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The Siegel norm of algebraic numbers

by
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Abstract

In this paper we investigate connections between the Siegel norm and the spectral norm on the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and their extensions to the spectral completion $\widetilde{\overline{\mathbb{Q}}}$ of $\overline{\mathbb{Q}}$.

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1 Introduction

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and denote by $\|\cdot\|$ the spectral norm on $\overline{\mathbb{Q}}$, defined by

$$\|\alpha\| = \max_{\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})} |\sigma(\alpha)|,$$

for any algebraic number α. We consider the map $A : \overline{\mathbb{Q}} \longrightarrow [0, \infty)$ given by

$$A(\alpha) = \frac{1}{[K : \mathbb{Q}]} \sum_{\sigma} |\sigma(\alpha)|^2,$$

where σ runs over all the embeddings of K into $\mathbb{C}$. Here $A(\alpha)$ depends only on α and not on the field K containing α. Note that if $\alpha = \beta^2$, where α is totally real and positive, then β is totally real and

$$A(\beta) = \frac{Tr\alpha}{\deg \alpha}.$$

For a totally real and positive algebraic integer $\alpha \in O_{\overline{\mathbb{Q}}}$, let $n = \deg \alpha$ be the degree of α over $\mathbb{Q}$, and let $\alpha_1 = \alpha, \alpha_2, \cdots, \alpha_n$ be the conjugates of α over $\mathbb{Q}$. Then

$$\frac{Tr\alpha}{\deg \alpha} = \frac{\alpha_1 + \cdots + \alpha_n}{n} \geq \sqrt[n]{\alpha_1 \cdots \alpha_n} \geq 1.$$

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The well known trace problem of Siegel asks for the best possible constant λ_0 for which given $\lambda < \lambda_0$, the trace of a totally real and positive algebraic integer γ is at least λ times its degree, except for finitely many γ's.

The best result to date is $\lambda_0 \geq 1.78702$. On the other hand, as Siegel pointed out, for every odd prime p, the number $4\cos^2\frac{\pi}{p}$ is a totally real and positive algebraic integer of degree $\frac{p-1}{2}$ and its trace is $p-2$. So the best possible constant λ_0 is at most 2.

In [3], and more recently in [9], the restriction of the map $A(\cdot)$ to the ring of cyclotomic integers $O_{\mathbb{Q}^{ab}}$ is studied. It is shown in [9] that the set $T = \{A(\alpha) : \alpha \in O_{\mathbb{Q}^{ab}}\}$ is closed under addition, that T is topologically closed in $\mathbb{R}$, and that for any $0 \leq r < 1$, $r \in \mathbb{Q}$, there is a $t_r \in \mathbb{Q}$ such that

$$T \cap (r + \mathbb{Z}) = \{t_r, t_r + 1, t_r + 2, \cdots\}.$$

We also mention that from Siegel's work [8] it follows that the intersection $T \cap [0, \frac{3}{2})$ consists of only two elements: 0 and 1, and they are attained when $\alpha = 0$, respectively when α is a root of unity. A striking application of this result is provided in an unpublished theorem of Thompson. Recall that the values of a linear character of a finite group are roots of unity. A classical theorem of Burnside [1] says that a non-linear irreducible character of a finite group has at least one zero. Thompson's theorem, whose proof also implies Burnside's result, states that an irreducible character of a finite group is zero or a root of unity at more than a third of the group elements.

In the present paper we extend $A(\alpha)$ to a larger set. The completion $\tilde{\overline{\mathbb{Q}}}$ of $\overline{\mathbb{Q}}$ with respect to the spectral norm (see [4] and [5]), provides a natural setting for such an extension. The map $A(\cdot) : \overline{\mathbb{Q}} \longrightarrow [0, \infty)$ is continuous with respect to the spectral norm, and it naturally extends to a map, which we will still denote by $A(\cdot)$, from $\tilde{\overline{\mathbb{Q}}}$ to $[0, \infty)$. For any algebraic number α, we define its Siegel norm $\|\alpha\|_{Si}$ by $\|\alpha\|_{Si} = \sqrt{A(\alpha)}$. In this paper we investigate connections between the Siegel norm and the spectral norm on $\overline{\mathbb{Q}}$, and their extensions to the spectral completion $\tilde{\overline{\mathbb{Q}}}$ of $\overline{\mathbb{Q}}$. In the last section we show how one can explicitly compute the Siegel norm for some classes of elements of $\tilde{\overline{\mathbb{Q}}}$.

2 Construction of the Siegel norm

Let us first remark that the function $\|\cdot\|_{Si} : \overline{\mathbb{Q}} \longrightarrow [0, \infty)$ has the following properties.

1. $\|\alpha + \beta\|_{Si} \leq \|\alpha\|_{Si} + \|\beta\|_{Si}$, for any $\alpha, \beta \in \overline{\mathbb{Q}}$.
2. $\|c\alpha\|_{Si} = |c|\|\alpha\|_{Si}$, for any $c \in \mathbb{Q}$, $\alpha \in \overline{\mathbb{Q}}$.

Indeed, let $\alpha, \beta \in \overline{\mathbb{Q}}$. Let K be a number field such that $\alpha, \beta \in K$. Then

$$\sum_{\sigma} |\sigma(\alpha) + \sigma(\beta)|^2 \leq \sum_{\sigma} |\sigma(\alpha)|^2 + \sum_{\sigma} |\sigma(\beta)|^2 + |\sum_{\sigma} \sigma(\alpha)\overline{\sigma(\beta)}| + |\sum_{\sigma} \overline{\sigma(\alpha)}\sigma(\beta)|.$$

Employing Cauchy's inequality, we derive that

$$|\sum_{\sigma}\sigma(\alpha)\overline{\sigma(\beta)}| = |\sum_{\sigma}\overline{\sigma(\alpha)}\sigma(\beta)| \leq \left(\sum_{\sigma}|\sigma(\alpha)|^2\right)^{\frac{1}{2}}\left(\sum_{\sigma}|\sigma(\beta)|^2\right)^{\frac{1}{2}}.$$

Hence $A(\alpha+\beta) \leq A(\alpha)+A(\beta)+2\sqrt{A(\alpha)}\sqrt{A(\beta)}$, so $\sqrt{A(\alpha+\beta)} \leq \sqrt{A(\alpha)}+\sqrt{A(\beta)}$, and the first remark follows.

The second part follows easily from the definition of the map A.

The above properties show that $\|\cdot\|_{Si}$ is a vector space norm on $\overline{\mathbb{Q}}$. Note that $A(\alpha) \leq \|\alpha\|^2$, hence $\|\alpha\|_{Si} \leq \|\alpha\|$, for any $\alpha \in \overline{\mathbb{Q}}$.

For $x \in \widetilde{\overline{\mathbb{Q}}}$ and $\delta > 0$, consider the open ball $B(x,\delta) = \{y \in \widetilde{\overline{\mathbb{Q}}} : \|y-x\| < \delta\}$, and let $n(x,\delta) = \min\{\deg\alpha : \alpha \in B(x,\delta)\}$. We can now state the following theorem.

Theorem 1. *i) The map $A : \overline{\mathbb{Q}} \longrightarrow [0,\infty)$ is continuous with respect to the spectral norm and it extends canonically to a map, which we will still denote by A, from $\widetilde{\overline{\mathbb{Q}}}$ to $[0,\infty)$.*

ii) Let $x \in \widetilde{\overline{\mathbb{Q}}}, x \neq 0$. Then $\|x\|_{Si} \geq \frac{\|x\|}{4\sqrt{n(x,\frac{\|x\|}{4})}}$.

iii) $\|\cdot\|_{Si}$ is a vector space norm on $\widetilde{\overline{\mathbb{Q}}}$.

Proof: i) Let $(x_n)_{n\geq 0}$ be a convergent sequence in $\overline{\mathbb{Q}}$. Then $(x_n)_n$ is Cauchy and let $M > 0$ be such that $|x_n| \leq M$, for any $n \geq 0$. From the proof of the remarks at the beginning of this section it follows that for all $m, n \geq 0$ we have

$$|\sqrt{A(x_n)} - \sqrt{A(x_m)}| \leq \sqrt{A(x_n - x_m)}. \tag{1}$$

On the other hand, since $\sqrt{A(\alpha)} \leq \|\alpha\|$, for any algebraic number α, we derive that

$$\sqrt{A(x_n - x_m)} \leq \|x_n - x_m\|. \tag{2}$$

Combining relations (1) and (2), we obtain

$$|\sqrt{A(x_n)} - \sqrt{A(x_m)}| \leq \|x_n - x_m\|. \tag{3}$$

We have

$$|A(x_n) - A(x_m)| = |\sqrt{A(x_n)} - \sqrt{A(x_m)}||\sqrt{A(x_n)} + \sqrt{A(x_m)}| \leq$$

$$\leq 2M|\sqrt{A(x_n)} - \sqrt{A(x_m)}| \leq 2M\|x_n - x_m\|,$$

and hence

$$|A(x_n) - A(x_m)| \leq 2M\|x_n - x_m\|. \tag{4}$$

Since $(x_n)_n$ is Cauchy with respect to the spectral norm, it follows from inequality (4) that the sequence $(A(x_n))_n$ is Cauchy in $[0,\infty)$, hence it is convergent. Another consequence of relation (4) is that the map A can be extended to $\widetilde{\overline{\mathbb{Q}}}$ as follows. Let $\alpha \in \widetilde{\overline{\mathbb{Q}}}$. Then $\alpha = \lim_{n\to\infty} \alpha_n$, with $\alpha_n \in \overline{\mathbb{Q}}$, and $A(\alpha) := \lim_{n\to\infty} A(\alpha_n) \in [0,\infty)$.

ii) Let $x \in \widetilde{\overline{\mathbb{Q}}}, x \neq 0$, and let $0 < \delta = \frac{\|x\|}{4}$. Also let $\alpha, \alpha_0 \in \overline{\mathbb{Q}} \cap B(x,\delta)$ and denote $n = \deg \alpha$, $n_0 = \deg \alpha_0$. Let K be a finite field extension of $\mathbb{Q}$ such that $\alpha, \alpha_0 \in K$. Denote by m the degree of K over $\mathbb{Q}(\alpha_0)$. From the above choice of α we have $\|\alpha_0 - x\| < \frac{\|x\|}{4}$. It follows that

$$\|\alpha_0\| \geq \|x\| - \|x-\alpha_0\| > \|x\| - \frac{\|x\|}{4} = \frac{3}{4}\|x\|. \tag{5}$$

Let $\sigma_1, \dots, \sigma_{mn_0}$ be the embeddings of K in $\mathbb{C}$. From relation (5) we deduce that

$$\max_{1\leq j\leq n_0 m} |\sigma_j(\alpha_0)| = \|\alpha_0\| > \frac{3}{4}\|x\|.$$

There exist $j_1, \dots, j_m$ such that $|\sigma_{j_1}(\alpha_0)| = |\sigma_{j_2}(\alpha_0)| = \cdots = |\sigma_{j_m}(\alpha_0)| > \frac{3}{4}\|x\|$.

We have $\|\alpha_0 - \alpha\| \leq \|\alpha_0 - x\| + \|x - \alpha\| < \frac{1}{4}\|x\| + \frac{1}{4}\|x\| = \frac{1}{2}\|x\|$.

It follows that $|\sigma_j(\alpha_0) - \sigma_j(\alpha)| < \frac{1}{2}\|x\|$, for any $j \in \{1, 2, \dots, n_0 m\}$. In particular, $|\sigma_{j_1}(\alpha_0) - \sigma_{j_1}(\alpha)| < \frac{1}{2}\|x\|$. Since $|\sigma_{j_1}(\alpha_0)| > \frac{3}{4}\|x\|$, we obtain

$$|\sigma_{j_1}(\alpha)| \geq |\sigma_{j_1}(\alpha_0)| - |\sigma_{j_1}(\alpha_0) - \sigma_{j_1}(\alpha)| > \frac{3}{4}\|x\| - \frac{1}{2}\|x\| = \frac{1}{4}\|x\|.$$

We derive that

$$A(\alpha) = \frac{1}{n_0 m} \sum_{1\leq j\leq n_0 m} |\sigma_j(\alpha)|^2 \geq \frac{1}{n_0 m}\Big(|\sigma_{j_1}(\alpha)|^2 + \cdots + |\sigma_{j_m}(\alpha)|^2\Big)$$

$$\geq \frac{1}{n_0 m} \cdot \frac{\|x\|^2}{16} m = \frac{\|x\|^2}{16 n_0}.$$

Hence

$$\|\alpha\|_{Si} \geq \frac{\|x\|}{4\sqrt{n_0}}, \text{ for any } \alpha \in B\Big(x, \frac{\|x\|}{4}\Big).$$

iii) From the two remarks at the beginning of this section, by continuity it follows that $\|\alpha + \beta\|_{Si} \leq \|\alpha\|_{Si} + \|\beta\|_{Si}$, for any $\alpha, \beta \in \widetilde{\overline{\mathbb{Q}}}$ and also that $\|c\alpha\|_{Si} = |c|\|\alpha\|_{Si}$, for any $c \in \mathbb{Q}$, $\alpha \in \widetilde{\overline{\mathbb{Q}}}$. Moreover, part ii) shows that for $x \in \widetilde{\overline{\mathbb{Q}}}$, $\|x\|_{Si} = 0$ if and only if $x = 0$, which completes the proof of the theorem. □

Recall ([2]) that a *Pisot-Vijayaraghavan* number (or simply a *Pisot* number or a *PV* number) is a real algebraic integer $\alpha > 1$ such that all its conjugates are in absolute value < 1.

We remark that, apart from the constant $1/4$ on its right hand side, the inequality from Theorem 1 part ii) is best possible. Indeed let us choose a *PV* number, β say, a positive integer m, and put $x = \beta^m$. Let d denote the degree of β over $\mathbb{Q}$. Since all the conjugates $\sigma(\beta)$ are in absolute value < 1, it is easy to see (using the fact that a natural power of a *PV* number β of degree d over $\mathbb{Q}$ also has degree d over $\mathbb{Q}$) that if we keep β fixed and let m tend to infinity, the ratio $||x||_{Si}/||x||$ will approach $1/\sqrt{d}$. On the other hand $n(x, ||x||/4) \leq d$. Therefore, for any fixed $\epsilon > 0$, if m is large enough then $||x||_{Si} < (1+\epsilon)\frac{||x||}{\sqrt{n(x,||x||/4)}}$.

3 Explicit computations

In this section we obtain a formula that gives the value of the map $A(\cdot)$ on a large class of elements of $\widetilde{\overline{\mathbb{Q}}}$. To proceed, we introduce a few notations and recall some of the results from [6].

Let $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$, endowed with the Krull topology, and let $C(G_{\mathbb{Q}})$ be the $\mathbb{C}$-Banach algebra of all continuous functions defined on $G_{\mathbb{Q}}$ with values in $\mathbb{C}$ ($\|f\| = \sup\{|f(\sigma)|, \sigma \in G_{\mathbb{Q}}\}$ for any $f \in C(G_{\mathbb{Q}})$). Denote by μ the Haar measure on $G_{\mathbb{Q}}$, normalized such that $\mu(G_{\mathbb{Q}}) = 1$, and let $\int_{G_{\mathbb{Q}}} f(\sigma)d\sigma$ be the corresponding Haar integral of any continuous function $f : G_{\mathbb{Q}} \to \mathbb{C}$.

Let x be an element of $\widetilde{\overline{\mathbb{Q}}}$ and let $\{x_n\}_n$ a Cauchy sequence in $\overline{\mathbb{Q}}$ (relative to the spectral norm $\|\cdot\|$) in the class of x, i.e. $\lim\limits_{n\to\infty} x_n \overset{\|.\|}{=} x$. Since $|\sigma(x_{n+p}) - \sigma(x_n)| \leq \|x_{n+p} - x_n\|$, for all $\sigma \in G_{\mathbb{Q}}$, $\{\sigma(x_n)\}_n$ is also a Cauchy sequence in $\mathbb{C}$. Let $x_{(\sigma)}$ be the limit of $\{\sigma(x_n)\}_n$ in $\mathbb{C}$. It can be shown that $\|x\| = \sup\left\{\left|x_{(\sigma)}\right|, \sigma \in G_{\mathbb{Q}}\right\}$ and that $\|x\| = \|\varphi_x\| = \sup\{|\varphi_x(\sigma)|, \sigma \in G_K\}$, where $\varphi_x : G_{\mathbb{Q}} \to \mathbb{C}$, $\varphi_x(\sigma) = x_{(\sigma)}$. In [6] it is shown that for any $x \in \widetilde{\overline{\mathbb{Q}}}$ the function φ_x is continuous and that the mapping

$$\Phi : \widetilde{\overline{\mathbb{Q}}} \to C(G_{\mathbb{Q}}), \qquad \Phi(x) = \varphi_x \tag{6}$$

is an isomorphism between the $\mathbb{C}-$Banach algebras $\widetilde{\overline{\mathbb{Q}}}$ and $C(G_{\mathbb{Q}})$.

Following [6], we introduce a continuous function $H : G_{\mathbb{Q}} \to [0, 1]$ with a special property: it is a measure preserving function, in the sense that it takes a Haar measurable subset of $G_{\mathbb{Q}}$ to a Lebesgue measurable subset of $[0, 1]$.

We begin by fixing a tower of subgroups of finite index $G_{\mathbb{Q}} \supset G_1 \supset \cdots \supset G_n \supset \cdots \supset \{e\}$ for $G_{\mathbb{Q}}$, where $\bigcap\limits_{i=0}^{\infty} G_i = \{e\}$ and e is the identity of $G_{\mathbb{Q}}$, and for this tower we consider a complete set of left cosets $\{\Delta_{G_i}\}_{i\geq 1}$ of $G_{\mathbb{Q}}$ relative to the subgroup G_i, of the form $\Delta_{G_i} = \left\{G_i, \sigma_2^{(i)} G_i, ..., \sigma_{k_i}^{(i)} G_i\right\}$ (where $k_i = [G_{\mathbb{Q}} : G_i]$, i.e. k_i is the index of G_i in $G_{\mathbb{Q}}$). We choose an ordering in every Δ_{G_i} such that if $n \le m$, $\sigma_n^{(i)} G_i \subset \sigma_s^{(i-1)} G_{i-1}$ and $\sigma_m^{(i)} G_i \subset \sigma_l^{(i-1)} G_{i-1}$, then $s \le l$.

Consider the partitions $G_{\mathbb{Q}} = G_i \cup \sigma_2^{(i)} G_i \cup \cdots \cup \sigma_{k_i}^{(i)} G_i$ and respectively $[0,1] = \left[0, \frac{1}{k_i}\right) \cup \left[\frac{1}{k_i}, \frac{2}{k_i}\right) \cup \cdots \cup \left[\frac{k_i - 1}{k_i}, 1\right]$, for every $i = 1, 2, \ldots$. For an $i \in \{1, 2, \ldots\}$ we define the step function H_i such that $H_i(\sigma) = \frac{2j-1}{2k_i}$ if and only if $\sigma \in \sigma_j^{(i)} G_i$, $j = 1, 2, \ldots, k_i$. It is easy to see that the function $H = \lim_{n\to\infty} H_n$ is a continuous function and

$$H^{-1}\left(\left[\frac{j-1}{k_i}, \frac{j}{k_i}\right]\right) \supseteq \sigma_j^{(i)} G_i \supseteq H^{-1}\left(\left(\frac{j-1}{k_i}, \frac{j}{k_i}\right)\right) \text{ for every } j = 1, 2, \ldots, k_i.$$

The function H depends not only on the tower $G_{\mathbb{Q}} \supset G_1 \supset \cdots \supset G_n \supset \cdots \supset \{e\}$, but also on the chosen ordering in every Δ_{G_i}, $i = 1, 2, \ldots$

Next, we find a formula for evaluating the map A at elements of $\tilde{\tilde{\mathbb{Q}}}$ that correspond, via the isomorphism (6), to functions of the form $f \circ H$, for a continuous $f : [0,1] \to \mathbb{C}$.

Proposition 1. *Let $H : G_{\mathbb{Q}} \to [0,1]$ be the above defined map, and let $f : [0,1] \to \mathbb{C}$ be a continuous function. Let $x_f \in \tilde{\tilde{\mathbb{Q}}}$ be the element corresponding to the function $f \circ H$ under the isomorphism* (6)*. Then*

$$A(x_f) = \int_0^1 |f(t)|^2 dt.$$

Proof: Let $x_f \in \tilde{\tilde{\mathbb{Q}}}$ be such an element. Let $(\alpha_j)_j$ be a sequence of algebraic numbers that converges in the spectral norm to x_f. Let $M_j = \{\sigma \in G_{\mathbb{Q}} : \sigma(\alpha_j) = \alpha_j\}$, and let $n_j = [G_{\mathbb{Q}} : M_j] = \deg_{\mathbb{Q}}(\alpha_j)$.

Consider a coset decomposition of $G_{\mathbb{Q}}$ with respect to M_j:

$$G_{\mathbb{Q}} = \bigcup_{s=1}^{n_j} \sigma_s M_j\,,$$

and let $\alpha_1^{(j)}, \alpha_2^{(j)}, \cdots, \alpha_{n_j}^{(j)}$ be the conjugates of α_j over $\mathbb{Q}$, where $\sigma_s(\alpha_j) = \alpha_s^{(j)}$.

For $\sigma \in \sigma_s M_j$, we have $\sigma = \sigma_s m_j$, with $m_j \in M_j$, and deduce that

$$\varphi_{\alpha_j}(\sigma) = \varphi_{\alpha_j}(\sigma_s m_j) = (\sigma_s m_j)(\alpha_j) = \sigma_s(\alpha_j) = \alpha_s^{(j)}.$$

Hence

$$A(\alpha_j) = \frac{1}{n_j}|\alpha_1^{(j)}|^2 + \frac{1}{n_j}|\alpha_2^{(j)}|^2 + \cdots + \frac{1}{n_j}|\alpha_{n_j}^{(j)}|^2 =$$

$$= \sum_{s=1}^{n_j} \int_{\sigma_s M_j} |\varphi_{\alpha_j}(\sigma)|^2 d\sigma = \int_{G_{\mathbb{Q}}} |\varphi_{\alpha_j}(\sigma)|^2 d\sigma.$$

Let $M > 0$ be such that $|\varphi_{\alpha_j}(\sigma)|$, $|\varphi_{x_f}(\sigma)| \le M$ for any $\sigma \in G_{\mathbb{Q}}$. For $\sigma \in G_{\mathbb{Q}}$ one has:

$$\Big||\varphi_{\alpha_j}(\sigma)|^2 - |\varphi_{x_f}(\sigma)|^2\Big| = \Big||\varphi_{\alpha_j}(\sigma)| + |\varphi_{x_f}(\sigma)|\Big| \cdot \Big||\varphi_{\alpha_j}(\sigma)| - |\varphi_{x_f}(\sigma)|\Big|$$

$$\le 2M\Big|\varphi_{\alpha_j}(\sigma) - \varphi_{x_f}(\sigma)\Big| \le 2M \left\|\varphi_{\alpha_j} - \varphi_{x_f}\right\| = 2M \left\|\alpha_j - x_f\right\|.$$

This shows that the sequence $\left(|\varphi_{\alpha_j}(\cdot)|^2\right)_j$ converges uniformly to $|\varphi_{\alpha_j}(\cdot)|^2$, and hence

$$\int_{G_{\mathbb{Q}}} |\varphi_{\alpha_j}(\sigma)|^2 d\sigma \longrightarrow \int_{G_{\mathbb{Q}}} |\varphi_{x_f}(\sigma)|^2 d\sigma.$$

Since $A(x_f) = \lim_{j\to\infty} A(\alpha_j)$, and $\varphi_{x_f} = f \circ H$ we obtain that

$$A(x_f) = \int_{G_{\mathbb{Q}}} |(f \circ H)(\sigma)|^2 d\sigma. \tag{7}$$

In [6] it is proven that for any continuous function $g : [0,1] \to \mathbb{C}$ one has

$$\int_0^1 g(t)dt = \int_{G_{\mathbb{Q}}} (g \circ H)(\sigma) d\sigma.$$

Choosing $g(t) = |f(t)|^2$, we obtain

$$\int_0^1 |f(t)|^2 dt = \int_{G_{\mathbb{Q}}} |(f \circ H)(\sigma)|^2 d\sigma \tag{8}$$

Combining relations (7) and (8) we conclude that

$$A(x_f) = \int_0^1 |f(t)|^2 dt, \tag{9}$$

which completes the proof of the proposition. □

We end this paper with a couple of examples.

Example 1. Let $f : [0,1] \to \mathbb{C}$,

$$f(t) = \exp(-2\pi i n t) + \exp(2\pi i n t),$$

and let $x_f := \Phi^{-1}(f \circ H) \in \widetilde{\overline{\mathbb{Q}}}$. Then

$$A(x_f) = \int_0^1 |2\cos(2\pi n t)|^2 dt = 2.$$

Note that this example may be interpreted in some sense as a limiting case in $\widetilde{\overline{\mathbb{Q}}}$ of the sequence of examples in $\overline{\mathbb{Q}}$ provided by Siegel, mentioned above in the introduction, which showed that the best possible constant λ_0 in Siegel's trace problem is at most 2.

Example 2. Let $f : [0,1] \to \mathbb{C}$, $f(t) = 2t$ and let $x_f := \Phi^{-1}(f \circ H) \in \tilde{\overline{\mathbb{Q}}}$. Then

$$A(x_f) = \int_0^1 4t^2 dt = \frac{4}{3}\,.$$

As an element of $\tilde{\overline{\mathbb{Q}}}$, we know that x_f is a limit of a Cauchy sequence $\{x_n\}_n$ in $\overline{\mathbb{Q}}$, which must then satisfy $\lim\limits_{n\to\infty} x_n = \frac{4}{3}$. Let us remark that x_f cannot be a limit of algebraic integers, since by Siegel's result, for any algebraic integer x_n we must have $A(x_n) \in \{0,1\}$ or $A(x_n) \geq \frac{3}{2}$.

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