

On Oscillation of a Certain Class of Third-Order Nonlinear Functional Dynamic Equations on Time Scales

by
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Abstract

In this paper, we establish some new sufficient conditions for oscillation of the third order nonlinear functional dynamic equation

$$\left[p(t) \left[(r(t)x^\Delta(t))^\Delta \right]^\gamma \right]^\Delta + q(t)f(x(\tau(t))) = 0, \quad \text{for } t \in [t_0, \infty)_\mathbb{T},$$

on a time scale \mathbb{T} , where $\gamma > 0$ is the quotient of odd positive integers, p , q , r and τ are positive rd-continuous functions defined on \mathbb{T} and $f \in C(\mathbb{R}, \mathbb{R})$, $uf(u) > 0$ and $f(u)/u^\gamma \geq K > 0$, for $u \neq 0$. The results provided substantial improvement over those obtained by Yu and Wang [J. Comp. Appl. Math. 225 (2009), 531-540] and Hassan [Oscillation of third order nonlinear delay dynamic equations on time scales, Math. Comp. Modelling 49 (2009), 1573-1586], in the sense that our results can be applied when $0 < \gamma < 1$, $(\tau \circ \sigma)(t) \neq (\sigma \circ \tau)(t)$, and do not require that $\int_{t_0}^\infty q(t)\Delta t = \infty$. Some examples illustrating the main results are given.

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1 Introduction

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [16], is an area of mathematics that has recently received a lot of attention. It has been created in order to unify the study of differential and difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale \mathbb{T} , which may be an arbitrary closed subset of the real numbers \mathbb{R} . The book on the subject of time scale, i.e., measure chains, by Bohner and Peterson [5] summarizes and organizes much of time scale calculus. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum

calculus (see Kac and Cheung [18]), i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. There are applications of dynamic equations on time scales to *quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics*. A recent cover story article in New Scientist [25] discusses several possible applications. Since then several authors have expounded on various aspects of this new theory [6].

In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations on time scales, we refer the reader to the papers [1, 3, 7, 8, 21, 22] and the references cited therein. To the best of the author’s knowledge, there are few papers dealing with oscillation of third order ordinary and delay dynamic equations on time scales [12], [13], [14], [15] and [26]. Following this trend, in this paper, we are concerned with the third order nonlinear functional dynamic equation

$$\left(p(t) \left[(r(t)x^\Delta(t))^\Delta \right]^\gamma \right)^\Delta + q(t)f(x(\tau(t))) = 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \quad (1.1)$$

on an arbitrary time scale \mathbb{T} . Our interest is to establish some sufficient conditions which guarantee that the equation has oscillatory solutions or the solutions tend to zero as $t \rightarrow \infty$. Our results do not require that

$$\gamma \geq 1, \quad (\tau \circ \sigma)(t) = (\sigma \circ \tau)(t), \quad \tau(t) < t \quad \text{and} \quad \int_{t_0}^\infty q(t)\Delta t = \infty.$$

For completeness, we recall the following concepts that are related to the notion of time scales. The forward jump operator and the backward jump operator are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, where $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, is right-dense if $\sigma(t) = t$, is left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. A function $g : \mathbb{T} \rightarrow \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided g is continuous at right-dense points and at left-dense points in \mathbb{T} , left hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale \mathbb{T} is defined by $\mu(t) := \sigma(t) - t$, and for any function $g : \mathbb{T} \rightarrow \mathbb{R}$ the notation $g^\sigma(t)$ denotes $g(\sigma(t))$.

Equation (1.1) is called a delay dynamic equation if $\tau(t) < t$, and is called an advance dynamic equation if $\tau(t) > t$, and ordinary if $\tau(t) = t$. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. By a solution of (1.1), we mean a function $x^{[0]}(t) \in C_r^1([t_0, \infty), \mathbb{R})$ such that $x^{[1]}(t), x^{[2]}(t) \in C_r^1([t_0, \infty), \mathbb{R})$ satisfying (1.1) for all $t \geq t_0$, where C_r is the space of rd-continuous functions, and

$$x^{[0]} := x, \quad x^{[1]} := r \left(x^{[0]} \right)^\Delta, \quad x^{[2]} := p \left[\left(x^{[1]} \right)^\Delta \right]^\gamma, \quad x^{[3]} := \left(x^{[2]} \right)^\Delta. \quad (1.2)$$

Any solution of (1.1) is said to be a proper solution if it is defined on the interval $[t_0, \infty)_{\mathbb{T}}$ and is nontrivial in any neighborhood of infinity. So the solutions vanishing in some neighborhood of infinity will be excluded from our consideration and we are interested only in oscillation and asymptotic behavior of the proper solutions. A proper solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory in case there exists at least one oscillatory solution and is nonoscillatory if all its solutions are nonoscillatory. Throughout this paper, we will assume the following hypotheses:

(h_1). p, q and r are positive real-valued rd -continuous functions defined on \mathbb{T} , $\tau : \mathbb{T} \rightarrow \mathbb{T}$ and satisfies $\lim_{t \rightarrow \infty} \tau(t) = \infty$,

(h_2). $f \in C(\mathbb{R}, \mathbb{R})$, $uf(u) > 0$ and $f(u)/u^\gamma \geq K > 0$, for $u \neq 0$.

In the following, we recall some of the related results that have been established for third order dynamic equations on time scales that serve to motivate the contents of this paper. Erbe, Peterson and Saker [12] considered the third order nonlinear dynamic equation

$$(p(t) [(r(t)x^\Delta(t))^\Delta]^\Delta + q(t)f(x(t)) = 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \quad (1.3)$$

on a time scale \mathbb{T} , where p, r and q are positive rd -continuous functions, $f \in C(\mathbb{R}, \mathbb{R})$, $uf(u) > 0$, $f(u)/u \geq K > 0$, for $u \neq 0$, and established some sufficient conditions which guarantee that (1.3) has oscillatory solutions or the solutions tend to zero as $t \rightarrow \infty$. Erbe, Peterson and Saker [13] considered the third order nonlinear dynamic equation

$$\left[p(t) [(r(t)x^\Delta(t))^\Delta]^\gamma \right]^\Delta + f(t, x(t)) = 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \quad (1.4)$$

on a time scale \mathbb{T} , where $\gamma \geq 1$ is the quotient of odd integers, p and r are positive rd -continuous functions defined on \mathbb{T} , and there exists a positive rd -continuous function $q(t)$ such that $|f(t, u)| \geq q(t)|u^\gamma|$ and $uf(t, u) > 0$, for $u \neq 0$, and extended the results established in [12] and obtained some sufficient conditions which guarantee that the equation has oscillatory solutions or the solutions tend to zero as $t \rightarrow \infty$. Erbe, Peterson and Saker [14] considered the third order linear dynamic equation

$$x^{\Delta\Delta\Delta}(t) + q(t)x^\sigma(t) = 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \quad (1.5)$$

on a time scale \mathbb{T} , where q is positive rd -continuous function defined on \mathbb{T} and established some oscillation criteria of Hille [17] and Nehari [20] types. Yu and Wang [26] considered the third order dynamic equation

$$\left(\frac{1}{a_2(t)} \left(\left(\frac{1}{a_1(t)} (x^\Delta(t))^{\alpha_1} \right)^\Delta \right)^{\alpha_2} \right)^\Delta + q(t)f(x(t)) = 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \quad (1.6)$$

on a time scale \mathbb{T} , where α_i is a quotient of odd positive integers for $i = 1, 2$ such that $\alpha_1\alpha_2 = 1$, a_1 and a_2 are positive rd -continuous functions satisfying

$$\int_{t_0}^{\infty} (a_i(t))^{\frac{1}{\alpha_i}} \Delta t = \infty, \text{ for } i = 1, 2, \tag{1.7}$$

and $f \in C(\mathbb{R}, \mathbb{R})$, $uf(u) > 0$ and $f(u)/u \geq K > 0$, for $u \neq 0$, or $f'(u) > C > 0$. The authors employed the Riccati substitution and followed the technique used in [12, 13] and established some sufficient conditions which ensure that the solution $x(t)$ for (1.6) is oscillatory or $\lim_{t \rightarrow \infty} x(t)$ exists (finite). To prove that $\lim_{t \rightarrow \infty} x(t) = 0$ (see [26, Lemma 3.2]) the condition $\int_{t_0}^{\infty} q(t)\Delta t = \infty$ is required which is satisfied only when $q(t) = \beta/t^\gamma$ for $0 < \gamma \leq 1$. When $f(u)/u \geq M$ for $M > 0$, the authors in [26] proved that if there exists a positive rd -continuous function $\zeta(t)$ such that

$$\limsup_{t \rightarrow \infty} \int_{t_2}^t \left[M\zeta(s)q(s) - \frac{(r^\Delta(s))^2}{4Q(s)} \right] \Delta s = \infty, \tag{1.8}$$

where $Q(t) = \zeta(t) [a_1(t)\delta(t, t_1)]^{\frac{1}{\alpha_1}}$ and $\delta(t, t_1) = \int_{t_1}^t (a_2(s))^{\frac{1}{\alpha_2}} ds$, then every solution $x(t)$ of (1.6) is either oscillatory or $\lim_{t \rightarrow \infty} x(t)$ exists (finite). Note that the condition (1.8), cannot be applied on the equation

$$x'''(t) + \frac{1}{\sqrt{3}t^3}x(t) = 0, \quad t \geq 1, \tag{1.9}$$

even if the function $\zeta(t) = t^3$, since

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_{t_2}^t \left[M\zeta(s)q(s) - \frac{(r^\Delta(s))^2}{4Q(s)} \right] \Delta s &= \\ \limsup_{t \rightarrow \infty} \int_2^t \left[s^3 \frac{1}{\sqrt{3}s^3} - \frac{(3s^2)^2}{4s^3(s-1)} \right] ds &= \\ \limsup_{t \rightarrow \infty} \int_2^t \left[\frac{1}{\sqrt{3}} - \frac{9}{4} \frac{s}{(s-1)} \right] ds &\neq \infty \end{aligned} .$$

This means that the results established in [26] cannot be applied on the third order Euler equations when $q(t) = \beta/t^3$ for $\beta > 0$ and the conclusion $\lim_{t \rightarrow \infty} x(t) = 0$ cannot be obtained, since $\int_{t_0}^{\infty} (\beta/t^3)\Delta t < \infty$. This will be treated in our paper by proving that the solution $x(t)$ of (1.9) is oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$. Note that the roots of the characteristic equation of (1.9) are given by $m = 1.6073 - 0.32634i$, $m = 1.6073 + 0.32634i$, and $m = -0.21463$ (see Example 4 below).

Hassan [15], considered the nonlinear functional dynamic equation

$$\left[p(t) \left[(r(t)x^\Delta(t))^\Delta \right]^\gamma \right]^\Delta + f(t, x(\tau(t))) = 0, \quad t \geq t_0, \tag{1.10}$$

on a time scale \mathbb{T} , where $\gamma \geq 1$ is a quotient of odd positive integers, p and r are rd -continuous positive functions, $f(t, u) : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with $uf(t, u) > 0$ for all $u \neq 0$ and there exists a positive rd -continuous function $q(t)$ defined on \mathbb{T} such that $|f(t, u)| \geq q(t)|u^\gamma|$ and $\tau : \mathbb{T} \rightarrow \mathbb{T}$ is continuous and $\tau(t) \leq t$, $\tau^\Delta(t) \geq 0$ for $t \in \mathbb{T}$, $\lim_{t \rightarrow \infty} \tau(t) = \infty$. The author in [15] assumed that $(\tau \circ \sigma)(t) = (\sigma \circ \tau)(t)$ and followed the technique used in [2, 4] and established some sufficient conditions for oscillation of (1.10) by using the chain rule

$$(x(\tau(t)))^\Delta = x^\Delta(\tau(t))\tau^\Delta(t). \tag{1.11}$$

In fact this chain rule is not satisfied on many time scales, and required that $\mathbb{T}^* = g(\mathbb{T})$ is a time a scale and $x^{\Delta^*}(\tau(t))$ should exist and be defined, which in general is not satisfied (see [5, Example, 1.85, Exccercise 1.86, Exccercise 1.96]). Of course trivially (1.11) is satisfied if $\tau(t) = t - \delta \in \mathbb{T}$ for $t \in \mathbb{T}$ and δ is a constant which is satisfied when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$. On the other hand the condition $(\tau \circ \sigma)(t) = (\sigma \circ \tau)(t)$ that has been used in [15] is a restrictive condition and cannot be satisfied on many time scales (see [2]). For instance, if the time scale $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$, we have $\sigma(t) = (\sqrt{t} + 1)^2$ and if $\tau(t) = (t - 1)^2 \geq 2t > t$, then

$$\sigma(\tau(t)) = (\sqrt{(t - 1)^2 + 1})^2 = t^2 \neq \tau(\sigma(t)) = ((\sqrt{t} + 1)^2 - 1)^2 = t + 2\sqrt{t}.$$

So the results in [15] cannot be applied on the case when $\tau(t) > t$ in the time scale $\mathbb{T} = \mathbb{N}^2$. We remark that the results that have been obtained in [12], [13], [14], [15] and [26] are given only when $\tau(t) \leq t$, $\gamma \geq 1$ and

$$\int_{t_0}^\infty \frac{1}{p^{\frac{1}{\gamma}}(t)} \Delta t = \infty, \quad \int_{t_0}^\infty \frac{1}{r(t)} \Delta t = \infty, \tag{1.12}$$

and nothing is known regarding the oscillation of (1.1), when $\tau(t) > t$, $\gamma < 1$ and

$$\int_{t_0}^\infty \frac{1}{p^{\frac{1}{\gamma}}(t)} \Delta t < \infty, \quad \int_{t_0}^\infty \frac{1}{r(t)} \Delta t < \infty. \tag{1.13}$$

So the natural question now is: *If one can find new oscillation conditions which improve the results established in above mentioned papers and can be applied when $\tau(t) > t$, $\gamma < 1$ and when (1.13) holds?* One of our aims in this paper is to give an affirmative answer to this question and establish some sufficient conditions for oscillation by employing the Riccati substitution and the Pötzsche chain rule [5, Theorem 1.90].

The paper is organized as follows: In Section 2, we consider the case when (1.12) holds and establish some sufficient conditions which guarantee that the equation has oscillatory solutions or the solutions tend to zero as $t \rightarrow \infty$. The results in the Subsection 2.1 cover the case when $\tau(t) > t$ and the results in the Subsection 2.2 cover the case when $\tau(t) \leq t$. The results when $\tau(t) \leq t$ improve the obtained results in the above mentioned papers. In Section 3, we consider the

case when (1.13) holds and establish some sufficient conditions which guarantee that the equation (1.1) has oscillatory solutions or the solutions tend to zero as $t \rightarrow \infty$. Some examples are considered in Section 4 to illustrate the main results.

2 Oscillation of (1.1) when (1.12) holds

In this section, we establish some sufficient conditions which guarantee that the equation (1.1) oscillates or the solutions tend to zero as $t \rightarrow \infty$. We start with the following lemma which provides the signs of the Δ -quasi derivatives of $x(t)$ that are defined as in (1.2).

Lemma 2.1. *If $x(t)$ is a nonoscillatory solution of (1.1), then there exists $T > t_0$ such that $x^{[i]}(t) \neq 0$ for $i = 0, 1, 2$ and $t \geq T$.*

Proof: Without loss of generality, we may assume that the solution $x(t)$ is positive and there exists $t_1 > t_0$ so that $x(t) > 0$ and $x(\tau(t)) > 0$, for $t \geq t_1$. Then from (1.1), (h_1) and (h_2) , we see that $x^{[3]}(t) = -q(t)f(x(\tau(t))) < 0$ for $t \geq t_1$. Then there exists $t_2 > t_1$ such that $x^{[2]}(t)$ is decreasing and either positive or negative for $t \geq t_2$. Thus, $x^{[1]}$ is either increasing or decreasing for $t \geq t_2$ and so that there exists $T > t_2$ such that $x^{[1]}(t)$ is either positive or negative for $t \geq T$. The proof is complete. \square

Remark 1. *In view of Lemma 2.1, all nonoscillatory solutions of (1.1) belong to the following classes:*

$$\begin{aligned} C_0 &= \{x : \exists T \text{ such that } x(t)x^{[1]}(t) < 0, x(t)x^{[2]}(t) > 0, \text{ for } t \geq T\}, \\ C_1 &= \{x : \exists T \text{ such that } x(t)x^{[1]}(t) > 0, x(t)x^{[2]}(t) < 0, \text{ for } t \geq T\}, \\ C_2 &= \{x : \exists T \text{ such that } x(t)x^{[1]}(t) > 0, x(t)x^{[2]}(t) > 0, \text{ for } t \geq T\}, \\ C_3 &= \{x : \exists T \text{ such that } x(t)x^{[1]}(t) < 0, x(t)x^{[2]}(t) < 0, \text{ for } t \geq T\}. \end{aligned}$$

We need the following lemmas in the proofs of the main results. The proofs are similar to the proofs of the results in [13, Lemma 1, Lemma 2, Theorem 2.2] and hence are omitted.

Lemma 2.2. *Assume that (h_1) and (h_2) hold. Let $x(t)$ be a nonoscillatory solution of (1.1). If (1.12) holds, then $x(t) \in C_0 \cup C_2$.*

Lemma 2.3. *Assume that (h_1) and (h_2) . Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(t) \in C_0$. If*

$$(h_3). \int_{t_0}^{\infty} r^{-1}(t) \int_t^{\infty} (p^{-1}(u) \int_u^{\infty} q(s) \Delta s)^{\frac{1}{\gamma}} \Delta u \Delta t = \infty,$$

then $\lim_{t \rightarrow \infty} x^{[0]}(t) = 0$.

Remark 2. *Note that the condition (h_3) improves the condition $\int_{t_0}^{\infty} q(t) \Delta t = \infty$ that has been used in [26, Lemma 3.2] to prove that $\lim_{t \rightarrow \infty} x(t) = 0$.*

Lemma 2.4. *Assume that (h_1) and (h_2) hold. Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(t) \in C_2$. Then there exists $T > t_0$ such that*

$$x^{[1]}(t) \geq \delta(t, T) \left(x^{[2]}(t)\right)^{\frac{1}{\gamma}}, \text{ where } \delta(t, T) := \int_T^t \left(p^{-\frac{1}{\gamma}}(s)\right) \Delta s, \text{ for } t \geq T.$$

2.1 The case when $\tau(t) > t$.

In this subsection, we consider the case when $\tau(t) > t$. We introduce the following notations:

$$Q(t) : = Kq(t) \left(\frac{R(\tau, t)p^{\frac{1}{\gamma}}(t)P(t, T)}{p^{\frac{1}{\gamma}}(t)P(t, T) + \sigma(t) - t} \right)^{\gamma}, P(t, T) := \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s,$$

$$R(\tau, t) : = \int_t^{\tau(t)} \frac{1}{r(s)} \Delta s, \text{ for } T \geq t_0.$$

Now, we are ready to state and prove the main results in this subsection. In the following theorem we derive the Riccati inequality (see (2.13)) which plays an important role in the proof of the main results as well as establish our first oscillation criterion of Leighton-Wintner type.

Theorem 2.1. *Assume that $(h_1) - (h_3)$, (1.12) hold and $x(t)$ is a solution of (1.1). If*

$$\int_{t_0}^{\infty} Q(s) \Delta s = \infty, \tag{2.1}$$

then $x(t)$ is oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof: Suppose to the contrary and assume that $x(t)$ is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq T$ (where T is chosen so large as in Lemma 2.1). We consider only this case, because the proof when $x(t) < 0$ is similar, since $uf(u) > 0$. Now, since (1.12) holds, then from Lemma 2.2, we see that $x(t) \in C_0 \cup C_2$. If $x(t) \in C_0$, then by Lemma 2.3, we get $\lim_{t \rightarrow \infty} x(t) = 0$. Next, we consider the case when $x(t) \in C_2$ and define the function $w(t)$ by the Riccati substitution

$$w(t) := \frac{x^{[2]}(t)}{(x^{[1]}(t))^{\gamma}}. \tag{2.2}$$

Now, since $x(t) \in C_2$, we see that $w(t) > 0$ and by the quotient rule [5, Theorem 1.20], we obtain

$$w^{\Delta} = \left(\frac{x^{[2]}}{(x^{[1]})^{\gamma}} \right)^{\Delta} = \frac{x^{[3]} ((x^{[0]})^{\tau})^{\gamma}}{((x^{[0]})^{\tau})^{\gamma} ((x^{[1]})^{\sigma})^{\gamma}} - \frac{\left((x^{[1]})^{\gamma} \right)^{\Delta} x^{[2]}}{(x^{[1]})^{\gamma} ((x^{[1]})^{\sigma})^{\gamma}}.$$

From (1.1), (h_1) and (h_2) , we get

$$w^\Delta \leq -Kq \frac{((x^{[0]})^\tau)^\gamma}{((x^{[1]})^\sigma)^\gamma} - \frac{((x^{[1]})^\gamma)^\Delta x^{[2]}}{(x^{[1]})^\gamma ((x^{[1]})^\sigma)^\gamma}. \quad (2.3)$$

By the Pötzsche chain rule ([5, Theorem 1.90]), if $f^\Delta(t) > 0$ and $\gamma > 1$, we obtain

$$(f^\gamma(t))^\Delta = \gamma \int_0^1 [f + \mu h f^\Delta]^{\gamma-1} f^\Delta(t) dh \geq \gamma \int_0^1 (f)^{\gamma-1} f^\Delta(t) dh = \gamma (f)^{\gamma-1} f^\Delta(t). \quad (2.4)$$

Also by the Pötzsche chain rule ([5, Theorem 1.90]), if $f^\Delta(t) > 0$ and $0 < \gamma \leq 1$, we obtain

$$(f^\gamma(t))^\Delta \geq \gamma \int_0^1 (f^\sigma(t))^{\gamma-1} dh f^\Delta(t) = \gamma (f^\sigma(t))^{\gamma-1} f^\Delta(t). \quad (2.5)$$

So that from (2.4) and (2.5), using $f(t) = x^{[1]}(t)$, and the fact that $x^{[1]}$ is increasing and $x^{[2]}$ is decreasing, we have

$$\frac{((x^{[1]})^\gamma)^\Delta x^{[2]}}{(x^{[1]})^\gamma ((x^{[1]})^\sigma)^\gamma} \geq \frac{\gamma x^{[2]} (x^{[2]})^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}} (x^{[1]}) ((x^{[1]})^\sigma)^\gamma} \geq \gamma p^{-\frac{1}{\gamma}} (w^\sigma)^{\frac{1}{\gamma}+1}, \text{ for } \gamma > 1, \quad (2.6)$$

and

$$\frac{((x^{[1]})^\gamma)^\Delta x^{[2]}}{(x^{[1]})^\gamma ((x^{[1]})^\sigma)^\gamma} \geq \frac{\gamma x^{[2]} ((x^{[1]})^\sigma)^{\gamma-1} (x^{[2]})^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}} (x^{[1]})^\gamma ((x^{[1]})^\sigma)^\gamma} \geq \gamma \frac{1}{p^{\frac{1}{\gamma}}} (w^\sigma)^{1+\frac{1}{\gamma}}, \text{ for } 0 < \gamma \leq 1. \quad (2.7)$$

Thus combining (2.6) and (2.7), we have

$$\frac{((x^{[1]})^\gamma)^\Delta x^{[2]}}{(x^{[1]})^\gamma ((x^{[1]})^\sigma)^\gamma} \geq \gamma \frac{1}{p^{\frac{1}{\gamma}}} (w^\sigma)^{1+\frac{1}{\gamma}}, \text{ for } \gamma > 0. \quad (2.8)$$

Substituting (2.8) in (2.3), we have

$$w^\Delta \leq -Kq \left(\frac{x(\tau)}{(x^{[1]})^\sigma} \right)^\gamma - \gamma \frac{1}{p^{\frac{1}{\gamma}}} (w^\sigma)^{1+\frac{1}{\gamma}}, \text{ for } t \geq T. \quad (2.9)$$

Next we consider the coefficient of Kq in (2.9). Since $(x^{[1]})^\sigma = x^{[1]}(t) + \mu(t)(x^{[1]})^\Delta$, we have

$$\frac{(x^{[1]})^\sigma}{x^{[1]}(t)} = 1 + \mu(t) \frac{(x^{[1]})^\Delta}{x^{[1]}(t)} = 1 + \frac{\mu(t)}{p^{\frac{1}{\gamma}}(t)} \frac{(x^{[2]}(t))^{\frac{1}{\gamma}}}{x^{[1]}(t)}.$$

Also since $x^{[2]}(t)$ is decreasing, we have

$$x^{[1]}(t) = x^{[1]}(T) + \int_T^t \left(x^{[2]}(s)\right)^{\frac{1}{\gamma}} \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s \geq \left(x^{[2]}(t)\right)^{\frac{1}{\gamma}} \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s.$$

It follows that

$$\frac{x^{[1]}(t)}{\left(x^{[2]}(t)\right)^{\frac{1}{\gamma}}} \geq \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s = P(t, T). \tag{2.10}$$

Hence

$$\frac{(x^{[1]})^\sigma}{x^{[1]}(t)} = 1 + \mu(t) \frac{(x^{[1]})^\Delta}{x^{[1]}(t)} = 1 + \frac{\mu(t)}{p^{\frac{1}{\gamma}}(t)} \frac{\left(x^{[2]}(t)\right)^{\frac{1}{\gamma}}}{x^{[1]}(t)} \leq \frac{p^{\frac{1}{\gamma}}(t)P(t, T) + \mu(t)}{p^{\frac{1}{\gamma}}(t)P(t, T)}.$$

Hence, we have

$$\frac{x^{[1]}(t)}{(x^{[1]})^\sigma} \geq \frac{p^{\frac{1}{\gamma}}(t)P(t, T)}{p^{\frac{1}{\gamma}}(t)P(t, T) + \sigma(t) - t}, \text{ for } t \geq T.$$

So that

$$\frac{x(\tau)}{(x^{[1]})^\sigma} = \frac{x(\tau)}{x^{[1]}} \frac{x^{[1]}}{(x^{[1]})^\sigma} \geq \frac{x(\tau)}{x^{[1]}} \frac{p^{\frac{1}{\gamma}}(t)P(t, T)}{p^{\frac{1}{\gamma}}(t)P(t, T) + \sigma(t) - t}, \text{ for } t \geq T. \tag{2.11}$$

Now, since $\tau(t) > t$ and $x^{[1]}$ is increasing, we have

$$x(\tau(t)) > x(\tau(t)) - x(t) = \int_t^{\tau(t)} \frac{x^{[1]}(s)}{r(s)} \Delta s \geq x^{[1]}(t)R(\tau, t).$$

This and (2.11) lead to

$$\frac{x(\tau)}{(x^{[1]})^\sigma} \geq \frac{R(\tau, t)P(t, T)p^{\frac{1}{\gamma}}(t)}{\left(p^{\frac{1}{\gamma}}(t)P(t, T) + \sigma(t) - t\right)}, \text{ for } t \geq T. \tag{2.12}$$

Substituting (2.12) into (2.9), we have the Riccati inequality

$$w^\Delta(t) + Q(t) + \frac{\gamma}{p^{\frac{1}{\gamma}}(t)} (w^\sigma)^{1+\frac{1}{\gamma}}(t) \leq 0, \text{ for } t \geq T. \tag{2.13}$$

This implies that

$$-w^\Delta(t) \geq Q(t) + \frac{\gamma}{p^{\frac{1}{\gamma}}(t)} (w^\sigma(t))^{\frac{\gamma+1}{\gamma}}, \text{ for } t \geq T. \tag{2.14}$$

From the definition of $x^{[2]}(t)$, we see that $(x^{[1]}(t))^\Delta = (x^{[2]}(t)/p(t))^{\frac{1}{\gamma}}$. Integrating from T to t we obtain

$$x^{[1]}(t) = x^{[1]}(T) + \int_T^t \left(\frac{1}{p(s)}x^{[2]}(s)\right)^{\frac{1}{\gamma}} \Delta s.$$

Taking into account that $x^{[2]}(t)$ is positive and decreasing, we get $x^{[1]}(t) \geq x^{[1]}(T) + (x^{[2]}(t))^{1/\gamma} \int_T^t (1/p(s))^{1/\gamma} \Delta s$. It follows that

$$w(t) = \frac{x^{[2]}}{(x^{[1]})^\gamma} \leq \left(\int_{t_0}^t \left(\frac{1}{p(s)} \right)^{\frac{1}{\gamma}} \Delta s \right)^{-\gamma}, \quad \text{for } t \geq T,$$

which implies using (1.12) that $\lim_{t \rightarrow \infty} w(t) = 0$. Integrating (2.14) from T to ∞ and using $\lim_{t \rightarrow \infty} w(t) = 0$, we have $w(T) \geq \int_T^\infty Q(s) \Delta s$, which contradicts (2.1). The proof is complete. \square

In the following, we establish some sufficient conditions for oscillation of (1.1) when

$$\int_{t_0}^\infty Q(s) \Delta s < \infty. \tag{2.15}$$

Theorem 2.2. *Assume that $(h_1) - (h_3)$, (1.12) hold and $x(t)$ is a solution of (1.1). If there exists a positive rd-continuous Δ -differentiable function $\phi(t)$ such that*

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\phi(s)Q(s) - \frac{p(s)(\phi^\Delta(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^\gamma(s)} \right] \Delta s = \infty, \tag{2.16}$$

then $x(t)$ is oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof: Suppose to the contrary and assume that $x(t)$ is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq T$ (where T is chosen so large as in Lemma 2.1). We consider only this case, because the proof when $x(t) < 0$ is similar, since $uf(u) > 0$. Now, since (1.12) holds, we see from Lemma 2.2 that $x(t) \in C_0 \cup C_2$. If $x(t) \in C_0$, then by Lemma 2.3, we have $\lim_{t \rightarrow \infty} x(t) = 0$. Next, we consider the case when $x(t) \in C_2$ and define the function $w(t)$ by the Riccati substitution (2.2). Proceeding as in the proof of Theorem 2.1, we obtain

$$w^\Delta(t) + Q(t) + \frac{\gamma}{p^{\frac{1}{\gamma}}(t)} (w^\sigma)^{\frac{\gamma+1}{\gamma}}(t) \leq 0, \quad \text{for } t \geq T. \tag{2.17}$$

Multiplying (2.17) by $\phi(s)$ and integrating from T to t ($t \geq T$), we have after using integration by parts, that

$$\int_T^t \phi(s)Q(s) \Delta s \leq w(T)\phi(T) + \int_T^t \left[\phi^\Delta(s)w^\sigma(s) - \frac{\gamma\phi(s)}{p^{\frac{1}{\gamma}}(s)} (w^\sigma)^{\frac{\gamma+1}{\gamma}}(s) \right] \Delta s. \tag{2.18}$$

Using the inequality

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}, \tag{2.19}$$

with $B = \phi^\Delta(s)$ and $A = \gamma\phi(s)p^{-\frac{1}{\gamma}}(s)$ and $u = w^\sigma$, we have by (2.18) that

$$\int_T^t \left[\phi(s)Q(s) - \frac{1}{(\gamma + 1)^{\gamma+1}} \frac{p(s)(\phi^\Delta(s))^{\gamma+1}}{\phi^\gamma(s)} \right] \Delta s < w(T)\phi(T),$$

for all large t . This contradicts (2.16). The proof is complete. □

The following theorem gives the Philos-type oscillation criterion for equation (1.1). First, let us introduce now the class of functions \mathfrak{R} which will be extensively used in the sequel. Let $\mathbb{D}_0 \equiv \{(t, s) \in \mathbb{T}^2 : t > s \geq t_0\}$ and $\mathbb{D} \equiv \{(t, s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$. The function $H \in C_{rd}(\mathbb{D}, \mathbb{R})$ is said to belong to the class \mathfrak{R} if

(i) $H(t, t) = 0, \quad t \geq t_0, \quad H(t, s) > 0, \quad \text{on } \mathbb{D}_0,$

(ii) H has a continuous Δ -partial derivative $H^{\Delta_s}(t, s)$ on \mathbb{D}_0 with respect to the second variable. (H is rd -continuous function if H is an rd -continuous function in t and s). The proof is similar to that of the proof of Theorem 3.3 in [22] if one uses the inequality (2.17) and hence is omitted.

Theorem 2.3. *Assume that $(h_1) - (h_3)$, (1.12) hold, $H \in \mathfrak{R}$ and $x(t)$ is a solution of (1.1). If for $t > s$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s)Q(s) - \frac{p(s)(H^{\Delta_s}(t, s))^{\gamma+1}}{(\gamma + 1)^{\gamma+1}H^\gamma(t, s)} \right] \Delta s = \infty, \quad (2.20)$$

then $x(t)$ is oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

In the following theorems we will assume that $\tau(t) \geq \sigma(t)$ and establish some sufficient conditions which guarantee that the equation (1.1) oscillates or the solutions tend to zero as $t \rightarrow \infty$.

Theorem 2.4. *Assume that $(h_1) - (h_3)$, (1.12) hold, $\tau(t) \geq \sigma(t)$ and $x(t)$ is a solution of (1.1). If there exist positive rd -continuous Δ -differentiable functions $\alpha(t)$ and $\phi(t)$ such that*

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[K\alpha(s)\phi(s)q(s) - \frac{(\alpha^\sigma)^{\gamma+1} r^\gamma(s)C^{\gamma+1}(s)}{(\gamma + 1)^{\gamma+1}\alpha^\gamma(s)\phi^\gamma(s)\delta^\gamma(s, T)} \right] \Delta s = \infty, \quad (2.21)$$

where $T \geq t_0$, and $C(s) := (\phi(s)\alpha^\Delta(s)/\alpha^\sigma) + \phi^\Delta(s)$, then $x(t)$ is oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof: Suppose to the contrary and assume that $x(t)$ is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq T$ (where T is chosen so large as in Lemma 2.1). We consider only this case, because the proof when $x(t) < 0$ is similar, since $uf(u) > 0$. Now, since (1.12) holds, we see from Lemma 2.2 that $x(t) \in C_0 \cup C_2$. If $x(t) \in C_0$, then by Lemma 2.3, we have $\lim_{t \rightarrow \infty} x(t) = 0$. Next, we assume that $x(t) \in C_2$ and define the function $\omega(t)$ by the Riccati substitution

$$\omega(t) := \alpha(t) \frac{x^{[2]}(t)}{x^\gamma(t)}, \quad \text{for } t \geq T. \quad (2.22)$$

Noting that $\omega(t) > 0$, and using the product rule, we obtain

$$\omega^\Delta = \left(x^{[2]}\right)^\sigma \left[\frac{\alpha}{x^\gamma}\right]^\Delta + \frac{\alpha}{x^\gamma} \left(x^{[2]}\right)^\Delta. \quad (2.23)$$

Using the quotient rule, we have

$$\omega^\Delta = \left(x^{[2]}\right)^\sigma \left[\frac{\alpha^\Delta}{(x^\sigma)^\gamma} - \frac{\alpha(x^\gamma)^\Delta}{x^\gamma(x^\sigma)^\gamma}\right] + \frac{\alpha}{x^\gamma} \left(x^{[2]}\right)^\Delta. \quad (2.24)$$

Since $x(t) \in C_2$, we see that $x^\Delta(t) > 0$, which implies that $x^\tau(t) > x^\sigma(t)$. Proceeding as in the proof of Theorem 2.1, to get

$$\omega^\Delta \leq -K\alpha q + \frac{\alpha^\Delta}{\alpha^\sigma} \omega^\sigma - \alpha \frac{\left(x^{[2]}\right)^\sigma (x^\gamma)^\Delta}{x^\gamma(x^\sigma)^\gamma}. \quad (2.25)$$

From Pötzsche chain rule, we have

$$(x^\gamma(t))^\Delta \geq \gamma(x(t))^{\gamma-1} x^\Delta(t), \text{ for } \gamma \geq 1, \quad (2.26)$$

and

$$(x^\gamma(t))^\Delta \geq \gamma(x^\sigma(t))^{\gamma-1} x^\Delta(t), \text{ for } \gamma < 1. \quad (2.27)$$

From (2.25), (2.26) and (2.27), we get

$$\omega^\Delta \leq -K\alpha q + \frac{\alpha^\Delta}{\alpha^\sigma} \omega^\sigma - \gamma \left(x^{[2]}\right)^\sigma \frac{\alpha x^\Delta}{(x^\sigma)^{\gamma+1}}, \quad (2.28)$$

where we used the fact that $x^\sigma(t) \geq x(t)$. From Lemma 2.4, since $x^\Delta(t) \geq \frac{\delta(t,T)}{r(t)} \left(x^{[2]}(t)\right)^{\frac{1}{\gamma}}$, we obtain from (2.28) that

$$\omega^\Delta \leq -K\alpha q + \frac{\alpha^\Delta}{\alpha^\sigma} \omega^\sigma - \gamma \alpha \left(x^{[2]}\right)^\sigma \frac{\delta p^{\frac{1}{\gamma}} (rx^\Delta)^\Delta}{r (x^\sigma)^{\gamma+1}}. \quad (2.29)$$

Also since $x^{[2]}(t)$ is decreasing, we have $\left(x^{[2]}\right)^{\frac{1}{\gamma}} \geq \left(\left(x^{[2]}\right)^\sigma\right)^{\frac{1}{\gamma}}$. This and (2.29) imply that

$$\omega^\Delta(t) \leq -K\alpha(t)q(t) + \frac{\alpha^\Delta(t)}{\alpha^\sigma(t)} \omega^\sigma(t) - \frac{\gamma\alpha(t)\delta(t,T)}{(\alpha^\sigma)^\lambda(t)r(t)} (\omega^\sigma)^{\frac{\gamma+1}{\gamma}}(t), \text{ for } t \geq T. \quad (2.30)$$

Multiplying (2.30) by $\phi(s)$ and integrating from T to t ($t \geq T$), we have

$$\begin{aligned} \int_T^t K\phi(s)\alpha(s)q(s)\Delta s &\leq -\int_T^t \phi(s)\omega^\Delta(s)\Delta s \\ &\quad + \int_T^t \left[\phi(s) \frac{\alpha^\Delta(t)}{\alpha^\sigma(t)} \omega^\sigma - \frac{\gamma\phi(s)\alpha(s)\delta(s,t_1)}{(\alpha^\sigma)^\lambda(s)r(s)} (\omega^\sigma)^{\frac{\gamma+1}{\gamma}} \right] \Delta s. \end{aligned}$$

Using integration by parts, we get

$$\begin{aligned} & \int_T^t K\phi(s)\alpha(s)q(s)\Delta s \leq \omega(T)\phi(T) + \\ & \int_T^t \left[\frac{\phi(s)\alpha^\Delta(s)}{\alpha^\sigma} + \phi^\Delta(s) \right] \omega^\sigma(s)\Delta s - \\ & \int_{t_1}^t \frac{\gamma\phi(s)\alpha(s)\delta(s,t_1)}{(\alpha^\sigma)^\lambda(s)r(s)} (\omega^\sigma)^\frac{\gamma+1}{\gamma} \Delta s. \end{aligned} \tag{2.31}$$

Setting $B = C(s)$, $A = \frac{\gamma\phi(s)\alpha(s)\delta(s,T)}{(\alpha^\sigma)^\lambda(s)r(s)}$, $u = \omega^\sigma$, and using the inequality (2.19), we have by (2.31), for all large t , that

$$\int_T^t \left[K\phi(s)\alpha(s)q(s) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{(\alpha^\sigma)^{\gamma+1}(s)r^\gamma(s)C^{\gamma+1}(s)}{(\alpha(s)\phi(s)\delta(s,T))^\gamma} \right] \Delta s < \omega(T)\phi(T), \tag{2.32}$$

which contradicts (2.21). The proof is complete. \square

The proofs of the following theorems are similar to that of the proof of Theorem 3.3 in [22] if one uses the inequality (2.30) and hence are omitted.

Theorem 2.5. *Assume that $(h_1) - (h_3)$, (1.12) hold, $\tau(t) \geq \sigma(t)$ and $x(t)$ is a solution of (1.1). Let $\alpha(t)$ is defined as in Theorem 2.4, and $H \in \mathfrak{R}$. If for $t > s$*

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t,t_0)} \int_{t_0}^t [KH(t,s)\alpha(s)q(s) - \\ & \frac{1}{(\gamma+1)^{\gamma+1}} \frac{(\alpha^\sigma)^{\gamma+1}r^\gamma(s)D^{\gamma+1}(t,s)}{\alpha^\gamma(s)\delta^\gamma(s,T)H^\gamma(t,s)}] \Delta s = \infty, \end{aligned} \tag{2.33}$$

where $T \geq t_0$, and $D(t,s) := (H(t,s)\alpha^\Delta(s)/\alpha^\sigma) - H^{\Delta_s}(t,s)$, then $x(t)$ is oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

Theorem 2.6. *Assume that $(h_1) - (h_3)$, (1.12) hold, $\tau(t) \geq \sigma(t)$ and $x(t)$ is a solution of (1.1). Let $\phi(t)$ is defined as in Theorem 2.4, $H \in \mathfrak{R}$. If for $t > s$*

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t,t_0)} \int_{t_0}^t [KH(t,s)\phi(s)q(s) - \\ & \frac{r^\gamma(s)((\phi^\Delta(s))^{\gamma+1}(H^{\Delta_s}(t,s))^{\gamma+1})}{(\gamma+1)^{\gamma+1}\delta^\gamma(s,T)\phi^\gamma(s)H^\gamma(t,s)}] \Delta s = \infty, \end{aligned} \tag{2.34}$$

then $x(t)$ is oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

Remark 3. We note that with an appropriate choice of the functions H , ϕ and α one can derive a number of oscillation criteria for equation (1.1) on different types of time scales. For example if the function $H(t, s) = (t - s)^\lambda$, $(t, s) \in \mathbb{D}$ with $\lambda > 1$, then we see that H belongs to the class \mathfrak{R} and then we obtain the Kamenev-type oscillation criteria.

2.2 The case when $\tau(t) \leq t$.

In this subsection, we consider the case when $\tau(t) \leq t$. To prove the main results we will use the class of the functions $\{h_n(t, s)\}_{n=0}^\infty$. These functions are defined recursively by

$$h_0(t, s) = 1, \quad h_{n+1}(t, s) = \int_s^t h_n(\tau, s) \Delta s, \quad n \geq 1. \tag{2.35}$$

A simple formula is when $n = 1$, where $h_1(t, s) = t - s$, but in general this formula does not hold for $n \geq 2$. However, if $\mathbb{T} = \mathbb{R}$, then $h_n(t, s) = (t - s)^n/n!$, if $\mathbb{T} = \mathbb{N}_0$, then $h_n(t, s) = (t - s)^{\underline{n}}/n!$, where $t^{\underline{n}} = t(t - 1) \cdots (t - n + 1)$ is the so-called falling function (cf. Kelley and Peterson [19]), and if $\mathbb{T} = q^{\mathbb{N}_0}$, then $h_n(t, s) = \prod_{\nu=0}^{n-1} (t - q^\nu s) / \sum_{r=0}^\nu q^r$. Note that $h_n(t, s) \leq h_n(t, 0)$.

To prove the main results in this subsection we need the following lemma.

Lemma 2.5. *If*

$$z(t) > 0, \quad z^\Delta(t) > 0, \quad z^{\Delta\Delta}(t) > 0 \text{ and } z^{\Delta\Delta\Delta}(t) \leq 0, \text{ for } t \geq t_0, \tag{2.36}$$

then there exists $T \geq t_0$ such that

$$\frac{z(t)}{z^\Delta(t)} \geq \frac{h_2(t, T)}{(t - T)}, \text{ for } t > T, \tag{2.37}$$

where $h_2(t, T)$ is a polynomial of degree 2 and be defined as in (2.35).

Proof: The proof is similar to the proof of Lemma 2.5 in [23] and hence is omitted. □

Remark 4. From Remark 1, it is clear that if $x(t)$ is a positive solution of (1.1) and belongs to C_2 , then there exists $T \geq t_0$ such that

$$x(t) > 0, \quad x^{[1]}(t) > 0, \quad x^{[2]}(t) > 0 \text{ and } x^{[3]} < 0, \quad \text{for } t \geq T. \tag{2.38}$$

Assuming that $r(t) = 1$, and $p^\Delta(t) \geq 0$, we have from the definitions of $x^{[1]}$ and $x^{[2]}$ that $x^\Delta > 0$ and $x^{\Delta\Delta}(t) > 0$. To apply Lemma 2.5, we put $z(t) = x(t)$. Then we have $z^\Delta(t) = x^{[1]}(t) > 0$ and

$$z^{\Delta\Delta}(t) = \left(x^{[1]}(t)\right)^\Delta = \left(\frac{x^{[2]}(t)}{p(t)}\right)^{\frac{1}{\gamma}} > 0.$$

This implies that $(z^{\Delta\Delta}(t))^\gamma = x^{[2]}(t)/p(t)$. Then, since $p^\Delta(t) \geq 0$, $x^{[2]}(t) > 0$ and $x^{[3]}(t) < 0$, we have

$$\left((z^{\Delta\Delta}(t))^\gamma\right)^\Delta = \frac{p(t)x^{[3]}(t) - p^\Delta(t)x^{[2]}(t)}{p(t)p^\sigma(t)} < 0. \tag{2.39}$$

Using the chain rule when $\gamma \geq 1$

$$(f \circ g)^\Delta(t) = \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))dhg^\Delta(t), \text{ for } 0 < h < 1,$$

we have from (2.39) that

$$0 > \left((z^{\Delta\Delta}(t))^\gamma\right)^\Delta = \gamma z^{\Delta\Delta\Delta}(t) \int_0^1 \left[h((z^{\Delta\Delta}(t))^\sigma + (1-h)(z^{\Delta\Delta}(t)))\right]^{\gamma-1} dh. \tag{2.40}$$

So that $z^{\Delta\Delta\Delta}(t) < 0$ provided we can prove that

$$\gamma \int_0^1 \left[h((z^{\Delta\Delta}(t))^\sigma + (1-h)(z^{\Delta\Delta}(t)))\right]^{\gamma-1} dh > 0.$$

In fact since $0 < h < 1$, we see that $0 < 1-h < 1$ and this implies that

$$\gamma \int_0^1 \left[h((z^{\Delta\Delta}(t))^\sigma + (1-h)(z^{\Delta\Delta}(t)))\right]^{\gamma-1} dh > \left[(z^{\Delta\Delta}(t))^\sigma\right]^\gamma > 0.$$

(Note that the trivial case is when $\mathbb{T} = \mathbb{R}$, $\left((z''(t))^\gamma\right)' = \gamma z'''(t) (z''(t))^{\gamma-1}$). Also by using the chain rule when $0 < \gamma < 1$, we see that

$$0 > \left((z^{\Delta\Delta}(t))^\gamma\right)^\Delta = \gamma z^{\Delta\Delta\Delta}(t) \int_0^1 \frac{1}{\left[h((z^{\Delta\Delta}(t))^\sigma + (1-h)(z^{\Delta\Delta}(t)))\right]^{1-\gamma}} dh.$$

Again we obtain that $z^{\Delta\Delta\Delta}(t) < 0$ provided we can prove that

$$\int_0^1 \frac{1}{\left[h((z^{\Delta\Delta}(t))^\sigma + (1-h)z^{\Delta\Delta}(t))\right]^{1-\gamma}} dh > 0.$$

In this case, since $1-h < 1$, we have

$$\begin{aligned} & \int_0^1 \frac{1}{\left[h((z^{\Delta\Delta}(t))^\sigma + (1-h)(z^{\Delta\Delta}(t)))\right]^{1-\gamma}} dh \\ & > \frac{1}{((z^{\Delta\Delta}(t))^\sigma)^\gamma} \left(\left[1 + \frac{(z^{\Delta\Delta}(t))}{((z^{\Delta\Delta}(t))^\sigma)}\right]^\gamma - \left[\frac{(z^{\Delta\Delta}(t))}{((z^{\Delta\Delta}(t))^\sigma)}\right]^\gamma \right) > 0. \end{aligned}$$

Then the condition (2.36) in Lemma 2.5 holds with $z(t) = x(t)$, and thus

$$\frac{x(t)}{x^\Delta(t)} \geq \frac{h_2(t, T)}{(t - T)}, \text{ for } t > T.$$

This implies, since $x^\Delta(t) > 0$, that

$$x(t) \geq \frac{h_2(t, T)}{(t - T)} x^\Delta(t), \text{ for } t > T.$$

Then since $\lim_{t \rightarrow \infty} \tau(t) = \infty$, we can choose $T_1 \geq T$ such that

$$x^\tau \geq \frac{h_2(\tau, T)}{(\tau - T)} (x^\Delta)^\tau, \text{ for } t > T_1. \tag{2.41}$$

Now, we are ready to prove the main results for the delay case. We will use the notations of Theorem 2.1 and also introduce the following:

$$A(t) := Kq(t) \left(\frac{p^{\frac{1}{\gamma}}(t)P(\tau(t), T)}{p^{\frac{1}{\gamma}}(t)P(t, T) + \mu(t)} \right)^\gamma \eta^\gamma(\tau(t)), \text{ where } \eta(\tau(t)) := \frac{h_2(\tau(t), T)}{(\tau(t) - T)}.$$

We start with an oscillation criterion of Leighton-Wintner type .

Theorem 2.7. Assume that $(h_1) - (h_3)$, (1.12) hold and $x(t)$ is a solution of (1.1). Furthermore, assume that $r(t) = 1$, and $p^\Delta(t) \geq 0$. If

$$\int_{t_0}^\infty A(s) \Delta s = \infty, \tag{2.42}$$

then $x(t)$ is oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof: Suppose to the contrary and assume that $x(t)$ is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_2$ where t_2 is chosen so large. We consider only this case, because the proof when $x(t) < 0$ is similar, since $uf(u) > 0$ for $u \neq 0$. Now, since (1.12) holds, we see from Lemma 2.2 that $x(t) \in C_0 \cup C_2$. If $x(t) \in C_0$, then by Lemma 2.3, we have $\lim_{t \rightarrow \infty} x(t) = 0$. Next, we assume that $x(t) \in C_2$ and define the function $u(t)$ by the Riccati substitution

$$u(t) := \frac{x^{[2]}}{(x^{[1]})^\gamma}. \tag{2.43}$$

Then $u(t) > 0$ and by the quotient rule [5, Theorem 1.20] and proceeding as in the proof of Theorem 2.1, to get

$$u^\Delta(t) \leq -Kq(t) \left(\frac{x(\tau)}{(x^{[1]})^\sigma} \right)^\gamma - \gamma \frac{1}{p^{\frac{1}{\gamma}}(t)} (u^\sigma)^{1+\frac{1}{\gamma}}. \tag{2.44}$$

Now, we consider the coefficient of $q(t)$ in (2.44). Setting $y = x^{[1]} = x^\Delta$, we have

$$y > 0, \quad y^\Delta > 0, \quad \text{and} \quad \left(p [y^\Delta]^\gamma \right)^\Delta < 0. \tag{2.45}$$

Then the coefficient of $q(t)$ can be written in the form

$$\left(\frac{x^\tau}{(x^{[1]})^\sigma} \right)^\gamma = \left(\frac{x^\tau}{y^\tau} \right)^\gamma \left(\frac{y^\tau}{y^\sigma} \right)^\gamma. \tag{2.46}$$

Since $y^{[1]}(t) = p (y^\Delta)^\gamma(t)$ is decreasing for $t \geq T$, then we have

$$y^\sigma(t) - y(\tau(t)) = \int_{\tau(t)}^{\sigma(t)} \frac{y^{[1]}(s)}{p^{\frac{1}{\gamma}}(s)} \Delta s \leq y^{[1]}(\tau(t)) \int_{\tau(t)}^{\sigma(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)} \Delta s,$$

and this implies that

$$\frac{y^\sigma(t)}{y(\tau(t))} \leq 1 + \frac{y^{[1]}(\tau(t))}{y(\tau(t))} \int_{\tau(t)}^{\sigma(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)} \Delta s. \tag{2.47}$$

On the other hand, we have that

$$y(\tau(t)) > y(\tau(t)) - y(T) = \int_T^{\tau(t)} \frac{y^{[1]}(s)}{p^{\frac{1}{\gamma}}(s)} \Delta s \geq (y^{[1]})(\tau(t)) \int_T^{\tau(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)} \Delta s,$$

which leads to

$$\frac{y^{[1]}(\tau(t))}{y(\tau(t))} < \left(\int_T^{\tau(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)} \Delta s \right)^{-1}.$$

Using the last inequality in (2.47), we obtain for $t \geq T$ that

$$\frac{y^\sigma(t)}{y(\tau(t))} < 1 + \frac{\int_{\tau(t)}^{\sigma(t)} p^{-\frac{1}{\gamma}}(s) \Delta s}{\int_T^{\tau(t)} p^{-\frac{1}{\gamma}}(s) \Delta s} = \frac{\int_T^{\sigma(t)} p^{-\frac{1}{\gamma}}(s) \Delta s}{\int_T^{\tau(t)} p^{-\frac{1}{\gamma}}(s) \Delta s} = \frac{\int_T^t p^{-\frac{1}{\gamma}}(s) \Delta s + \mu(t) p^{-\frac{1}{\gamma}}(t)}{\int_T^{\tau(t)} p^{-\frac{1}{\gamma}}(s) \Delta s}.$$

Hence, we get

$$y(\tau(t)) \geq \frac{p^{\frac{1}{\gamma}}(t) P(\tau(t), T)}{p^{\frac{1}{\gamma}}(t) P(t, T) + \mu(t)} y^\sigma(t), \quad \text{for } t \geq T,$$

which implies that

$$\frac{(y^\tau(t))^\gamma}{(y^\sigma(t))^\gamma} \geq \left(\frac{p^{\frac{1}{\gamma}}(t) P(\tau(t), T)}{p^{\frac{1}{\gamma}}(t) P(t, T) + \mu(t)} \right)^\gamma, \quad \text{for } t \geq T. \tag{2.48}$$

Then, from (2.46), (2.48) and (2.41), we have

$$\left(\frac{x^\tau}{(x^{[1]})^\sigma}\right)^\gamma \geq \left(\frac{p^{\frac{1}{\gamma}}(t)P(\tau(t), T)}{p^{\frac{1}{\gamma}}(t)P(t, T) + \mu(t)}\right)^\gamma \eta^\gamma(\tau(t)), \text{ for } t \geq T. \tag{2.49}$$

Substituting (2.49) into (2.44), we have the inequality

$$u^\Delta(t) + A(t) + \gamma \frac{1}{p^{\frac{1}{\gamma}}(t)} (u^\sigma)^{1+\frac{1}{\gamma}}(t) \leq 0, \text{ for } t \geq T. \tag{2.50}$$

The remainder of the proof is similar to the proof of Theorem 2.1 and hence is omitted. The proof is complete. \square

Remark 5. (1). In the definition of $Q(t)$ there exists a function $R(t, \tau)$. This function is replaced by the function $\eta(\tau(t))$ in $A(t)$ which includes the function $h_2(\tau, T)$.

(2). We note the difference between the condition $\int_{t_0}^\infty q(s)\Delta s = \infty$ that has been proved by Yu and Wang [26, Corollary 2.1] and the condition $\int_{t_0}^\infty A(s)\Delta s = \infty$ of Theorem 2.7.

In the following, we establish some sufficient conditions for oscillation of (1.1) and since the proofs are similar to the proofs of theorems in the subsection 2.1 by using the inequality (2.50), we omitted the details. We assume that

$$\int_{t_0}^\infty A(s)\Delta s < \infty.$$

Theorem 2.8. Assume that $(h_1) - (h_3)$, (1.12) hold and let $x(t)$ be a solution of (1.1). Furthermore assume that $r(t) = 1$, $p^\Delta(t) \geq 0$. If there exists positive rd-continuous Δ -differentiable function $\phi(t)$ such that

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\phi(s)A(s) - \frac{p(s)(\phi^\Delta(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^\gamma(s)} \right] \Delta s = \infty, \tag{2.51}$$

then $x(t)$ is oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

Theorem 2.9. Assume that $(h_1) - (h_3)$, (1.12) hold and let $x(t)$ be a solution of (1.1). Assume that $r(t) = 1$, $p^\Delta(t) \geq 0$, and let $H \in \mathfrak{R}$. If for $t > s$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s)A(s) - \frac{p(s)(H^{\Delta_s}(t, s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}H^\gamma(t, s)} \right] \Delta s = \infty, \tag{2.52}$$

then $x(t)$ is oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

3 Oscillation of (1.1) when (1.13) holds

Throughout this section, we assume that (1.13) holds and start with the following lemma.

Lemma 3.1. *Assume that $(h_1) - (h_2)$, (1.13) hold. Let $x(t)$ be a nonoscillatory solution of (1.1). If*

$$\int_T^\infty \frac{1}{r(s)} \int_T^s \left(\frac{1}{p(u)} \right)^{\frac{1}{\gamma}} \Delta u \Delta s = \infty, \tag{3.1}$$

then C_3 is empty.

Proof: The proof is similar to the proof of Lemma 2.2 in [23] and hence is omitted. □

Remark 6. *According to Lemma 3.1, if (1.13) and (3.1) hold, then the class C_3 is empty and then the solution $x(t)$ of (1.1) belongs to $C_0 \cup C_1 \cup C_2$.*

First, we consider the case when $\tau(t) > t$.

Theorem 3.1. *Assume that $(h_1) - (h_3)$, (1.13), (3.1) hold and $\tau(t) > t$. Furthermore, assume that (2.1) holds and there exist $T > t_0$ such that $\tau(t) > T$ and $x(t)$ be a solution of (1.1). If*

$$\int_T^\infty \frac{1}{r(\vartheta)} \int_T^\vartheta \frac{1}{p(s)} \left[\int_T^s B(u) \left(\int_{\tau(u)}^\infty \frac{1}{p(\theta)} \Delta \theta \right)^\gamma \Delta u \right]^{\frac{1}{\gamma}} \Delta s \Delta \vartheta = \infty, \tag{3.2}$$

where $B(t) := Kq(t) \left(\int_T^{\tau(t)} \frac{\Delta s}{r(s)} \right)^\gamma$. Then either $x(t)$ oscillates or $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof: The proof is similar to the proof of Theorem 4.1 in [23] and hence is omitted. □

Theorem 3.2. *Assume that $(h_1) - (h_3)$, (1.13), (3.1) hold and $\tau(t) > t$. Furthermore, assume that (2.16) holds and there exists $T > t_0$ such that $\tau(t) > T$ and (3.2) holds. If $x(t)$ is a solution of (1.1), then either $x(t)$ oscillates or $\lim_{t \rightarrow \infty} x(t) = 0$.*

Proof: Suppose the contrary and assume that $x(t)$ is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_1$ where t_1 is chosen so large. We consider only this case, because the proof when $x(t) < 0$ is similar, since $uf(u) > 0$. If $x(t) \in C_0$, then we have from Lemma 2.3 that $\lim_{t \rightarrow \infty} x(t) = 0$. If $x(t) \in C_2$ then we proceed as in the proof of Theorem 2.2 to get a contradiction. If $x(t) \in C_1$, then we are back to the proof of Theorem 3.1 to get a contradiction. The proof is complete. □

Remark 7. One can obtain similar results as in Theorems 3.1 and 3.2 by using the conditions in Theorems 2.3, 2.4 and 2.5 and due to the limited space the details are left to the interested reader.

Next, we consider the case when $\tau(t) \leq t$. For the delay case, we proceed as in the proofs of Theorems 3.1-3.2 to prove the following results. The details are omitted.

Theorem 3.3. Assume that $(h_1)-(h_3)$, (1.13), (3.1) hold, $r(t) = 1, p^\Delta(t) \geq 0$ and $\tau(t) \leq t$. Furthermore, assume that (2.42) or (2.51) holds and there exists $T > t_0$ such that $\tau(t) > T$ and (3.2) holds. If $x(t)$ is a solution of (1.1), then either $x(t)$ oscillates or $\lim_{t \rightarrow \infty} x(t) = 0$.

Theorem 3.4. Assume that $(h_1)-(h_3)$, (1.13), (3.1) hold, $r(t) = 1, p^\Delta(t) \geq 0$ and $\tau(t) \leq t$. Furthermore, assume that (2.52) holds and there exists $T > t_0$ such that $\tau(t) > T$ and (3.2) holds. If $x(t)$ is a solution of (1.1), then either $x(t)$ oscillates or $\lim_{t \rightarrow \infty} x(t) = 0$.

Remark 8. We note that the condition (3.2) can be replaced by one of the following conditions:

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[sB(s) \left(\frac{\tau(s)}{\sigma(s)} \right)^\gamma - \frac{p(s)}{s^{\gamma(\gamma+1)\gamma+1}} \right] \Delta s = \infty, \tag{3.3}$$

$$\liminf_{t \rightarrow \infty} t^\gamma \int_{\sigma(t)}^\infty P(s) \Delta s > \frac{\gamma^\gamma}{l^{\gamma^2(\gamma+1)\gamma+1}}, \tag{3.4}$$

$$\liminf_{t \rightarrow \infty} t^\gamma \int_{\sigma(t)}^\infty P(s) \Delta s + \liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t s^{\gamma+1} P(s) \Delta s > \frac{1}{l^{\gamma(\gamma+1)}}, \tag{3.5}$$

where $P(t) = \left(\frac{\tau(t)}{\sigma(t)} \right)^\gamma B(t), l := \liminf_{t \rightarrow \infty} \frac{t}{\sigma(t)}$ and $B(t)$ is defined as in Theorem 3.1.

4 Examples

In this section, we give some examples to illustrate the main results. To obtain the conditions for oscillation we will use the following facts: $\int_{t_0}^\infty \frac{\Delta s}{s^\nu} = \infty$, if $0 \leq \nu \leq 1$, and $\int_{t_0}^\infty \frac{\Delta s}{s^\nu} < \infty$, if $\nu > 1$. For more details, we refer the reader to ([5, Theorem 5.68 and Corollary 5.71]). First, we give some examples to illustrate the main results when $\tau(t) > t$.

Example 1. Consider the dynamic equation

$$x^{\Delta\Delta\Delta}(t) + \frac{\alpha}{t^2} x(2t) = 0, \text{ for } t \in [1, \infty)_{\mathbb{T}}, \tag{4.1}$$

on a time scale \mathbb{T} such that $\int_{t_0}^{\infty} \frac{1}{\sigma(s)} \Delta s = \infty$ and $2t \in \mathbb{T}$. Here $\gamma = 1$, $K = 1$, $r(t) = p(t) = 1$, $q(t) = \alpha/t^2$ and $\tau(t) = 2t > t$. So that

$$P(t, T) = \int_1^t \left(\frac{1}{p(\tau)} \right)^{\frac{1}{\gamma}} \Delta \tau = (t - 1), \quad R(2t, t) = \int_t^{2t} \frac{1}{r(s)} \Delta s = t.$$

Thus

$$Q(t) = q(t) \left(\frac{R(2t, t)p^{\frac{1}{\gamma}}(t)P(t, 1)}{p^{\frac{1}{\gamma}}(t)P(t, 1) + \sigma(t) - t} \right)^{\gamma} = \frac{\alpha}{t^2} \left(\frac{t(t - 1)}{\sigma(t) - 1} \right).$$

Now, we apply Theorem 2.1. In this case it is clear that the conditions $(h_1) - (h_3)$, and (1.12) hold. It remains to prove that (2.1) is satisfied. In this case, note that

$$\begin{aligned} \int_{t_0}^{\infty} Q(s) \Delta s &= \int_1^{\infty} \frac{\alpha}{s^2} \left(\frac{s(s - 1)}{\sigma(s) - 1} \right) \Delta s = \int_1^{\infty} \frac{\alpha}{s} \left(\frac{s - 1}{\sigma(s) - 1} \right) \Delta s \\ &\geq \int_1^{\infty} \frac{\alpha}{s} \left(\frac{s - 1}{\sigma(s)} \right) \Delta s = \int_1^{\infty} \left(\frac{\alpha}{\sigma(s)} - \frac{\alpha}{s\sigma(s)} \right) \Delta s = \infty. \end{aligned}$$

Then (2.1) is satisfied and by Theorem 2.1 the solution $x(t)$ of (4.1) oscillates or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$. In fact if $\mathbb{T} = \mathbb{R}$ and $\alpha = 3/8 > 1/4$, the equation reduces to the advanced differential equation

$$x'''(t) + \frac{3}{8t^3}x(2t) = 0, \quad \text{for } t \geq 1.$$

Setting $\theta = 2t$ the equation reduces to the third order differential equation

$$y'''(\theta) + \frac{24}{\theta^3}y(\theta) = 0, \quad \text{for } \theta \in [2, \infty),$$

and the basis of the solution is given by $\{\theta^{-2}, \theta^{\frac{5}{2}} \cos \frac{1}{2}\sqrt{23} \ln \theta, \theta^{\frac{5}{2}} \sin \frac{1}{2}\sqrt{23} \ln \theta\}$.

Example 2. We consider $\mathbb{T} = \mathbb{R}$ and consider the differential equation

$$\left(t \left(\frac{1}{t} x'(t) \right)' \right)' + \frac{\beta}{t^2} x(\tau(t)) = 0, \quad \text{for } t \geq 1. \tag{4.2}$$

Here $\gamma = 1$, $K = 1$, $r(t) = 1/t$, $p(t) = t$, $q(t) = \beta/t^2$ and $\tau(t) > t$. Now, we apply Theorem 2.4 when $\mathbb{T} = \mathbb{R}$. In this case it is clear that the conditions $(h_1) - (h_2)$ hold, and

$$\delta(t, T) := \int_1^t \frac{ds}{p^{\frac{1}{\gamma}}(s)} = \int_1^t \frac{ds}{s} \geq \int_1^t \frac{ds}{s^2} = \int_1^t \left(\frac{-1}{s} \right)' ds = \frac{t - 1}{t}.$$

It remains to prove that the conditions (h_3) and (2.21) hold. In this case the condition (h_3) reads

$$\int_{t_0}^{\infty} r^{-1}(t) \int_t^{\infty} \left(p^{-1}(u) \int_u^{\infty} q(s) ds \right)^{\frac{1}{\gamma}} dudt = \int_1^{\infty} t \int_t^{\infty} \left(\frac{1}{u} \int_u^{\infty} \frac{\beta}{s^2} ds \right) dudt = \infty.$$

Then (h_3) is satisfied. Now, we discuss (2.21). If we choose $\phi(t) = t$ and $\alpha(t) = 1$, we have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\phi(s)q(s) - \frac{r(s)(\phi'(s))^2}{(\gamma + 1)^{\gamma+1}\phi(s)} \right] ds \\ & \geq \limsup_{t \rightarrow \infty} \int_1^t \left[s \frac{\beta}{s^2} - \frac{1}{4} \frac{s-1}{s^2} \right] ds = \infty, \text{ for } \beta > 1/4. \end{aligned}$$

Then the conditions of Theorem 2.4 are satisfied provided that $\beta > 1/4$. Then the solution $x(t)$ of (4.2) is oscillatory or converges to zero.

In the following, we give some examples to illustrate the main results when $\tau(t) \leq t$.

Example 3. Consider the third order dynamic equation

$$x^{\Delta\Delta\Delta}(t) + \frac{\beta(\sigma(t) - 1)}{\tau^2(t)h_2(\tau(t), 1)}x(\tau(t)) = 0, \quad \text{for } t \in [1, \infty)_{\mathbb{T}}, \quad (4.3)$$

on a time scale \mathbb{T} . Here $p(t) = 1$, $r(t) = 1$, $\gamma = 1$, $K = 1$, $\tau(t) \leq t$, such that $\lim_{t \rightarrow \infty} \tau(t) = \infty$, and $q(t) = \beta(\sigma(t) - 1)/(\tau^2(t)h_2(\tau(t), 1))$. So that

$$A(t) := \frac{\beta(\sigma(t) - 1)}{\tau^2(t)} \frac{1}{\sigma(t) - 1} = \frac{\beta}{\tau^2(t)}.$$

Now, we apply Theorem 2.8. It is clear that $(h_1) - (h_3)$, (1.12) hold. It remains to prove that (2.51) holds. Note

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\phi(s)A(s) - \frac{p(s)(\phi^\Delta(s))^{\gamma+1}}{(\gamma + 1)^{\gamma+1}\phi^\gamma(s)} \right] \Delta s \\ & = \limsup_{t \rightarrow \infty} \int_{t_0}^t \left[s \frac{\beta}{\tau^2(s)} - \frac{1}{4s} \right] \Delta s = \limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\frac{\beta s}{\tau^2(s)} - \frac{1}{4s} \right] \Delta s \\ & > \limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\frac{\beta}{s} - \frac{1}{4s} \right] \Delta s = \infty, \text{ if } \beta > 1/4. \end{aligned}$$

Then (2.51) is satisfied and then by Theorem 2.8, the solution $x(t)$ of (4.3) is oscillatory or converges to zero if $\beta > 1/4$.

Example 4. To prove the main results that we have proved are sharp we consider the third order differential equation

$$x'''(t) + \frac{1}{\sqrt{3}t^3}x(t) = 0, \quad t \geq 1. \quad (4.4)$$

Here $q(t) = \frac{1}{\sqrt{3}t^3}$ and using the definition of $h_2(t, 1) = \frac{(t-1)^2}{2}$, we see that $A(t) = \frac{1}{2\sqrt{3}t^2} - \frac{1}{2\sqrt{3}t^3}$. We apply Theorem 2.8. It is clear that $(h_1) - (h_3)$, (1.12) hold

and $\int_1^\infty A(s)ds < \infty$. It remains to prove that (2.51) holds. In this case if we choose $\phi(t) = \frac{8}{3\sqrt{3}}t$, we see that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\phi(s)A(s) - \frac{p(s)(\phi'(s))^2}{4\phi(s)} \right] ds \\ &= \limsup_{t \rightarrow \infty} \int_1^t \left[\frac{4}{9s} - \frac{2}{3\sqrt{3}s} \right] ds = \limsup_{t \rightarrow \infty} \int_1^t \frac{5.9544 \times 10^{-2}}{s} ds = \infty. \end{aligned}$$

Then (2.51) is satisfied and then by Theorem 2.8, the solution $x(t)$ of (4.4) is oscillatory or converges to zero. For more illustration and prove that the results are sharp, we apply the results by Škerlik [24] on (4.4). In this case the Škerlik condition reads

$$\int_{t_0}^\infty \left(s^2 q(s) - \frac{2}{3\sqrt{3}s} \right) ds = \int_1^\infty \left(\frac{3s^2}{3\sqrt{3}s^3} - \frac{2}{3\sqrt{3}s} \right) ds = \int_1^\infty \frac{0.19245}{s} ds = \infty.$$

Note the difference between our results and the results that have been established by Škerlik [24] for differential equations. The roots of the characteristic equation of (2.46) are given by $m = 1.6073 - 0.32634i$, $m = 1.6073 + 0.32634i$, and $m = -0.21463$. This means that the results in this paper are sharp and cannot be weakened.

Remark 9. We mentioned here that none of the results established in [12], [13], [14], [15] and [26] can be applied on (4.1)-(4.4).

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