Theorem 3 on the ray

$$\theta = \text{ constant},$$

we obtain

***Corollary* 3.** *Let*

$$\frac{-a_k}{a_0} = B_k e^{i\alpha_k}, k = 1, 2, \ldots, n \quad \& \quad a_0 a_n \ne 0, \tag{1.32}$$

$$\cos_+ t = \max(0, \cos t), \textit{ for real } t, \tag{1.33}$$

$$A_1(\theta) = B_1 \cos_+(\alpha_1 + \theta) \tag{1.34}$$

and

$$\delta(\theta) = \max_{2 \le j \le n} B_j cos_+(\alpha_j + j\theta). \tag{1.35}$$

Then the angle-independent zero free bound $R(A_1, \delta)$ for $p(z)$, of Theorem 3 can be replaced by $R(A_1(\theta), \delta(\theta))$, except when $\delta(\theta) = 0$, in which case $R(A_1, \delta)$ will be replaced by $1/A_1(\theta)$.

2 Proofs of the theorems

Proof of Theorem 1. For each zero

$$z(\neq 0) = re^{i\theta}, \tag{2.1}$$

of $q(z)$, we have

$$-z^n = a_p z^{n-p} + a_{p+1} z^{n-p-1} + \ldots + a_n, \text{(by (1.4))},$$

and therefore

$$r^n \leq D(r^{n-p} + r^{n-p-1} + \ldots + 1), \text{(by (2.1) and (1.5))}, \tag{2.2}$$

which implies that

$$D > 0.$$

Further let

$$\phi(r) = r^n - D(r^{n-p} + r^{n-p-1} + \ldots + 1). \tag{2.3}$$

Then the equation

$$\phi(r) = 0 \tag{2.4}$$

has a unique positive root β, with

$$\beta = 1, \text{ if } (n-p+1)D = 1, \tag{2.5}$$

$$\beta < 1, \text{ if } (n-p+1)D < 1, \tag{2.6}$$

$$\beta > 1, \text{ if } (n-p+1)D > 1, \tag{2.7}$$

thereby helping us to write

$$\phi(r) = (r-\beta)\psi(r), \tag{2.8}$$

with

$$\psi(r) > 0, r > 0. \tag{2.9}$$

Now by (2.2), (2.3), (2.8) and (2.9) we can say that

$$r \leq \beta. \tag{2.10}$$

But as β is a root of (2.4), we also have

$$\beta^n = D(\beta^{n-p} + \beta^{n-p-1} + \ldots + \beta + 1), \text{(by (2.3))}. \tag{2.11}$$

Therefore if

$$(n-p+1)D < 1$$

then by using (2.6) in (2.11), we get

$$\beta < \{D(n-p+1)\}^{1/n},$$

which on being used in (2.11), gives us

$$\beta^n < D\left\{((n-p+1)D)^{(n-p)/n} + ((n-p+1)D)^{(n-p-1)/n} + \ldots + 1\right\},$$

i.e.

$$\beta < \left[D\frac{((n-p+1)D)^{(n-p+1)/n} - 1}{((n-p+1)D)^{1/n} - 1}\right]^{1/n}. \qquad (2.12)$$

Now we assume that

$$D(n-p+1) > 1.$$

Therefore on using (2.7) in (2.11), we get firstly

$$\beta > \{D(n-p+1)\}^{1/n} > \{D(n-p+1)\}^{1/(n+1)},$$

which implies

$$\beta = \{D(n-p+1)\}^{1/(n+1)}\, s, \text{ for certain } s(>1) \qquad (2.13)$$

and secondly

$$\beta^n < D(\beta^{n-p} + \ldots + \beta^{n-p}),$$

i.e.

$$\beta \le \beta^p < (n-p+1)D. \qquad (2.14)$$

Further by (2.11) we get

$$\begin{aligned} \beta^{n+1} + D &= (D + \beta^{p-1})\beta^{n-p+1}, \\ &< \left\{D + ((n-p+1)D)^{p-1}\right\}\beta^{n-p+1}, \text{(by (2.14))}, \end{aligned}$$

which by (2.13), implies

$$\begin{aligned} s^p + (s^{-n+p-1}/(n-p+1)) &< \left\{D + ((n-p+1)D)^{p-1}\right\}((n-p+1)D)^{-(p/(n+1))}, \\ &= B, \text{ (say)}. \end{aligned} \qquad (2.15)$$

The function

$$\psi(s) = s^p + (s^{-n+p-1}/(n-p+1)), s > 1 \qquad (2.16)$$

has positive and strictly increasing derivative and as

$$s < B^{1/p}, \text{(by (2.15))},$$

we have

$$\psi(B^{1/p}) - \psi(s) < (B^{1/p} - s)\psi'(B^{1/p}),$$

i.e.

$$\begin{aligned} (s - B^{1/p})\psi'(B^{1/p}) + \psi(B^{1/p}) &< \psi(s), \\ &< B, \text{(by (2.16) and (2.15))}, \end{aligned}$$

i.e.

$$s < B^{1/p}\left[1 - \left\{(pB^{(n+1)/p} - 1)^{-1}/(n-p+1)\right\}\right], \text{(by (2.16))},$$

which by (2.13) and (2.15) implies that

$$\beta < \{D+((n-p+1)D)^{p-1}\}^{1/p}\{1-D(p(D+(n-p+1)D)^{(n+1)/p}-(n-p+1)D)^{-1}\}. \quad (2.17)$$

For the function

$$g(r) = 1/(r^{-p} + r^{-p-1} + \ldots + r^{-n}), r > 0, \quad (2.18)$$

we have

$$\begin{aligned} g^{'}(r) &= (pr^{-p-1} + (p+1)r^{-p-2} + \ldots + nr^{-n-1})/(r^{-p} + r^{-p-1} + \ldots + r^{-n})^2, \\ &\quad r > 0, && (2.19) \\ &> 0, (\text{ and strictly increasing }), r > 0 && (2.20) \end{aligned}$$

and therefore

$$g(s) + (r-s)g^{'}(s) < g(r); r \neq s, r > 0 \ \&\ s > 0,$$

i.e

$$\begin{aligned} g(s) + (r-s)g^{'}(s) &\leq g(r), r > 0 \ \&\ s > 0, \\ &\leq D, \text{ (by (2.18) and (2.2))}. && (2.21) \end{aligned}$$

By taking

$$s = 1,$$

in (2.21) and using (2.18), (2.19) and (2.20), we get

$$r \leq [1 + 2\{(D(n-p+1)-1)/(p+n)\}]$$

and now, on combining (2.10) with (2.12), (2.17) and (2.5), Theorem 1 follows.

Proof of Corollary 1. For each zero

$$z(\neq 0) = re^{i\theta}, \quad (2.22)$$

of $q(z)$, we have

$$\begin{aligned} r^n &\leq \sum_{k=p}^{n} A_k r^{n-k} \cos_+(\alpha_k - k\theta), \text{(by (1.7), (2.22) and (1.6))}, \\ &\leq D(\theta)\left\{r^{n-p} + r^{n-p-1} + \ldots + 1\right\}. \end{aligned}$$

Now Corollary 1 follows, by following the line of proof of Thereom 1.

Proof of Theorem 2. For each zero

$$z = re^{i\theta}, \quad (2.23)$$

of $s(z)$, we have

$$E \leq r + r^2 + \ldots + r^{n-p} + r^n, \text{(by (2.23) and (1.17))}. \tag{2.24}$$

Further let

$$\chi(r) = r^n + r^{n-p} + \ldots + r^2 + r - E. \tag{2.25}$$

Then the equation

$$\chi(r) = 0 \tag{2.26}$$

has a unique positive root γ, with

$$\gamma = 1, \text{ if } \quad n - p + 1 = E, \tag{2.27}$$

$$\gamma < 1, \text{ if } \quad n - p + 1 > E, \tag{2.28}$$

$$\gamma > 1, \text{ if } \quad n - p + 1 < E, \tag{2.29}$$

thereby helping us to write

$$\chi(r) = (r - \gamma)g(r), \tag{2.30}$$

with

$$g(r) > 0, r > 0. \tag{2.31}$$

By (2.24), (2.25), (2.30) and (2.31) we can say that

$$r \geq \gamma. \tag{2.32}$$

But as γ is a root of (2.26) we also have

$$E = \gamma + \gamma^2 + \ldots + \gamma^{n-p} + \gamma^n, \text{(by (2.25))}. \tag{2.33}$$

Now firstly we assume that

$$E < n - p + 1.$$

Therefore on using (2.28) in (2.33), we get

$$\gamma > \max(E/(n - p + 1), E/(1 + E)), (= \rho), \text{(by (1.18))}, \tag{2.34}$$

which, by (2.28) and (2.31), implies

$$(\rho - 1)(\rho - \gamma)g(\rho) > 0,$$

i.e.

$$E/(1 + E - \rho^{n-p} + \rho^{n-1} - \rho^n) > \rho, \text{(by (2.30) and (2.25))}.$$

Further for the function

$$f(x) = -x^n + x^{n-1} - x^{n-p}, 1 < p < n, \tag{2.35}$$

we have

$$\begin{aligned} f'(x) &= -x^{n-p-1}\left\{nx^p - (n-1)x^{p-1} + (n-p)\right\}, \\ &= -x^{n-p-1}\{g(x)\}, \text{ say,} \end{aligned} \tag{2.36}$$

with

$$g(0) > 0, \tag{2.37}$$

$$g^{'}(x) = npx^{p-2}\{x - ((n-1)(p-1)/(np))\}, \tag{2.38}$$

and

$$g\Big(\frac{(n-1)(p-1)}{np}\Big) > 0. \tag{2.39}$$

By (2.38), (2.39) and (2.37) we can say that $g(x)$ is positive in $[0, \infty)$ and therefore by (2.36) and (2.35), $f(x)$ is strictly decreasing in $[0, \infty)$. Now by (2.34) and (2.28) we get

$$\begin{aligned} E/(1 + E - \rho^{n-p} + \rho^{n-1} - \rho^n) &< E/(1 + E - \gamma^{n-p} + \gamma^{n-1} - \gamma^n), \\ &= \gamma, \text{(by (2.33))}. \end{aligned} \tag{2.40}$$

Secondly we assume that

$$E > n - p + 1. \tag{2.41}$$

Therefore on using (2.29) in (2.33), we get

$$\gamma > \{E/(n - p + 1)\}^{1/n}. \tag{2.42}$$

Again by (2.33), we have

$$\begin{aligned} \gamma^n &= E/\Big\{1 + (1/\gamma)^p + \ldots + (1/\gamma)^{n-1}\Big\}, \\ &> E/\{1 + ((1/\gamma^p)/(1 - (1/\gamma)))\}, \text{(by (2.29))}, \\ &> E/\Big\{1 + ((E/(n-p+1))^{-p/n}/(1 - (E/(n-p+1))^{-1/n}))\Big\}, \end{aligned}$$

thereby implying by (2.42) and (1.19) that

$$\gamma > \rho, \tag{2.43}$$

with

$$\rho > 1, \text{(by (2.41) and (1.19))}. \tag{2.44}$$

Now by (2.43), (2.44) and (2.31), we have

$$(\rho - 1)(\rho - \gamma)g(\rho) < 0,$$

i.e.

$$(1 + E - \rho^{n-p} + \rho^{n-1} - (E/\rho))^{1/n} > \rho, \text{(by (2.30) and (2.25))}. \tag{2.45}$$

Further by (2.43) and (2.44), we get

$$\rho^{n-p}\{(\gamma/\rho)^{n-p} - 1\} \leq \rho^{n-1}\{(\gamma/\rho)^{n-1} - 1\},$$

which implies

$$\begin{aligned} (1 + E - \rho^{n-p} + \rho^{n-1} - (E/\rho))^{1/n} &< (1 + E - \gamma^{n-p} + \gamma^{n-1} - (E/\gamma))^{1/n}, \text{(by (2.43))}, \\ &= \gamma, \quad \text{(by (2.33))}. \end{aligned} \tag{2.46}$$

Now let

$$\delta = \left(1 + E - \rho^{n-p} + \rho^{n-1} - (E/\rho)\right)^{1/n}.$$

Then

$$\gamma > \delta, \qquad \text{(by (2.46))}$$

and

$$\delta > 1, \qquad \text{(by (2.45) and (2.44))}.$$

Hence on repeating all steps, after (2.44) and upto (2.46), we will get

$$\left(1 + E - \delta^{n-p} + \delta^{n-1} - (E/\delta)\right)^{1/n} < \gamma,$$

i.e.

$$\begin{aligned} \Big\{1 + E - (1 + E - \rho^{n-p} + \rho^{n-1} \quad &- \quad (E/\rho))^{(n-p)/n} + \\ (1 + E - \rho^{n-p} + \rho^{n-1} - (E/\rho))^{(n-1)/n} \quad &- \quad E(1 + E - \rho^{n-p} + \rho^{n-1} - (E/\rho))^{-1/n}\Big\}^{1/n} \\ &< \gamma. \end{aligned} \tag{2.47}$$

Finally by using (2.32), (2.40), (2.47) and (2.27), Theorem 2 follows.

Proof of Corollary 2. For each zero

$$z = re^{i\theta} \tag{2.48}$$

of $s(z)$, we have

$$1 \le \sum_{k=p}^{n-1} A_k r^{n-k} \cos_+(\alpha_k + (n-k)\theta) + A_0 r^n \cos_+(\alpha_0 + n\theta), \text{(by (1.20), (2.48) and (1.21))},$$

which, by (1.22) , implies

$$E(\theta) \le r + r^2 + \ldots + r^{n-p} + r^n.$$

Now Corollary 2 follows, by following the line of proof of Theorem 2.

Proof of Theorem 3. For each zero

$$z = re^{i\theta}, \tag{2.49}$$

of $p(z)$, we have

$$1 \le A_1 r + \delta(r^2 + r^3 + \ldots + r^n), \text{(by (2.49), (1.24) and (1.25))}. \tag{2.50}$$

Further let

$$\psi_1(r) = \delta(r^n + r^{n-1} + \ldots + r^2) + A_1 r - 1. \tag{2.51}$$

Then the equation

$$\psi_1(r) = 0 \tag{2.52}$$

has a unique positive root α, with

$$\alpha = 1,\ \text{if } (n-1)\delta + A_1 = 1, \tag{2.53}$$
$$\alpha < 1,\ \text{if } (n-1)\delta + A_1 > 1, \tag{2.54}$$
$$\alpha > 1,\ \text{if } (n-1)\delta + A_1 < 1, \tag{2.55}$$

thereby helping us to write

$$\psi_1(r) = (r-\alpha)\phi_1(r), \tag{2.56}$$

with

$$\phi_1(r) > 0, r > 0. \tag{2.57}$$

By (2.50), (2.51), (2.56) and (2.57) we can say that

$$r \geq \alpha. \tag{2.58}$$

But as α is a root of (2.52), we also have

$$1 = \delta(\alpha^n + \alpha^{n-1} + \ldots + \alpha^2) + A_1\alpha, (\text{by } (2.51)). \tag{2.59}$$

Now firstly we assume that

$$(n-1)\delta + A_1 > 1. \tag{2.60}$$

Therefore on using (2.54) in (2.59), we get firstly

$$\alpha > 1/((n-1)\delta + A_1) \tag{2.61}$$

and secondly

$$(\delta - A_1)\alpha^2 + (A_1 + 1)\alpha - 1 > 0,$$

which implies

$$\alpha > G(A_1, \delta), (\text{by } (1.27) \text{ and } (1.28)). \tag{2.62}$$

On combining (2.61) and (2.62) we get

$$\alpha > \max\{1/((n-1)\delta + A_1), G(A_1, \delta)\}, \tag{2.63}$$
$$= \rho, A_1 \geq \delta \text{ or } \delta > \max(A_1, A_0) \text{ or } A_1 = 0, (\text{by } (1.29)). \tag{2.64}$$

And if

$$A_1 < \delta \leq A_0 (\& \text{ so } A_1 > 0, \text{by } (1.26)) \text{ and } t = ((\delta - A_1)/\delta)^{1/(n-1)} \tag{2.65}$$

then

$$t \leq 1/(1 + A_1), \qquad (\text{by } (1.26)),$$

i.e.

$$A_1 \leq (1 - A_1 t)(1-t)/t^2,$$

i.e.

$$\delta(1 - t^{n-1}) \leq (1 - A_1 t)(1-t)/t^2,$$

i.e.

$$\psi_1(t) \le 0, \qquad \text{(by (2.56) and (2.57))},$$

which implies

$$\begin{aligned} \alpha &\ge ((\delta - A_1)/\delta)^{1/(n-1)}, \\ &> ((\delta - A_1)/(n\delta))^{1/(n-1)} \end{aligned} \tag{2.66}$$

and therefore

$$\begin{aligned} \alpha &> \max\left\{((\delta - A_1)/(n\delta))^{1/(n-1)}, 1/((n-1)\delta + A_1), G(A_1,\delta)\right\}, \text{(by (2.63))}, &(2.67)\\ &= \rho, \text{(by (1.30))}. &(2.68) \end{aligned}$$

Further for the function

$$h(x) = 1/(1 + A_1 + (\delta - A_1)x - \delta x^n), x \in (0, \alpha], \tag{2.69}$$

we have

$$h(\rho) > \rho, A_1 \ge \delta \text{ or } A_1 < \delta \le A_0, \text{(by (2.64) and (2.68))}, \tag{2.70}$$

$$h^{'}(x) > 0, x^{n-1} > (\delta - A_1)/(n\delta)$$

and therefore, if

$$A_1 \ge \delta$$

then $h(x)$ will be strictly increasing in $(0, \alpha]$, thereby implying

$$h(\rho) < 1/(1 + A_1 + (\delta - A_1)\alpha - \delta\alpha^n) = \alpha, \text{(by (2.59))}, \tag{2.71}$$

and if

$$A_1 < \delta \le A_0$$

then $h(x)$ will be strictly increasing in $[((\delta - A_1)/(n\delta))^{1/(n-1)}, \alpha]$, (by (2.66)), thereby implying

$$h(\rho) < 1/(1 + A_1 + (\delta - A_1)\alpha - \delta\alpha^n) = \alpha, \text{(by (2.59))}. \tag{2.72}$$

Secondly we assume that

$$(n-1)\delta + A_1 < 1. \tag{2.73}$$

Therefore on using (2.53) in (2.59), we get firstly

$$\alpha \ge 1/((n-1)\delta + A_1)^{1/n}, \tag{2.74}$$

$$> 1 \quad , \text{(by (2.73))}, \tag{2.75}$$

and secondly

$$1 < \alpha^n \left\{\delta(1 - (1/\alpha))^{-1} + (A_1 - \delta)(1/\alpha)^{n-1}\right\},$$

which, by (2.74), implies

$$\alpha > 1/(\delta\{1 - ((n-1)\delta + A_1)^{1/n}\}^{-1} + (A_1 - \delta)\{(n-1)\delta + A_1\}^{(n-1)/n})^{1/n}. \tag{2.76}$$

On combining (2.74) and (2.76) we get

$$\alpha \geq \rho, \text{(by (1.31))}, \tag{2.77}$$
$$> 1, \text{(by (2.75))}. \tag{2.78}$$

Further if

$$A_1 \leq \delta$$

then by (2.77) and (2.78), we have

$$\rho \leq \{(A_1/\delta) + (1/\delta)(1 - (1/\rho)) + (1 - (A_1/\delta))\rho\}^{1/n}, \tag{2.79}$$
$$\leq \{(A_1/\delta) + (1/\delta)(1 - (1/\alpha)) + (1 - (A_1/\delta))\alpha\}^{1/n},$$
$$= \alpha, \text{(by (2.59))}. \tag{2.80}$$

Finally on combining (2.58) along with ((2.64), (2.68), (2.69), (2.70), (2.71), (2.72)), ((2.77), (2.79), (2.80)) and (2.53), Theorem 3 follows.

Proof of Corollary 3. For each zero

$$z = re^{i\theta} \tag{2.81}$$

of $p(z)$, we have

$$1 \leq \sum_{j=1}^{n} B_j r^j \cos_+(\alpha_j + j\theta), \text{(by (1.32), (2.81) and (1.33))},$$

which, by (1.34) and (1.35), implies

$$1 \leq A_1(\theta)r + \delta(\theta)\left\{r^2 + r^3 + \ldots + r^n\right\}. \tag{2.82}$$

Now if

$$\delta(\theta) > 0$$

then Corollary 3 follows, by following the line of proof of Theorem 3, and if

$$\delta(\theta) = 0$$

then

$$r \geq 1/A_1(\theta), \text{(by (2.82))},$$

thereby completing the proof of Corollary 3.

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