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Sequentially Cohen-Macaulay path ideals of cycles

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Abstract

Let $R = k[x_1, \ldots, x_n]$, where k is a field. The path ideal (of length $t \geq 2$) of a directed graph G is the monomial ideal, denoted by $I_t(G)$, whose generators correspond to the directed paths of length t in G. Let C_n be an n-cycle. We determine when $I_t(C_n)$ is unmixed. Moreover, We show that $R/I_t(C_n)$ is sequentially Cohen-Macaulay if and only if $n = t$ or $t + 1$ or $2t + 1$.

Key Words: Path ideals, sequentially Cohen-Macaulay.
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1 Introduction

The path ideal of a graph was first introduced by Conca and De Negri in [3]. Fix an integer $n \geq t \geq 2$ and let G be a directed graph. A sequence $x_{i_1}, \ldots, x_{i_t}$ of distinct vertices, is called a **path** of length t if there are $t - 1$ distinct directed edges $e_1, \ldots, e_{t-1}$ where e_j is a directed edge from x_{i_j} to $x_{i_{j+1}}$. Then the **path ideal** of G of length t is the monomial ideal

$$I_t(G) = (x_{i_1} \cdots x_{i_t} : x_{i_1}, \ldots, x_{i_t} \text{ is a path of length t in G})$$

in the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k. In [3], it is shown that the Rees algebra $\mathcal{R}(I_t(G))$ is normal and Cohen-Macaulay, when G is a directed tree. In [7], $R/(I_t(G))$ is shown to be sequentially Cohen-Macaulay when G is a directed tree. Moreover, in [1], it is shown that the path ideals of cycles have linear type and in [8], the path ideals of complete bipartite graphs are shown to be normal.

In this paper, we study some properties of the path ideals of cycles. Throughout the paper, we mean by C_n, the n-cycle with directed edges $e_1, \ldots, e_n$, where e_i is from x_i to x_{i+1} for $i = 1, \ldots, n-1$ and e_n is from x_n to x_1. In addition, we have $I_t(C_n) = (u_1, \ldots, u_n)$, where $u_i = \prod_{v=0}^{t-1} x_{i+v}$ for all $i = 1, \ldots, n$. Note that here the indices are considered in $\mathbb{Z}_n$. In [4, Proposition 4.1], the authors focused on the case $t = 2$ and determined when $R/I_2(C_n)$ is sequentially Cohen-Macaulay. Here we consider all $t > 2$ and study sequential Cohen-Macaulayness of $R/I_t(C_n)$ in general.

This paper is organized as follows. In the next section, we recall several definitions and terminology which we need later. In section 3, we prove that $I_t(C_n)$ is unmixed if and only if $t \leq n \leq \lfloor 3t/2 \rfloor + 1$ or $n = 2t + 1$. In section 4, we show that $R/I_t(C_n)$ is sequentially Cohen-Macaulay if and only if $n = t$ or $t + 1$ or $2t + 1$. To prove this, we use Alexander duality. Actually, we prove that just in these three cases, the Alexander dual of the path ideal is componentwise linear. Finally, in section 5, we deal with a conjecture of Bruns and Hibi. We present an example to show that the conjecture is not true.

2 Preliminaries

A **simplicial complex** Δ on the vertex set $V = \{x_1, \ldots, x_n\}$ is a collection of subsets of V such that if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.
An element in Δ is called a **face** of Δ, and $F \in \Delta$ is said to be a **facet** if F is maximal with respect to inclusion. Let $F_1, \ldots, F_q$ be all the facets of simplicial complex Δ. We sometimes write $\Delta = \langle F_1, \ldots, F_q \rangle$.
For a subset W of V the **restriction** of Δ on W is the subcomplex

$$\Delta_W = \{F \in \Delta : F \subseteq W\}.$$

The **dimension** of a face F is $|F| - 1$. Let $d = \max\{|F| \ : \ F \in \Delta\}$. Then the **dimension** of Δ, denoted by $\dim(\Delta)$, is $d - 1$.
We say that Δ is **pure** if all its facets have the same dimension.
The **facet ideal** of Δ is

$$I(\Delta) = (\prod_{x \in F} x : F \text{ is a facet of } \Delta).$$

Now we define the simplicial complex $\Delta_t(G)$ to be

$$\Delta_t(G) = \langle \{x_{i_1}, \ldots, x_{i_t}\} : x_{i_1}, \ldots, x_{i_t} \text{ is a path of length t in } G \rangle,$$

where G is a directed graph. So we have $I_t(G) = I(\Delta_t(G))$.
The **Stanley-Reisner ideal** of Δ is the monomial ideal

$$I_\Delta = (\prod_{x \in F} x : F \notin \Delta).$$

The **Stanley-Reisner ring** of Δ is $k[\Delta] = R/I_\Delta$.
The **Alexander dual** of Δ is the simplicial complex

$$\Delta^\vee = \{F^c : F \notin \Delta\}.$$

Let I be a squarefree monomial ideal. The **squarefree Alexander dual** of $I = (x_{1,1} \cdots x_{1,s_1}, \ldots, x_{t,1} \cdots x_{t,s_t})$ is the ideal

$$I^{\vee} = (x_{1,1}, \ldots, x_{1,s_1}) \cap \cdots \cap (x_{t,1}, \ldots, x_{t,s_t}).$$

Let $\Delta = \langle F_1, \ldots, F_q \rangle$. A **vertex cover** of Δ is a subset A of V, with the property that for every facet F_i there is a vertex $x_j \in A$ such that $x_j \in F_i$. A **minimal vertex cover** of Δ is a subset A of V such that A is a vertex cover and no proper subset of A is a vertex cover of Δ.
A simplicial complex Δ is **unmixed** if all of its minimal vertex covers have the same cardinality.

A graded R-module M is called sequentially Cohen-Macaulay (over k) if there exists a finite filtration of graded R-modules $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing, i.e.

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

Let I be a homogeneous ideal in R. Associated to R/I is a minimal free graded resolution of the form

$$0 \to \bigoplus_j R(-j)^{\beta_{p,j}(R/I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{1,j}(R/I)} \to R \to R/I \to 0$$

where $p \leq n$ and $R(-j)$ is the R-module obtained by shifting the degrees of R by j. The number $\beta_{i,j}$, the ij-th graded Betti number of M, is an invariant of R/I that equals the number of minimal generators of degree j in the i-th syzygy module. The projective dimension of R/I, denoted $\mathrm{pd}(R/I)$, is equal to p, the minimal length of all free resolutions of R/I. Suppose I is a homogeneous ideal of R whose generators all have degree d. Then R/I has a d**-pure resolution** if its minimal graded free resolution can be written in the form

$$0 \to R(-d_p)^{\beta_p(R/I)} \to \cdots \to R(-d_1)^{\beta_1(R/I)} \to R \to R/I \to 0,$$

where $d = d_1$. In addition, we say that R/I has a d**-linear resolution** if for all $i \geq 1$, $\beta_{i,j}(R/I) = 0$ for all $j \neq i + d - 1$. The **resolution degree** of R/I is defined to be $d_p - d_{p-1}, \ldots, d_2 - d_1, d = d_1$.

Theorem 2.1. *[4, Lemma 3.6] Let I be a squarefree monomial ideal in $R = k[x_1, \ldots, x_n]$. Then R/I is Cohen-Macaulay if and only if R/I is sequentially Cohen-Macaulay and I is unmixed.*

If I is a graded ideal of R, then we write $I_{\langle j \rangle}$ for the ideal generated by all homogeneous polynomials of degree j belonging to I. We say that a graded ideal $I \subset R$ is componentwise linear if $I_{\langle j \rangle}$ has a linear resolution for all j. Also, we write $I_{[j]}$ for the ideal generated by the squarefree monomials of degree j belonging to I. We say that I is squarefree componentwise linear if $I_{[j]}$ has a linear resolution for all j. Herzog and Hibi showed:

Proposition 2.2. *[5, Proposition 1.5] Suppose that $I \subset R$ is an ideal generated by squarefree monomials. Then I is componentwise linear if and only if I is squarefree componentwise linear.*

Theorem 2.3. *[5, Theorem 2.1] Let I be a squarefree monomial ideal of R. Then R/I is sequentially Cohen-Macaulay if and only if $I^\vee$ is componentwise linear.*

3 Unmixedness of the path ideal of C_n

In this section, we determine when $I_t(C_n)$ is unmixed.

Theorem 3.1. *Let $t \geq 3$. Then $I_t(C_n)$ is unmixed if and only if $t \leq n \leq \lfloor 3t/2 \rfloor + 1$ or $n = 2t+1$.*

We need the following lemmas to prove the above theorem.

First, note that the path ideal of length t of an n-cycle can be viewed as a Stanley-Reisner ideal of a simplicial complex.
Suppose that $\Delta_{n,t}$ is a simplicial complex such that $I_t(C_n) = I_{\Delta_{n,t}}$. Let $\Delta := \Delta_{n,t}$, $d_1 := \min\{|F| : F \text{ is a facet of } \Delta\}$ and $d_2 := \max\{|F| : F \text{ is a facet of } \Delta\}$. Note that Δ has no face containing t consecutive vertices.
Then we have:

Lemma 3.2. *Let $t \geq 3$. Then*
(i) If $n = q_1(t+1) + r_1$, where $0 \leq r_1 \leq t$, then

$$d_1 \leq \begin{cases} q_1(t-1) + r_1 & if \quad r_1 \leq t-2 \\ q_1(t-1) + r_1 - 1 & if \quad r_1 = t-1 \text{ or } t. \end{cases}$$

(ii) If $n = q_2 t + r_2$, where $0 \leq r_2 \leq t-1$, then $d_2 = q_2(t-1) + \max(r_2 - 1, 0)$.

Proof: (i) Let

$$d_1' := \begin{cases} q_1(t-1) + r_1 & \text{if} \quad r_1 \leq t-2 \\ q_1(t-1) + r_1 - 1 & \text{if} \quad r_1 = t-1 \text{ or } t, \end{cases}$$

and $W := \{x_1, x_t, x_{t+2}, x_{2t+1}, \ldots, x_{(q_1-1)(t+1)+1}, x_{(q_1-1)(t+1)+t}\}$. So, it is easy to see that

$$F := \begin{cases} V \setminus W & \text{if} \quad r_1 \leq t-2 \\ V \setminus (W \cup \{x_{q_1(t+1)+1}\}) & \text{if} \quad r_1 = t-1 \text{ or } t \end{cases}$$

is a facet of Δ with $|F| = d_1'$, since we set F such that it does not contain any t consecutive vertices and also we can not add any other vertex to it. Thus $d_1 \leq d_1'$.

(ii) Let $d_2' := q_2(t-1) + \max(r_2 - 1, 0)$ and $W := \{x_t, x_{2t}, \ldots, x_{q_2 t}\}$. Thus

$$F := \begin{cases} V \setminus W & \text{if} \quad r_2 = 0 \\ V \setminus (W \cup \{x_n\}) & \text{if} \quad r_2 \neq 0 \end{cases}$$

is a facet of Δ with $|F| = d_2'$, similar to part (i). Therefore $d_2 \geq d_2'$. Now suppose that there exists a face in Δ, say H, such that $|H| = d_2' + 1$. So $n - d_2' - 1$ other vertices do not appear in H. Thus there exists a face in Δ which contains $\lceil \frac{d_2'+1}{n-d_2'-1} \rceil$ consecutive vertices. We show that $\lceil \frac{d_2'+1}{n-d_2'-1} \rceil \geq t$, which is a contradiction, by definition of Δ. Then we deduce that there does not exist such face, H, in Δ and hence $d_2 = d_2'$, as desired.
We have

$$n - d_2' - 1 = \begin{cases} q_2 - 1 & \text{if} \quad r_2 = 0 \\ q_2 & \text{if} \quad r_2 \geq 1. \end{cases}$$

Now consider the following cases:

Case1. Let $r_2 = 0$. If $q_2 = 1$, then $n = t$ and so $d_2' = d_2$. Now suppose that $q_2 \neq 1$. Then

$\lceil \frac{d_2'+1}{n-d_2'-1} \rceil = \lceil \frac{q_2(t-1)+1}{q_2-1} \rceil = (t-1) + \lceil \frac{t}{q_2-1} \rceil \geq t$, since $\lceil \frac{t}{q_2-1} \rceil \geq 1$.

Case2. Let $r_2 \neq 0$. Then

$\lceil \frac{d_2'+1}{n-d_2'-1} \rceil = \lceil \frac{q_2(t-1)+r_2}{q_2} \rceil = (t-1) + \lceil \frac{r_2}{q_2} \rceil \geq t$, since $\frac{r_2}{q_2} > 0$. □

Note that throughout this section we use the above notations.

Remark 3.3. In Lemma 3.2, the assumption $t \geq 3$ is necessary, since the result is not true for the case $t = 2$. Consider the 4-cycle, then $\Delta = \Delta_{4,2}$ and we have $d_1' = 1$, but obviously $d_1 = 2$, so $d_1 > d_1'$.

Lemma 3.4. *Let $t \geq 3$. If $I_t(C_n)$ is unmixed, then $t \leq n \leq 2t-1$ or $n = 2t+1$.*

Proof: Suppose that $I_t(C_n)$ is unmixed. Then Δ is a pure simplicial complex, since $I_t(C_n) = I_\Delta$. So, $d_1 = d_2$. Moreover, we have $d_1 \leq d_1' \leq d_2$, thus

$$d_1' = d_2. \tag{3.1}$$

By the notations in Lemma 3.2,

$$n = q_1(t+1) + r_1 = q_2 t + r_2. \tag{3.2}$$

From (3.1), (3.2) and easy computation, we can see that $q_1 \leq q_2$. Thus there exists a nonnegative integer α such that $q_2 = q_1 + \alpha$. By Lemma 3.2 and (3.1), we have $\alpha = 0$ or 1. So, consider the following cases:

Case1. If $q_2 = q_1$, then by (3.2), we have $q_1 + r_1 = r_2$. Because $q_1 = q_2 \geq 1$, so $r_2 > r_1 \geq 0$. Also (3.1) yields

$$\max(r_2 - 1, 0) = \begin{cases} r_1 & \text{if} \quad r_1 \leq t-2 \\ r_1 - 1 & \text{if} \quad r_1 = t-1 \;\; or \;\; t, \end{cases} \tag{3.3}$$

but $r_2 > 0$, so $\max(r_2-1, 0) = r_2-1 > r_1-1$. Then by (3.3), $\max(r_2-1, 0) = r_1$, hence $r_2 = r_1+1$. So, by (3.2), we have $q_1 = q_2 = 1$. Thus $t+1 \leq n = t+1+r_1 \leq t+1+(t-2) = 2t-1$.

Therefore

$$t + 1 \leq n \leq 2t - 1.$$

Case2. If $q_2 = q_1+1$, then $n = q_1(t+1)+r_1 = (q_1+1)t+r_2$, by (3.2). Hence

$$q_1 + r_1 = t + r_2. \tag{3.4}$$

Also from (3.1), we get

$$(t-1) + \max(r_2 - 1, 0) = \begin{cases} r_1 & \text{if} \quad r_1 \leq t-2 \\ r_1 - 1 & \text{if} \quad r_1 = t-1 \;\; or \;\; t. \end{cases}$$

But only the case $r_1 = t$ occurs. So $\max(r_2 - 1, 0) = 0$. Thus $r_2 = 0$ or 1. If $r_2 = 0$, then $q_1 = 0$ and $q_2 = 1$, by (3.4). Hence $n = t$. If $r_2 = 1$, then $q_1 = 1$ and $q_2 = 2$, by (3.4). Thus $n = 2t + 1$.

□

Lemma 3.5. *Let $n = 2t + 1$. Then $R/I_t(C_n)$ is Cohen-Macaulay, and hence $I_t(C_n)$ is unmixed.*

Proof: If $n = 2t + 1$, then $q_2 = 2$ and $r_2 = 1$, hence $\dim(R/I_t(C_n)) = d_2 = 2t - 2 = n - 3$. So it suffices to show that $\operatorname{depth}(R/I_t(C_n)) = n - 3$.
By [1, Proposition 3.3], the minimal free resolution of $I_t(C_n)$ is of the form

$$0 \to R(-n) \to R(-t-1)^n \to R(-t)^n \to I_t(C_n) \to 0,$$

for $t+2 \leq n \leq 2t+1$. So $\mathrm{pd}(I_t(C_n)) = 2$, hence $\mathrm{pd}(R/I_t(C_n)) = \mathrm{pd}(I_t(C_n)) + 1 = 3$. Therefore $\mathrm{depth}(R/I_t(C_n)) = n-3$, by Auslander-Buchsbaum formula. □

Now, we are ready to prove Theorem 3.1:

Proof of Theorem 3.1. **"If"** If $n = t$, then clearly $I_t(C_n)$ is unmixed. If $n = 2t+1$, then by Lemma 3.5, $I_t(C_n)$ is unmixed. If $t+1 \leq n \leq \lfloor 3t/2 \rfloor + 1$, then $\dim(R/I_t(C_n)) = n-2$. Suppose that there exists a facet F of cardinality $n-\alpha$ in Δ such that $3 \leq \alpha \leq n-1$. Let $\{x_{i_1}, \ldots, x_{i_\alpha}\}$ be the set of vertices do not appear in F, where $i_1 < i_2 < \cdots < i_\alpha$. Let y_j be the number of consecutive vertices in F strictly between x_{i_j} and $x_{i_{j+1}}$ (in the direction of C_n) for $j = 1, \ldots, \alpha-1$ and y_α be the number of consecutive vertices in F between x_{i_α} and x_{i_1}. Thus $y_1+y_2+1 \geq t$, $y_2+y_3+1 \geq t$, ... , $y_{\alpha-1}+y_\alpha+1 \geq t$ and $y_\alpha+y_1+1 \geq t$, since F is a facet of Δ and adding a vertex to it implies that it should contain at least t consecutive vertices. Then $2(y_1+y_2+\ldots+y_\alpha)+\alpha \geq t\alpha$, so $n-\alpha = y_1+y_2+\ldots+y_\alpha \geq t\alpha/2-\alpha/2$, therefore we have $n \geq t\alpha/2+\alpha/2 \geq 3t/2+3/2 > \lfloor 3t/2 \rfloor + 1$. But it is a contradiction and there does not exist such a facet. Hence Δ is pure and so $I_t(C_n)$ is unmixed.

"Only if" Suppose that $I_t(C_n)$ is unmixed. Then by Lemma 3.4, we get $t \leq n \leq 2t-1$ or $n = 2t+1$. If $\lfloor 3t/2 \rfloor + 2 \leq n \leq 2t-1$, then $d_2 = n-2$. On the other hand, Δ is pure, so Δ does not contain any facets of cardinality $n-3$. Now consider the following cases:

Case 1. If t is even, then we have $n = 3t/2+\alpha$, where $2 \leq \alpha \leq t/2-1$. Set $F := \{x_1, x_{\frac{t}{2}+2}, x_{t+2}\}$. We can see that $V \setminus F$ is a facet of Δ with cardinality $n-3$, a contradiction.

Case 2. If t is odd, then we have $n = 3(t-1)/2+\alpha$, where $3 \leq \alpha \leq (t-1)/2+1$. Set $F := \{x_1, x_{\frac{(t-1)}{2}+2}, x_{t+2}\}$. Thus $V \setminus F$ is a facet of Δ with cardinality $n-3$, a contradiction.
Therefore $t \leq n \leq \lfloor 3t/2 \rfloor + 1$ or $n = 2t+1$. □

Corollary 3.6. *Let $t \geq 3$. Then $R/I_t(C_n)$ is Cohen-Macaulay if and only if $n = t$ or $t+1$ or $2t+1$.*

Proof: By Lemma 3.5, now it is enough to note that by [1, Proposition 3.3] and Lemma 3.2, we have $\mathrm{depth}(R/I_t(C_n)) = n-3$ but $\dim(R/I_t(C_n)) = n-2$, for $t \leq n \leq \lfloor 3t/2 \rfloor + 1$. □

4 Sequentially Cohen-Macaulayness of the path ideal of C_n

In this section we focus on sequentially Cohen-Macaulay property of the path ideals of cycles. Actually, we show that Cohen-Macaulay and sequentially Cohen-Macaulay properties of $R/I_t(C_n)$ coincide.

Theorem 4.1. *Let $t \geq 3$. Then $R/I_t(C_n)$ is sequentially Cohen-Macaulay if and only if $n = t$ or $t+1$ or $2t+1$.*

Proof: **"If"** If $n = t$ or $t+1$ or $2t+1$, then by Theorem 3.6, $R/I_t(C_n)$ is Cohen-Macaulay, and hence sequentially Cohen-Macaulay.

"Only if" If $t+2 \leq n \leq \lfloor 3t/2 \rfloor + 1$, then $R/I_t(C_n)$ is not sequentially Cohen-Macaulay, by Theorem 3.6, Theorem 3.1 and Theorem 2.1. So suppose that $\lfloor 3t/2 \rfloor + 2 \leq n \leq 2t$ or $n \geq 2t+2$. By Theorem 2.3, it is enough to show that $I_t(C_n)^\vee$ is not componentwise linear. So by Theorem 2.2, it suffices to show that $(I_t(C_n)^\vee)_{[j]}$ does not have any linear resolutions, for some j. To simplify the notation, we write simply $I_t(C_n)^\vee_{[j]}$ to denote the latter ideal. Suppose that $V = \{x_1, \ldots, x_n\}$ and also $n = qt + r$, where $0 \leq r \leq t-1$. We have the following cases:

Case 1. Suppose that $r = 0$. Then $q \geq 2$. We show that $\beta_{1,2q}(I_t(C_n)^\vee_{[q]}) \neq 0$. Let $X := \{x_1, x_t, x_{t+1}, x_{2t}, \ldots, x_{(q-1)t+1}, x_{qt} = x_n\}$ and Δ be the simplicial complex such that $I_\Delta = I_t(C_n)^\vee_{[q]}$. Also let C_{2q} be the $2q$-cycle on the vertex set X (see Figure 1) and Δ' be the simplicial complex such that $I_{\Delta'} = I_2(C_{2q})^\vee_{[q]}$. For every $F \subseteq X$ with $|F| = q$, F is a vertex cover of C_{2q} if and only if F is a vertex cover of $\Delta_t(C_n)$. Now we show that $\Delta_X = \Delta'$. If $F \in \Delta'$, then $\prod_{x \in F} x \notin I_2(C_{2q})^\vee_{[q]}$. So F does not contain any vertex covers of C_{2q} of cardinality q and hence it does not contain any vertex covers of $\Delta_t(C_n)$ of cardinality q. Thus $\prod_{x \in F} x \notin I_t(C_n)^\vee_{[q]}$, so $F \in \Delta_X$ and hence $\Delta' \subseteq \Delta_X$. Similarly, we have $\Delta_X \subseteq \Delta'$. By Hochster's formula (see [5, Theorem 8.1.1]), we have

$$\begin{aligned}
\beta_{1,2q}(I_t(C_n)^\vee_{[q]}) &= \sum_{Y \subseteq V, |Y| = 2q} \dim_k \widetilde{H}_{2q-3}(\Delta_Y, k) \\
&\geq \dim_k \widetilde{H}_{2q-3}(\Delta_X, k) \\
&= \dim_k \widetilde{H}_{2q-3}(\Delta', k) \\
&= \sum_{Y \subseteq X, |Y| = 2q} \dim_k \widetilde{H}_{2q-3}(\Delta'_Y, k) \\
&= \beta_{1,2q}(I_2(C_{2q})^\vee_{[q]}).
\end{aligned}$$

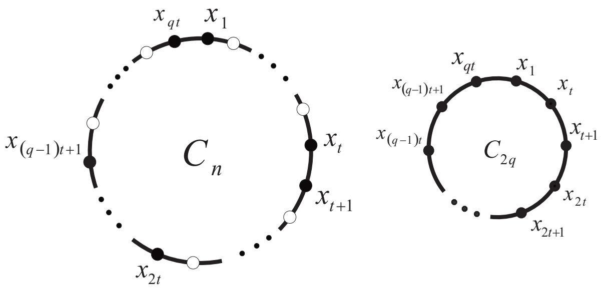

Figure 1:

By Remark 4.2, $\beta_{1,2q}(I_2(C_{2q})^{\vee}_{[q]}) \neq 0$. Thus $\beta_{1,2q}(I_t(C_n)^{\vee}_{[q]}) \neq 0$.

Case 2. Suppose that $r = 1$. So $q \geq 3$. We show that $\beta_{2,2q+1}(I_t(C_n)^{\vee}_{[q+1]}) \neq 0$. Let $X := \{x_1, x_t, x_{t+1}, x_{2t}, \ldots, x_{(q-2)t+1}, x_{(q-1)t}, x_{(q-1)t+1}, x_{qt}, x_{qt+1} = x_n\}$ and Δ be the simplicial complex such that $I_\Delta = I_t(C_n)^{\vee}_{[q+1]}$. Also let C_{2q+1} be the $(2q+1)$-cycle on the vertex set X and Δ' be the simplicial complex such that $I_{\Delta'} = I_2(C_{2q+1})^{\vee}_{[q+1]}$. For every $F \subseteq X$ with $|F| = q+1$, F is a vertex cover of C_{2q+1} if and only if F is a vertex cover of $\Delta_t(C_n)$. So, similar to the previous case, we have $\Delta_X = \Delta'$ and hence $\beta_{2,2q+1}(I_t(C_n)^{\vee}_{[q+1]}) \geq \beta_{2,2q+1}(I_2(C_{2q+1})^{\vee}_{[q+1]})$. But $\beta_{2,2q+1}(I_2(C_{2q+1})^{\vee}_{[q+1]}) \neq 0$, as was shown in the proof of [4, Proposition 4.1]. Hence $\beta_{2,2q+1}(I_t(C_n)^{\vee}_{[q+1]}) \neq 0$.

Case 3. Suppose that $2 \leq r \leq t-1$. Then we divide this case to the following cases:

(1) If $q = 1$, then $n = t + r$. We show that $\beta_{1,4}(I_t(C_n)^{\vee}_{[2]}) \neq 0$. Let $X := \{x_1, x_r, x_{t+1}, x_n\}$ and Δ be the simplicial complex such that $I_\Delta = I_t(C_n)^{\vee}_{[2]}$. Also let C_4 be the 4-cycle on the vertex set X and Δ' be the simplicial complex such that $I_{\Delta'} = I_2(C_4)^{\vee}_{[2]}$. For every $F \subseteq X$ with $|F| = 2$, F is a vertex cover of C_4 if and only if F is a vertex cover of $\Delta_t(C_n)$, since by assumption we have $\lfloor 3t/2 \rfloor + 2 \leq n$ and hence $t/2 + 1 < r$. Therefore, similar to the previous cases, we have $\Delta_X = \Delta'$ and hence $\beta_{1,4}(I_t(C_n)^{\vee}_{[2]}) \geq \beta_{1,4}(I_2(C_4)^{\vee}_{[2]})$. But, $\beta_{1,4}(I_2(C_4)^{\vee}_{[2]}) \neq 0$ (see Remark 4.2 below) so $\beta_{1,4}(I_t(C_n)^{\vee}_{[2]}) \neq 0$.

(2) If $q \geq 2$, then we show that $\beta_{1,2q+2}(I_t(C_n)^{\vee}_{[q+1]}) \neq 0$. Let $X := \{x_1, x_t, x_{t+1}$ $x_{2t}, \ldots, x_{(q-2)t+1}, x_{(q-1)t}, x_{(q-1)t+1}, x_{(q-1)t+r}, x_{qt+1}, x_{qt+r} = x_n\}$ and Δ be the simplicial complex such that $I_\Delta = I_t(C_n)^{\vee}_{[q+1]}$. Also let C_{2q+2} be the $(2q+2)$-cycle on the vertex set X and Δ' be the simplicial complex such that $I_{\Delta'} = I_2(C_{2q+2})^{\vee}_{[q+1]}$. For every $F \subseteq X$ with $|F| = q+1$, F is a vertex cover of C_{2q+2} if and only if F is a vertex cover of $\Delta_t(C_n)$. Therefore, like before, we have $\Delta_X = \Delta'$ and hence $\beta_{1,2q+2}(I_t(C_n)^{\vee}_{[q+1]}) \geq \beta_{1,2q+2}(I_2(C_{2q+2})^{\vee}_{[q+1]})$. But

$\beta_{1,2q+2}(I_2(C_{2q+2})^{\vee}_{[q+1]}) \neq 0$, by Remark 4.2. Then $\beta_{1,2q+2}(I_t(C_n)^{\vee}_{[q+1]}) \neq 0$.

So, by these cases, we get the desired result.

□

Remark 4.2. Note that for $m = 2s$, where $s \geq 2$, as was mentioned in the proof of [4, Proposition 4.1], we have $I_2(C_m)^{\vee}_{[s]} = (x_1x_3\cdots x_{2s-1}, x_2x_4\cdots x_{2s})$. It is not difficult to see that $0 \to R(-2s) \to R(-s)^2 \to I_2(C_m)^{\vee}_{[s]} \to 0$ is the graded minimal free resolution of $I_2(C_m)^{\vee}_{[s]}$. Thus $\beta_{1,2s}(I_2(C_m)^{\vee}_{[s]}) \neq 0$.

5 A counter example

In [2], the authors conjectured that there exists no 2-dimensional simplicial complex Δ on the vertex set V with $|V| \geq 9$ and $2 \not| |V|$ such that $k[\Delta]$ has a 3-pure, but not 3-linear resolution. (See [2, Theorem 3.1] and also [2, Remark 3.4 (d)]).

Now we have an example which shows that this conjecture is not true.

Example 5.1. Let Δ be the simplicial complex whose facets are $\{x_1, x_2, x_3\}$, $\{x_1, x_3, x_5\}$, $\{x_1, x_4, x_7\}$, $\{x_1, x_5, x_9\}$, $\{x_1, x_6, x_{10}\}$ and all their orbits by $\mathbb{Z}_{11}$ action. For example, considering $\{x_1, x_2, x_3\}$, we have the 10 other facets $\{x_2, x_3, x_4\}$, $\{x_3, x_4, x_5\}$, $\{x_4, x_5, x_6\}$, ... , $\{x_{11}, x_1, x_2\}$. Then I_Δ is generated by the monomials $x_1x_2x_4$, $x_1x_2x_5$, $x_1x_2x_6$, $x_1x_2x_8$, $x_1x_2x_9$, $x_1x_2x_{10}$, $x_1x_3x_6$, $x_1x_3x_7$, $x_1x_3x_8$, $x_1x_3x_9$ and their orbits with $\mathbb{Z}_{11}$ action. It means that by considering $x_1x_2x_4$, we then get the following generators: $x_2x_3x_5$, $x_3x_4x_6$, $x_4x_5x_7$, ... , $x_{11}x_1x_3$. Hence the number of generators is 110. Moreover, by *Macaulay 2*, the minimal free resolution of R/I_Δ is

$$0 \to R(-11)^{10} \to R(-10)^{44} \to R(-8)^{385} \to R(-7)^{1100} \to R(-6)^{1540}$$

$$\to R(-5)^{1232} \to R(-4)^{550} \to R(-3)^{110} \to R \to R/I_\Delta \to 0,$$

where $R = \mathbb{Q}[x_1, \ldots, x_{11}]$. So, $\mathbb{Q}[\Delta]$ has a pure resolution, but not a linear, since $\beta_{7,10}(R/I_\Delta) \neq 0$. Also its resolution degree is $1, 2, 1, \ldots, 1, 3$. Thus, all of the conditions of [2, Theorem 3.1] are satisfied. So, it makes the conjecture invalid.

Note that the path ideal of a cycle, which was discussed in this paper, is the simplest kind of the ideals with cyclic action (like the one mentioned in the above example).

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