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Jacobi-type vector fields on Ricci solitons

by
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Abstract

In this paper, we introduce the notion of Jacobi-type vector fields on Riemannian manifolds, which is a generalization of the Jacobi field along a geodesic. We study Ricci solitons with positive Ricci curvature whose potential vector field is a Jacobi-type vector field and show that if the metric on Ricci soliton is replaced by the Ricci tensor, then we get a Riemannian manifold that is an Einstein manifold. As a by-product, we get a criterion for compactness of a complete Ricci soliton using a Jacobi-type vector field. Finally it is shown that a Ricci soliton of positive Ricci curvature whose potential field is Jacobi-type vector field is necessarily an Einstein manifold.

Key Words: Jacobi-type vector fields, Killing vector fields, Ricci soliton, Einstein manifolds.
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1 Introduction

Let (M, g) be an n-dimensional Riemannian manifold. There are several important types of smooth vector fields, whose existence influence the geometry of the Riemannian manifold (M, g). A smooth vector field ξ on M is said to be Killing if its local flow consists of local isometries of the Riemannian manifold (M, g). The presence of a nonzero Killing vector field on a compact Riemannian manifold constrains its geometry as well as topology, for instance, it does not allow the Riemannian manifold to have negative Ricci curvature and its fundamental group contains a cyclic subgroup with constant index depending only on n (cf. [2], [16]). The geometry of Riemannian manifolds with Killing vector fields has been studied quite extensively (cf. [1], [2], [12], [16], [17]). Similarly a smooth vector field ξ on a Riemannian manifold (M, g) is said to be a conformal vector field if its

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local flow consists of local conformal transformations of the Riemannian manifold (M,g). Conformal vector fields are generalizations of Killing vector fields. Non-Killing conformal vector fields are used in characterizing spheres among compact Riemannian manifolds (cf. [2], [7], [8], [10], [13]-[15]). Recall that a Killing vector field ξ on a Riemannian manifold (M,g) is a Jacobi field along each geodesic $\gamma : I \to M$, that is, it satisfies the differential equation

$$\ddot{\gamma} + R(\xi,\dot{\gamma})\dot{\gamma} = 0,$$

where R is the curvature tensor field of the Riemannian manifold (M,g). In this paper, we extend the definition of Jacobi field along a geodesic and define Jacobi-type vector field on a Riemannian manifold. We say a smooth vector field ξ on a Riemannian manifold (M,g) is a Jacobi-type vector field if it satisfies

$$\nabla_X \nabla_X \xi - \nabla_{\nabla_X X}\xi + R(\xi,X)X = 0, \quad X \in \mathfrak{X}(M)$$

where ∇ is the Riemannian connection, R the curvature tensor field and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on (M,g). It is clear that a Jacobi-type vector field is a Jacobi field along each geodesic on the Riemannian manifold and it follows that each Killing vector field is a Jacobi-type vector field (see section-2). Thus, if we denote by $J(M)$ and $K(M)$ the sets of Jacobi-type vector fields and Killing vector fields on a Riemannian manifold (M,g), then we have

$$K(M) \subset J(M)$$

and there are Riemannian manifolds on which the above inclusion is strict. For instance, consider the Euclidean space $\mathbf{R}^n$ and ξ the position vector field on $\mathbf{R}^n$, then it is easy to verify that ξ is a Jacobi-type vector field and it is not a Killing vector field.

The next important type of smooth vector field on a Riemannian manifold (M,g) is the vector field which defines a Ricci soliton.

Definition. A smooth vector field ξ on a Riemannian manifold (M,g) is said to define a Ricci soliton if it satisfies

$$\frac{1}{2}\left(\pounds_\xi g\right)(X,Y) + Ric(X,Y) = \lambda g(X,Y), \quad X,Y \in \mathfrak{X}(M)$$

where $\pounds_\xi g$ is the Lie-derivative of the metric g with respect to ξ, Ric is the Ricci tensor of (M,g) and λ is a constant.

We shall denote a Ricci soliton by (M,g,ξ,λ) and call the vector field ξ the potential field of the Ricci soliton. It is not difficult to see that both the Jacobi-type vector field and the potential field of the Ricci soliton satisfy the following differential equation (see Remark 2.2)

$$\Delta\xi + Q(\xi) = 0,$$

where Δ is the Laplace operator acting on smooth vector fields (cf. [5]) and Q is the Ricci operator defined by

$$g(QX,Y) = Ric(X,Y), \quad X,Y \in \mathfrak{X}(M).$$

One of the important questions in the geometry of a Ricci soliton is to find conditions under which it is an Einstein manifold. In fact, Hamilton [9], conjectured that a compact gradient shrinking Ricci soliton with positive curvature operator must be Einstein and this conjecture is ultimately settled in [3]. However, the question of obtaining conditions under which a Ricci soliton is an Einstein manifold is an interesting question. After the Hamilton's conjecture was solved, a natural question arises, whether a compact gradient Ricci soliton of positive Ricci curvature is an Einstein manifold. In this paper, we consider a Ricci soliton (M, g, ξ, λ) with positive Ricci curvature, endowing M with another Riemannian metric namely $\overline{g} = Ric$, we show that if the potential field ξ is a Jacobi-type vector field, then the Riemannian manifold $(M, \overline{g})$ is an Einstein manifold (cf. Theorem 3.1). As a particular case of this result, we get a compactness criterion for the Ricci soliton (cf. Theorem 3.2). Finally, we show that a Ricci soliton with positive Ricci curvature and potential field a Jacobi-type vector field is necessarily an Einstein manifold (cf. Theorem 3.3).

2 Preliminaries

We start by proving that Killing fields provide examples for the notion we introduced:

Proposition 2.1. *A Killing vector field on a Riemannian manifold is a Jacobi-type vector field.*

Proof: Suppose ξ is a Killing vector field on a Riemannian manifold (M, g). Then we have

$$(\pounds_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0, \quad X,Y \in \mathfrak{X}(M) \tag{2.1}$$

which in particular gives

$$g(\nabla_X \xi, X) = 0, \quad X \in \mathfrak{X}(M). \tag{2.2}$$

We use equations (2.1) and (2.2) to compute

$$g(\nabla_X \nabla_X \xi, Y) = Xg(\nabla_X \xi, Y) + g\left(\nabla_{\nabla_X Y} \xi, X\right) \tag{2.3}$$

and

$$\begin{aligned} g(R(\xi,X)X,Y) &= g(R(X,Y)\xi,X) \\ &= Xg(\nabla_Y \xi, X) - g(\nabla_Y \xi, \nabla_X X) + g(\nabla_X \xi, \nabla_Y X) \\ &\quad -g\left(\nabla_{\nabla_X Y} \xi, X\right) + g\left(\nabla_{\nabla_Y X} \xi, X\right). \end{aligned}$$

Combining the above equation with equation (2.3), we get

$$g(\nabla_X \nabla_X \xi + R(\xi, X)X, Y) = g\left(\nabla_{\nabla_X X}\xi, Y\right),$$

which proves that ξ is a Jacobi-type vector field. ∎

Remark 2.1: (i) There are Jacobi-type vector fields on a Riemannian manifold which are not Killing vector fields. For example, consider the Euclidean space $(\mathbf{R}^n, g)$, where g is the Euclidean metric, then it is easy to check that the position vector field

$$\xi = \sum u^i \frac{\partial}{\partial u^i}$$

is a Jacobi-type vector field which satisfies

$$(\pounds_\xi g)(X, Y) = 2g(X, Y)$$

and therefore ξ is not a Killing vector field.

(ii) On a Lie group G with bi-invariant metric g, it is easy to check that each left invariant vector field is a Jacobi-type vector field.

Using the definition of a Jacobi-type vector field, by polarization we get the following:

Lemma 2.1. *Let ξ be a Jacobi-type vector field on a Riemannian manifold (M, g). Then*

$$\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi + R(\xi, X)Y = 0, \quad X, Y \in \mathfrak{X}(M).$$

Given a Jacobi-type vector field ξ on a Riemannian manifold (M, g), we denote by the same letter ξ the smooth 1-form dual to ξ and define a symmetric tensor field B of type $(1,1)$ and a skew-symmetric tensor field φ of type $(1,1)$ by

$$\frac{1}{2}(\pounds_\xi g)(X, Y) = g(BX, Y), \text{and} \frac{1}{2} d\xi(X, Y) = g(\varphi X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.4}$$

Then using Koszul's formula, we immediately arrive at

$$\nabla_X \xi = BX + \varphi X, \quad X \in \mathfrak{X}(M) \tag{2.5}$$

which together with Lemma 2.1 gives

$$(\nabla B)(X, Y) + (\nabla \varphi)(X, Y) + R(\xi, X)Y = 0 \tag{2.6}$$

where the covariant derivative $(\nabla A)(X, Y)$ of a $(1,1)$ tensor field A is defined as

$$(\nabla A)(X, Y) = \nabla_X AY - A\nabla_X Y.$$

Let (M, g, ξ, λ) be an n-dimensional Ricci soliton with potential field ξ. Then we have (cf. [4])

$$\frac{1}{2}\left(\pounds_{\xi}g\right)(X,Y) + Ric(X,Y) = \lambda g(X,Y), \quad X,Y \in \mathfrak{X}(M). \tag{2.7}$$

If we define a skew-symmetric tensor field ψ of type $(1,1)$ on the Ricci soliton (M, g, ξ, λ) by $d\xi(X,Y) = 2g(\psi X, Y)$, then using Koszul's formula we immediately get using (2.7) that the covariant derivative of the potential field ξ is given by

$$\nabla_X \xi = \lambda X - Q(X) + \psi X, \quad X,Y \in \mathfrak{X}(M). \tag{2.8}$$

The next Remark shows that Jacobi-type vector fields and potentials of Ricci solitons satisfy a same PDE.

Remark 2.2: Let ξ be a Jacobi-type vector field on an n-dimensional Riemannian manifold (M, g). Then using a local orthonormal frame $\{e_1, ..., e_n\}$ on M in the definition of Jacobi-type vector field, we get

$$\Delta\xi + Q(\xi) = 0, \tag{2.9}$$

where $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the Laplace operator acting on smooth vector fields defined by (cf. [5])

$$\Delta X = \sum_{i=1}^{n}\left(\nabla_{e_i}\nabla_{e_i}X - \nabla_{\nabla_{e_i}e_i}X\right), \quad X \in \mathfrak{X}(M).$$

Now let (M, g, ξ, λ) be an n-dimensional Ricci soliton with potential field ξ. Then the equation (2.8) gives

$$\Delta\xi = -\frac{1}{2}\nabla S + \sum_{i=1}^{n}(\nabla\psi)(e_i, e_i) \tag{2.10}$$

where we used $\sum_{i=1}^{n}(\nabla Q)(e_i, e_i) = \frac{1}{2}\nabla S$, ∇S being the gradient of the scalar curvature S of the Ricci soliton. Also using equation (2.8), we compute the following

$$R(X,Y)\xi = (\nabla Q)(Y,X) - (\nabla Q)(X,Y) + (\nabla\psi)(X,Y) - (\nabla\psi)(Y,X)$$

and which together with the symmetry of Q and skew-symmetry of ψ gives

$$Ric(Y,\xi) = Y(S) - \frac{1}{2}g(Y,\nabla S) - g\left(Y, \sum_{i=1}^{n}(\nabla\psi)(e_i, e_i)\right)$$

and consequently

$$Q(\xi) = \frac{1}{2}\nabla S - \sum_{i=1}^{n}(\nabla\psi)(e_i, e_i) \tag{2.11}$$

Combining equations (2.10) and (2.11), we see that the potential field ξ of the Ricci soliton satisfies the same differential equation as in (2.9) for a Jacobi-type vector field and this is the property shared by both the Jacobi-type vector field and the potential field of the Ricci soliton.

3 Jacobi-type vector fields on Ricci solitons

In this section, we study a n-dimensional Ricci soliton (M, g, ξ, λ) of positive Ricci curvature whose potential field ξ is a Jacobi-type vector field. It is known that if the Ricci soliton is compact, then the potential field ξ is a gradient of some smooth function and thus a compact Ricci soliton is a gradient Ricci soliton. Recall that in [9], Hamilton conjectured that a compact gradient Ricci soliton with positive curvature operator is an Einstein manifold, which is settled in [3]. The next important question is to find conditions under which a compact gradient Ricci soliton is an Einstein manifold. In this section first we show that a Ricci soliton of positive Ricci curvature and whose potential field is a Jacobi-type vector field is necessarily compact and therefore a gradient Ricci soliton. In doing this, we use the ingredient that Ricci curvature is positive, to get another Riemannian metric $\overline{g} = Ric$ on the Ricci soliton M and use the Myers theorem for the Riemannian manifold $(M, \overline{g})$, to prove M is compact, and therefore the Ricci soliton (M, g, ξ, λ) is a gradient Ricci soliton. Then we show that this gradient Ricci soliton is an Einstein manifold.

Theorem 3.1. *Let (M, g, ξ, λ) be a Ricci soliton of positive Ricci curvature and let the potential field ξ be a Jacobi-type vector field. Then the Riemannian manifold $(M, \overline{g})$, where $\overline{g} = Ric$, is an Einstein manifold.*

Proof: Since the Ricci soliton (M, g, ξ, λ) has positive Ricci curvature, we see that $\overline{g} = Ric$ is a Riemannian metric on M. Let ∇ and $\overline{\nabla}$ be the Riemannian connections with respect to the metrics g and $\overline{g}$ respectively. Then using Koszul's formula together with equation (2.7) of the Ricci soliton, after a straightforward calculation we arrive at the following expression for the covariant derivative with respect to the connection $\overline{\nabla}$:

$$2\overline{g}(\overline{\nabla}_X Y, Z) = 2Ric(\nabla_X Y, Z) + R(X, \xi; Y, Z) - g\left(\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi, Z\right),$$

$X, Y, Z \in \mathfrak{X}(M)$, where R is the curvature tensor of the Ricci soliton (M, g, ξ, λ).
Since $Ric = \overline{g}$, the above equation gives

$$2Q(\overline{\nabla}_X Y - \nabla_X Y) = -R(\xi, X)Y - \nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y}\xi. \tag{3.1}$$

Now, as ξ is also a Jacobi-type vector field, using Lemma 2.1, we get $Q(\overline{\nabla}_X Y - \nabla_Y X) = 0$ and as Ricci curvature is positive, it gives

$$\overline{\nabla}_X Y = \nabla_X Y, \quad X, Y \in \mathfrak{X}(M). \tag{3.2}$$

Consequently, the curvature tensors R and $\overline{R}$ of the Ricci soliton (M, g, ξ, λ) and the Riemannian manifold $(M, \overline{g})$ satisfy

$$\overline{R}(X,Y)Z = R(X,Y)Z, \quad X, Y, Z \in \mathfrak{X}(M) \tag{3.3}$$

which shows that the Ricci tensor $\overline{Ric}$ of the Riemannian manifold $(M, \overline{g})$ satisfies $\overline{Ric} = \overline{g}$, and hence the Riemannian manifold $(M, \overline{g})$ is an Einstein manifold. □

Theorem 3.2. *Let (M, g, ξ, λ) be a complete Ricci soliton of positive Ricci curvature and let the potential field ξ be a Jacobi-type vector field. Then M is compact.*

Proof: Recall that Myers theorem states that a complete Riemannian manifold whose Ricci curvature satisfies

$$Ric \geq \frac{1}{a^2} > 0$$

for a constant a, is compact. Note that, mere positive Ricci curvature of a complete Riemannian manifold does not guarantee the compactness as there are complete noncompact Riemannian manifolds of positive Ricci curvature. Now, as the Riemannian manifold (M, g) is complete, using equation (3.2), we see that the Riemannian manifold $(M, \overline{g})$ is also complete. Also, Theorem 3.1 implies that the complete Riemannian manifold $(M, \overline{g})$ satisfies the hypothesis of the Myers theorem and therefore M is compact. □

Let (M, g) be a Riemannian manifold and $f : M \to R$ be a smooth function. The Hessian operator A_f of the smooth function f is defined by

$$A_f(X) = \nabla_X \nabla f,$$

where ∇f is the gradient of the smooth function f. Then it follows that the trace of the operator A_f satisfies $Tr(A_f) = \Delta f$, where Δ is the Laplace operator acting on smooth functions on M. We shall need the following Lemma (cf. Lemma 2.3, [6], with a note that there is a sign difference in the definition of Δ).

Lemma 3.1. *[6] Let (M, g) be an n-dimensional Riemannian manifold and let f be a smooth function on M. Then the Hessian operator A_f satisfies*

$$\sum_{i=1}^{n} (\nabla A_f)(e_i, e_i) = Q(\nabla f) + \nabla(\Delta f)$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame, Δ is the Laplace operator on M acting on smooth functions.

Recalling that a compact Ricci soliton is necessarily a gradient Ricci soliton, now we prove the following:

Theorem 3.3. *Let (M, g, ξ, λ) be a complete connected Ricci soliton of positive Ricci curvature and the potential field ξ be a Jacobi-type vector field. Then M is an Einstein manifold.*

Proof: Using theorem 3.2, we see that the Ricci soliton (M, g, ξ, λ) is a compact gradient Ricci soliton, that is there is a smooth function $f : M \to R$ such that the potential field $\xi = \nabla f$, where ∇f is the gradient of the function f. Then as ξ is closed, using equation (2.5), we have $\varphi = 0$, $B = A_f$ and consequently equation (2.6) gives

$$(\nabla A_f)(X, Y) + R(\xi, X)Y = 0, \quad X, Y \in \mathfrak{X}(M). \tag{3.4}$$

Using Lemma 3.1 in the above equation, we arrive at

$$2Q(\nabla f) + \nabla(\Delta f) = 0. \tag{3.5}$$

Now the Ricci soliton equation (2.7) for gradient soliton takes the form

$$A_f(X) + Q(X) = \lambda X, \quad X \in \mathfrak{X}(M) \tag{3.6}$$

which gives

$$\Delta f = n\lambda - S \tag{3.7}$$

where $n = \dim M$ and S is the scalar curvature of the Ricci soliton. Then equation (3.6) gives $(\nabla A_f)(X, Y) + (\nabla Q)(X, Y) = 0$, which together with equation (3.4) gives

$$R(\xi, X)Y = (\nabla Q)(X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{3.8}$$

Taking the inner product in the above equation with Y, we arrive at

$$g((\nabla Q)(X, Y), Y) = 0, \quad X, Y \in \mathfrak{X}(M).$$

This confirms that the scalar curvature S is a constant and thus integrating equation (3.7) over the compact M, we shall get $S = n\lambda$. Thus $S = n\lambda$ together with equation (3.7) proves that f is a constant and hence the gradient Ricci soliton is an Einstein manifold with Einstein constant λ. □

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