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The Stanley conjecture on monomial almost complete intersection ideals

by
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Abstract

Let I be a monomial almost complete intersection ideal of a polynomial algebra S over a field. Then Stanley's Conjecture holds for S/I and I.

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Introduction

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring over K. Let M be a $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ as K-vector space, where $m_i \in M$, $Z_i \subset \{x_1, \dots, x_n\}$ such that $m_i K[Z_i]$ is a free $K[Z_i]$-module. We define $\operatorname{sdepth}(\mathcal{D}) = \min_{i=1}^r |Z_i|$ and $\operatorname{sdepth}_S(M) = \max\{\operatorname{sdepth}(\mathcal{D}) |\ \mathcal{D}$ is a Stanley decomposition of $M\}$. The number $\operatorname{sdepth}_S(M)$ is called the *Stanley depth* of M. It is conjectured by Stanley [7] that $\operatorname{depth}_S(M) \leq \operatorname{sdepth}_S(M)$ for all $\mathbb{Z}^n$-graded S-modules M. Herzog, Vladoiu and Zheng show in [5] that this invariant can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. In this paper, we prove that if $I \subset S$ is a monomial ideal generated by m monomials, then, there exists a variable x_j which appears in at least $\lceil \frac{m}{k} \rceil$ generators, where $k = \max\{|P| : P \in Ass(S/I)\}$, see Lemma 1.5. Using this lemma, we prove that Stanley's Conjecture holds for S/I and I, when I has a small number of generators, with respect to $\operatorname{depth}(S/I)$ and k, see Theorem 1.8, in particular this is the case when I is a monomial almost complete intersection ideal (see Corollary 1.9).

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1 Stanley depth

Firstly, we recall several results.

Proposition 1.1. *[3, Proposition 1.2] Let $I \subset S$ be a monomial ideal, minimally generated by m monomials. Then* $\operatorname{sdepth}(S/I) \geq n - m$.

Theorem 1.2. *[6, Theorem 2.3] Let $I \subset S$ be a monomial ideal, minimally generated by m monomials. Then* $\operatorname{sdepth}(I) \geq n - \lfloor m/2 \rfloor$.

Proposition 1.3. *[3, Theorem 1.4] Let $I \subset S$ be a monomial ideal such that $I = v(I : v)$, for a monomial $v \in S$. Then* $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(S/(I : v))$, $\operatorname{sdepth}(I) = \operatorname{sdepth}(I : v)$.

If $v \in S$ is a monomial, we define the *support of* v, to be $\operatorname{supp}(v) := \{x_j : x_j|v\}$. Also, we denote $\deg_{x_j}(v) := \max\{t : x_j^t|v\}$. Let $I = (v_1, \ldots, v_m) \subset S$, $I \neq S$ be a monomial ideal, where $G(I) = \{v_1, \ldots, v_m\}$ is a minimal system of monomial generators of I. We denote $t_j := |\{i : x_j|v_i\}|$ and $\mathcal{V} := \bigcup_{i=1}^m \operatorname{supp}(v_i)$.

Remark 1.4. It is well known that $\operatorname{depth}(S/I) \leq \min\{\dim(S/P) : P \in \operatorname{Ass}(S/I)\} = \min\{n - |P| : P \in \operatorname{Ass}(S/I)\}$ by [2, Proposition 1.2.13]. Denote $k = \max\{|P| : P \in \operatorname{Ass}(S/I)\}$. In particular, we get $k \leq n - \operatorname{depth}(S/I)$. We have $k \leq m$ because a prime ideal $P \in \operatorname{Ass}(S/I)$ has the form $I : u$ for some monomial $u \notin I$.

With these notations, we have the following lemma:

Lemma 1.5. *There exists a $j \in [n] := \{1, \ldots, n\}$ such that $t_j \geq \lceil m/k \rceil$.*

Proof: We use induction on $k \geq 1$ and $\varepsilon(I) = \sum_{i=1}^m \deg(v_i)$. If $k = 1$, it follows that I is principal, and therefore, we can assume that $I = (v_1)$ and $m = 1$. If we chose $x_j \in \operatorname{supp}(v_1)$, it follows that $t_j = 1 = \lceil m/1 \rceil$ and thus we are done. If $\varepsilon(I) = k$, it follows $\varepsilon(I) = k \leq m \leq \varepsilon(I)$ by Remark 1.4. Thus I is generated by $m = k$ variables, and there is nothing to prove. Assume $k \geq 2$ and $\varepsilon(I) > k$.

Assume that $(\mathcal{V}) \subset \sqrt{I}$. Since, for any monomial $v \in G(\sqrt{I})$ we have $\operatorname{supp}(v) \subset \mathcal{V}$ it follows that the prime ideal $P := (\mathcal{V})$ contains also $\sqrt{I}$. Thus $P = \sqrt{I}$ is a prime ideal and $P = (\mathcal{V})$. Therefore, I is P-primary. Since $k = |P|$, by reordering the variables, we may assume that $P = (x_1, \ldots, x_k)$. We may also assume that $v_1 = x_1^{a_1}, \ldots, v_r = x_k^{a_k}$ for some positive integers a_l, where $l \in [k]$. Since $\mathcal{V} = \{x_1, \ldots, x_k\}$, it follows that $t_j = 0$ for all $j > k$. Note that $\sum_{i=1}^m |\operatorname{supp}(v_i)| = \sum_{j=1}^k t_j$. Indeed, each variable x_j appear in the supports of exactly t_j monomials from the set $\{v_1, \ldots, v_m\}$. Now, we claim that there exists a $t_j \geq \lceil m/k \rceil$. Indeed, if this is not the case, then we get $m \leq \sum_{i=1}^m |\operatorname{supp}(v_i)| = \sum_{j=1}^k t_j < \sum_{j=1}^k (m/k) = m$, a contradiction.

If there exists a variable, let us say x_n, such that $x_n \in \mathcal{V}$ and $x_n \notin \sqrt{I}$, we consider the ideal $I' = (I : x_n)$. Obviously, $I' = (v_1', \ldots, v_m')$, where $v_i' = v_i/x_n$ if $x_n|v_i$ and $v_i' = v_i$ otherwise. For all $j \in [n]$, we denote $t_j' = |\{i : x_j|v_i'\}|$. Note that $t_j = t'_j$ for all $j \in [n-1]$, and $t_n \geq t'_n$. If we denote $\mathcal{V}' = \bigcup_{i=1}^m \operatorname{supp}(v'_i)$, we have $\mathcal{V}' \subset \mathcal{V}$. Note that $\operatorname{Ass}(S/I') \subset \operatorname{Ass}(S/I)$ because of the injection $S/I' \to S/I$ induced by the multiplication with x_n. It follows that $k' = \max\{|P'| : P' \in \operatorname{Ass}(S/I')\} \leq k$. Since $\varepsilon(I') = \sum_{i=1}^m \deg(v'_i) < \varepsilon(I)$, by induction hypothesis, there exists a $j \in [n]$, such that $t_j \geq t'_j \geq \lceil m/k' \rceil \geq \lceil m/k \rceil$. □

Example 1.6. Let $I = (x_1^3, x_1x_2, x_2x_3, x_3x_4, x_4^2) \subset S := K[x_1, x_2, x_3, x_4]$. Then $I = (x_1^3, x_2, x_4) \cap (x_1^3, x_2, x_3, x_4^2) \cap (x_1, x_3, x_4^2)$ is the primary decomposition of I. Therefore $\operatorname{Ass}(S/I) = \{(x_1, x_2, x_4), (x_1, x_3, x_4), (x_1, x_2, x_3, x_4)\}$ and $k = \max\{|P| : P \in \operatorname{Ass}(S/I)\} = 4$. The (minimal) number of monomial generators of I is $m = 5$. We have $\lceil m/k \rceil = 2$ and, indeed, x_1, for example, appears in two generators of I. This example also shows that the bound $\lceil m/k \rceil$ is, in general, the best possible.

Lemma 1.7. *Let $s \geq k \geq 2$ be two integers and let m be a positive integer. Then*

1. $m - \left\lceil \frac{m}{k} \right\rceil \leq s - 1$ *if and only if* $m \leq s - 1 + \left\lceil \frac{s}{k-1} \right\rceil$,
2. $\left\lfloor \frac{m - \lceil \frac{m}{k} \rceil}{2} \right\rfloor \leq s - 2$ *if and only if* $m \leq 2s - 3 + \left\lceil \frac{2s-2}{k-1} \right\rceil$.

Proof: Note that $m - \left\lceil \frac{m}{k} \right\rceil \leq s - 1$ if and only if $m - \frac{m}{k} < s$. This is equivalent with $m < \frac{sk}{k-1} = s + \frac{s}{k-1}$. Similarly, we get the second equivalence. □

Theorem 1.8. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal, minimally generated by m monomials, $k = \max\{|P| : P \in \operatorname{Ass}(S/I)\}$, and $s \geq k$ be an integer. Then*

1. *If* $m \leq s - 1 + \left\lceil \frac{s}{k-1} \right\rceil$, *then* $\operatorname{sdepth}(S/I) \geq n - s$.
2. *If* $m \leq 2s - 3 + \left\lceil \frac{2s-2}{k-1} \right\rceil$, *then* $\operatorname{sdepth}(I) \geq n - s + 1$.

If $\operatorname{depth}(S/I) = n - s$ *then* (1) *and* (2) *imply the Stanley Conjecture for S/I, respectively for I.*

Proof: If I is principal, then $k = m = 1$ and $\operatorname{sdepth}(S/I) \geq n-1$ by Proposition 1.1. Assume $m \geq 2$, $G(I) = \{v_1, \ldots, v_m\}$ and set $\varepsilon(I) := \sum_{i=1}^m \deg(v_i)$. We use induction on $\varepsilon(I)$. If $\varepsilon(I) = m$ it follows that I is generated by m variables. Therefore $k = |I| = m$ and so $\operatorname{sdepth}_S(S/I) = n - m = n - k \geq n - s$. Also by [1, Theorem 2.2] and [5, Lemma 3.6], $\operatorname{sdepth}(I) = n - \lfloor m/2 \rfloor \geq n - m + 1 = n - k + 1 \geq n - s + 1$.

Assume $\varepsilon(I) > m$. According to Lemma 1.5, we can assume that $r := t_n \geq \lceil m/k \rceil$ after renumbering of variables. If $r = m$, then $x_n | v_j$ for all all $i \in [m]$ and thus $I = x_n(I : x_n)$. According to Proposition 1.3, $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(S/(I : x_n))$ and $\operatorname{sdepth}(I) = \operatorname{sdepth}(I : x_n)$. As in the proof of Lemma 1.5, we have

$k' = \max\{|P'| : P' \in Ass(S/(I : x_n))\} \leq k$ and so our statement holds for $S/(I : x_n)$ by induction hypothesis. Thus $\text{sdepth}(S/I) = \text{sdepth}(S/(I : x_n)) \geq n - s$ and $\text{sdepth}(I) = \text{sdepth}(I : x_n) \geq n - s + 1$.

We consider now the case $r < m$. By reordering the generators of I, we may assume that $x_n|v_1,\ldots,x_n|v_r$ and $v_n \not| v_{r+1},\ldots, x_n \not| v_m$. Let $S' = K[x_1, \ldots, x_{n-1}]$. We write:

$$(*) \quad S/I = (S'/(I \cap S')) \oplus x_n(S/(I : x_n)), \text{ and } I = (I \cap S') \oplus x_n(I : x_n)$$

the direct sum being of linear spaces. By Proposition 1.1, Theorem 1.2 and Lemma 1.7, it follows that:

$$\text{sdepth}_{S'}(S'/(I \cap S')) \geq (n-1) - (m-r) \geq n - (m - \left\lceil \frac{m}{k} \right\rceil + 1) \geq n - s \text{ and}$$

$$\text{sdepth}_{S'}(I \cap S') \geq (n-1) - \left\lfloor \frac{m-r}{2} \right\rfloor \geq n - \left\lfloor \frac{m - \lceil \frac{m}{k} \rceil}{2} \right\rfloor + 1 \geq n - s + 1,$$

because $r \geq \lceil m/k \rceil$. Let m' be the minimal number of generators of $I : x_n$. In the first case, we have $m' \leq m \leq s - 1 + \left\lceil \frac{s}{k-1} \right\rceil \leq s - 1 + \left\lceil \frac{s}{k'-1} \right\rceil$ because $k' \leq k$. By induction hypothesis, we get $\text{sdepth}(S/(I : x_n)) \geq n - s$. Similarly, $\text{sdepth}(I : x_n) \geq n - s + 1$ in the second case. Using the decompositions $(*)$, we obtain Stanley decompositions of S/I, I with the Stanley depth $\geq n - s$, respectively $\geq n - s + 1$. □

Corollary 1.9. *Let I be a monomial almost complete intersection ideal. Then Stanley's Conjecture holds for S/I and I.*

Proof: Let $s := n - \text{depth}(S/I)$. Then $m \leq s + 1$. Since $s \geq k$ by Remark 1.4 we have $m \leq s - 1 + \left\lceil \frac{s}{k-1} \right\rceil$ and we see that S/I satisfies Stanley's Conjecture by (1) of the above theorem. If $s \geq 2$ then similarly $m \leq 2s - 3 + \left\lceil \frac{2s-2}{k-1} \right\rceil$ and so I satisfies Stanley's Conjecture by (2) of the above theorem. But if $s = 1$ then $k = 1$ and it follows that I is principal, in which case clearly Stanley's Conjecture holds. □

Remark 1.10. Note that the decomposition used in the proof of Theorem 1.8 is also useful to check Stanley's Conjecture for monomial ideals (and their quotient rings), which do not satisfy the hypothesis of the quoted theorem. For example, consider the ideal $I = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4) \subset S := K[x_1, \ldots, x_4]$. We have $(I : x_1) = (x_2, x_3, x_4)$. If we denote $S' := K[x_1, x_2, x_3]$, then $I' := I \cap S' = (x_1x_2, x_1x_3, x_2x_3)$. One can easily check that $\text{sdepth}_{S'}(S'/I') = 1$ and $\text{sdepth}_{S'}(I') = 2$. Using the decomposition $I = I' \oplus x_1(I : x_1)$, it follows, as in the proof of Theorem 1.8, that $\text{sdepth}(I) \geq 2$ and $\text{sdepth}(S/I) \geq 1$. On the other hand, it is well known that $\text{depth}(S/I) = 1$. Of course, I is a squarefree Veronese ideal, and we already know that I and S/I satisfy the Stanley conjecture, by [4, Corollary 1.2].

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