Conformally flat generalized globally framed f-space-forms

by

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Abstract

Conformally flat generalized globally framed f-space-forms are studied. In particular, in the case of corank $s > 2$, the φ-sectional curvature c, which is pointwise constant, determines the curvature tensor field. The constancy of c implies the flatness of the manifold. If c is not constant, a local classification of the considered spaces is obtained. This allows to produce explicit examples and to discuss the existence of those spaces whose underlying f-structure is of a particular type.

Key Words: Conformal flatness, f-structure, space-form, generalized space form.

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1 Introduction

The study of the curvature and the classification of Riemannian manifolds in the context of almost Hermitian or contact Geometry are classical problems. Analogously, interesting questions arise looking at the behavior of the curvature of manifolds carrying a metric f-structure with parallelizable kernel (briefly, $f.pk$-structure).

In particular, in [12], we introduced the concept of generalized $f.pk$-space form, requiring that the curvature of a metric $f.pk$-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, involves a set of smooth functions F_1, F_2, F_{ij}, with $F_{ij} = F_{ji}$ for $i, j \in \{1, \ldots, s\}$. Even if the problem of the classification of these spaces is quite far from being solved, several results are known. In particular one gets the pointwise constancy of the φ-sectional curvature c.

In this paper we study conformally flat generalized $f.pk$-space-forms of any dimension $2n + s$, with $n \geq 1$ and $s > 2$. The paper is organized as follows. In Sections 2, 3, several properties of these spaces are stated, showing that the curvature tensor depends on the function c, only. The constancy of c implies the

flatness of the manifold. In Section 4, which is devoted to the case c non constant, we prove that in an open neighborhood of a point p such that $c(p) \neq 0$, we have $c < 0$ and (M, g) is locally a warped product manifold. Moreover, a conformal change of g determines a new metric $\bar{g}$ so that $(M, \bar{g})$ is locally isometric with the Riemannian product of two manifolds, both with constant sectional curvature, one the opposite of the other. We also give explicit examples of conformally flat generalized $f.pk$-space-forms with negative φ-sectional curvature. Finally, in Section 5, we discuss the existence of conformally flat generalized $f.pk$-space-forms whose underlying structure is of special type.

All manifolds are assumed to be connected. Following [18], for the curvature of a Riemannian manifold we adopt the definitions $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, $R(X, Y, Z, W) = g(X, R(Z, W, Y))$ and, for any 1-form η, for any $X, Y \in \Gamma(TM)$, $2d\eta(X, Y) = X(\eta Y) - Y(\eta X) - \eta[X, Y]$. We also use the Einstein convention, omitting the sum symbol for repeated indexes, if there is no doubt.

2 Preliminaries

An $f.pk$-manifold, also called a globally framed f-manifold, is a manifold M^{2n+s} on which is defined an f-structure, that is a $(1, 1)$-tensor field φ of rank $2n$ and corank s, such that $\varphi^3 + \varphi = 0$ and the subbundle ker φ is parallelizable, [4, 9, 14]. Hence ker φ admits a global frame $\{\xi_i\}$, $i \in \{1, \ldots, s\}$, and 1-forms η^i, satisfying $\eta^i(\xi_j) = \delta^i_j$ and $\varphi^2 = -I + \eta^i \otimes \xi_i$, from which $\varphi(\xi_i) = 0$, $\eta^i \circ \varphi = 0$ follow. It is well-known that one can consider a Riemannian metric g on M^{2n+s} satisfying the compatibility condition $g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y)$, for any $X, Y \in \Gamma(TM^{2n+s})$, and the structure $(\varphi, \xi_i, \eta^i, g)$ is then called a *metric $f.pk$-structure*. So, TM^{2n+s} splits as orthogonal sum of its subbundles Im φ and ker φ. We denote their respective differentiable distributions by $\mathcal{D}$ and $\mathcal{D}^\perp$. Obviously, $\varphi_{|\mathcal{D}}$ determines an almost Hermitian structure on the distribution $\mathcal{D}$. Let Φ be the fundamental 2-form on M^{2n+s}, defined by $\Phi(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM^{2n+s})$. An $f.pk$-structure on M^{2n+s} is said to be *normal* if the tensor field $N = [\varphi, \varphi] + 2d\eta^i \otimes \xi_i$ vanishes, $[\varphi, \varphi]$ denoting the Nijenhuis torsion of φ.

Given a metric $f.pk$-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ let $\mathcal{F}$ denote any set of smooth functions F_{ij} on M^{2n+s} such that $F_{ij} = F_{ji}$ for any $i, j \in \{1, \ldots, s\}$. Then M^{2n+s} is called a generalized $f.pk$-space-form, if there exist smooth functions $F_1, F_2, \mathcal{F}$ such that the curvature tensor field satisfies:

$$
\begin{aligned}
R(X, Y, Z) = \; & F_1\{g(\varphi X, \varphi Z)\varphi^2 Y - g(\varphi Y, \varphi Z)\varphi^2 X\} \\
& + F_2\{g(Z, \varphi Y)\varphi X - g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z\} \\
& + \sum_{i,j=1}^{s} F_{ij}\{\eta^i(X)\eta^j(Z)\varphi^2 Y - \eta^i(Y)\eta^j(Z)\varphi^2 X \\
& + g(\varphi Y, \varphi Z)\eta^i(X)\xi_j - g(\varphi X, \varphi Z)\eta^i(Y)\xi_j\}.
\end{aligned} \tag{2.1}
$$

Then, for the Ricci tensor ρ, the Ricci operator Q, the scalar curvature τ, we get:

$$\rho(X,Y) = ((2n-1)F_1 + 3F_2 + \sum_{i=1}^{s} F_{ii})g(\varphi X, \varphi Y) + 2n \sum_{i,j=1}^{s} F_{ij}\eta^i(X)\eta^j(Y) \quad (2.2)$$

$$Q(X) = -((2n-1)F_1 + 3F_2 + \sum_{i=1}^{s} F_{ii})\varphi^2 X + 2n \sum_{i,j=1}^{s} F_{ij}\eta^i(X)\xi_j \quad (2.3)$$

$$\tau = 2n((2n-1)F_1 + 3F_2 + 2\sum_{i=1}^{s} F_{ii}) \quad (2.4)$$

Moreover, as proved in [12], from (2.1) we get that the φ-sectional curvature is p.c. $c = F_1 + 3F_2$ and the mixed sectional curvatures are $K(\xi_k, X) = F_{kk}$.
For $s = 1$, we obtain a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ with $f_1 = F_1$, $f_2 = F_2$ and $f_3 = F_1 - F_{11}$. Generalized Sasakian-space-forms, introduced and studied by Alegre, Blair and Carriazo [1], are locally described, under the conformal flatness hypothesis, by U.K. Kim in [17], as follows.

Theorem 1. *Given a generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$, one has:*
1) if $n > 1$, then M is conformally flat if and only if $f_2 = 0$,
2) if M is conformally flat and ξ is a Killing vector field, then M is flat, or of constant curvature, or locally the product $N^1 \times N^{2n}$, N^1 being a 1-dimensional manifold and N^{2n} an almost Hermitian manifold of constant curvature.
In any case, M is locally symmetric and has constant φ-sectional curvature.

In order to study conformally flat generalized $f.pk$-space-forms, we recall that, given a Riemannian manifold (M, g), $dim\, M = m \geq 3$, the Weyl curvature tensor field C is defined by

$$\begin{aligned} C(X,Y,Z) \;=\; & R(X,Y,Z) \\ & + \tfrac{1}{m-2}(\rho(X,Z)Y - \rho(Y,Z)X + g(X,Z)Q(Y) - g(Y,Z)Q(X)) \quad (2.5) \\ & + \tfrac{\tau}{(m-1)(m-2)}(g(Y,Z)X - g(X,Z)Y). \end{aligned}$$

If $m \geq 4$, then M is conformally flat if and only if $C = 0$. Moreover, if $C = 0$ the Schouten tensor $L = -\frac{1}{m-2}\left(Q - \frac{\tau}{2(m-1)}I\right)$ is a Codazzi tensor, that is it satisfies $(\nabla_X L)Y = (\nabla_Y L)X$, ∇ denoting the Levi-Civita connection, [23].

3 Algebraic properties

We begin this section considering $f.pk$-manifolds $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $s \geq 2$.

Proposition 1. *Let T_1, T_2 be the $(1,3)$-tensor fields defined by*

$$\begin{aligned} T_1(X,Y,Z) = \;& g(\varphi X, \varphi Z)\varphi^2 Y - g(\varphi Y, \varphi Z)\varphi^2 X \\ T_2(X,Y,Z) = \;& g(Z, \varphi Y)\varphi X - g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z \end{aligned}$$

Then, if $n = 1$, one has $T_2 = 3T_1$.

Proof: Assuming $n = 1$, since T_1 and T_2 both vanish on any triplet of vector fields such that one of them is in $\mathcal{D}^\perp$, it is enough to evaluate them on the triplets $(X, \varphi X, Z)$, with $X, Z \in \mathcal{D}$. Given $X, Z \in \mathcal{D}$ there exist smooth functions λ, μ such that $Z = \lambda X + \mu \varphi X$ and by direct calculation one has

$$T_1(X, \varphi X, Z) = -\lambda g(X, X)\varphi X + \mu g(X, X)X$$
$$T_2(X, \varphi X, Z) = -3\lambda g(X, X)\varphi X + 3\mu g(X, X)X = 3T_1(X, \varphi X, Z).$$

$\square$

Lemma 1. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a conformally flat generalized f.pk-space-form. If $n \geq 2$, then $F_2 = 0$ and $c = F_1$ is the φ-sectional curvature.*

Proof: Let X, Z be orthonormal vector fields in $\mathcal{D}$ such that $g(Z, \varphi X) = 0$. By (2.5), (2.1), (2.2) one has $0 = C(X, \varphi X, Z) = -2F_2 \varphi Z$, so $F_2 = 0$ and $c = F_1$. $\square$

Proposition 2. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a conformally flat generalized f.pk-space-form with φ-sectional curvature c. Then, one has:*

$$\begin{aligned} R(X, Y, Z) = \ & c\{g(\varphi X, \varphi Z)\varphi^2 Y - g(\varphi Y, \varphi Z)\varphi^2 X\} \\ & + \sum_{i,j=1}^s F_{ij}\{\eta^i(X)\eta^j(Z)\varphi^2 Y - \eta^i(Y)\eta^j(Z)\varphi^2 X \\ & + g(\varphi Y, \varphi Z)\eta^i(X)\xi_j - g(\varphi X, \varphi Z)\eta^i(Y)\xi_j\}, \end{aligned} \quad (3.1)$$

Proof: The statement easily follows by (2.1), Proposition 1 and Lemma 1. $\square$

Lemma 2. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a conformally flat generalized f.pk-space-form with φ-sectional curvature c. Then, one has: $sc = 2\sum_{i=1}^s F_{ii}$.*

Proof: Let $X \in \mathcal{D}$ be a unit vector field. By (2.1), (2.2) and (2.3) we have $R(X, \varphi X, X) = -(F_1 + 3F_2)\varphi X$, $\rho(X, X) = (2n-1)F_1 + 3F_2 + \sum_{i=1}^s F_{ii}$ and $\rho(X, \varphi X) = 0$. So, the condition $C(X, \varphi X, X) = 0$ implies:

$$s(1-s)(F_1 + 3F_2) - 12(n-1)(n+s-1)F_2 - 2(1-s)\sum_{i=1}^s F_{ii} = 0.$$

Since $s \geq 2$, assuming $n = 1$, we obtain $sc = s(F_1 + 3F_2) = 2\sum_{i=1}^s F_{ii}$. In the case $n \geq 2$, one gets the statement also applying Lemma 1. $\square$

Proposition 3. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$ be a generalized f.pk-space-form. Then M^{2n+s} is conformally flat and Einstein if and only is it is flat.*

Proof: Let M^{2n+s} be conformally flat and Einstein. Since $dim\, M^{2n+s} \geq 4$, M^{2n+s} has constant sectional curvature $c = F_1 + 3F_2$. On the other hand, for any unit $X \in \mathcal{D}$ and $k \in \{1, \ldots, s\}$ we have $K(X, \xi_k) = F_{kk}$. It follows $F_{kk} = c$ and, combining with Lemma 2 we get $c = 0$. Then, from (2.2) and (3.1), $R = 0$ follows. The converse statement is obvious. $\square$

3.1 The case of corank $s > 2$

From now on we assume $s > 2$, since the case $s = 2$ is very special and it will be discussed separately [13].

Proposition 4. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$, $s > 2$, be a conformally flat generalized f.pk-space-form with φ-sectional curvature c. Then, for any $h, k \in \{1, \ldots, s\}$, one has $F_{hk} = \frac{c}{2}\delta_{hk}$.*

Proof: Let $X \in \mathcal{D}$ be a unit vector field and $h \in \{1, \ldots, s\}$. By direct calculation, from $C(X, \xi_k, X) = 0$, also applying Lemma 2, we have:

$$0 = -(s-2)\sum_{j=i}^{s} F_{hj}\xi_j + \frac{1}{2n+s-1}(2(n-1)(s-1)F_1 + ((s-1)(\tfrac{s}{2}+1) - ns)c)\xi_h.$$

Being $s > 2$, taking the scalar product with ξ_k, $k \neq h$ we have $F_{hk} = 0$, while the scalar product with ξ_h gives

$$(s-2)F_{hh} = \frac{4(n-1)(s-1)F_1 + ((s-1)(s+2) - 2ns)c}{2(2n+s-1)}.$$

Hence, for any $h, k \in \{1, \ldots, s\}$ one has $F_{hh} = F_{kk}$ and, by Lemma 2, one gets $sc = 2sF_{hh}$ i.e. $F_{hh} = \frac{c}{2}$. □

Theorem 2. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$, $s > 2$, be a generalized f.pk-space-form with φ-sectional curvature c. Then M^{2n+s} is conformally flat if and only if R satisfies*

$$
\begin{aligned}
R(X, Y, Z) =\ & c(g(Y, Z)X - g(X, Z)Y) \\
& + \tfrac{c}{2}\sum_{i=1}^{s}(\eta^i(X)\eta^i(Z)Y - \eta^i(Y)\eta^i(Z)X \\
& - g(Y, Z)\eta^i(X)\xi_i + g(X, Z)\eta^i(Y)\xi_i),
\end{aligned}
\tag{3.2}
$$

for any X, Y, Z.
Moreover, if M^{2n+s} is conformally flat (or equivalently (3.2) holds), then the Ricci tensor ρ, the Ricci operator Q and the scalar curvature τ are given by:

$$\rho(X, Y) = c(2n + \tfrac{s}{2} - 1)g(X, Y) - c(n + \tfrac{s}{2} - 1)\sum_{i=1}^{s}\eta^i(X)\eta^i(Y) \tag{3.3}$$

$$Q(X) = -c(2n + \tfrac{s}{2} - 1)\varphi^2 X + nc\sum_{i=1}^{s}\eta^i(X)\xi_i \tag{3.4}$$

$$\tau = 2n(2n + s - 1)c \tag{3.5}$$

Proof: Assume that M^{2n+s} is conformally flat. By (3.1) and Proposition 4 we have

$$
\begin{aligned}
R(X, Y, Z) =\ & c(g(\varphi X, \varphi Z)\varphi^2 Y - g(\varphi Y, \varphi Z)\varphi^2 X) \\
& + \tfrac{c}{2}\sum_{i=1}^{s}(\eta^i(X)\eta^i(Z)\varphi^2 Y - \eta^i(Y)\eta^i(Z)\varphi^2 X \\
& + g(\varphi Y, \varphi Z)\eta^i(X)\xi_i - g(\varphi X, \varphi Z)\eta^i(Y)\xi_i).
\end{aligned}
\tag{3.6}
$$

By direct calculation we get (3.2) and then (3.3), (3.4), (3.5). Conversely, if R satisfies (3.2), then (3.3), (3.4), (3.5) hold and one easily gets $C = 0$. □

From Theorem 2, using also Lemma 1 and Proposition 4, one easily obtains the following characterizations.

Theorem 3. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$, $s > 2$, be a generalized f.pk-space-form with φ-sectional curvature c. Then one has:*
i) if $n \geq 2$, then M^{2n+s} is conformally flat if and only if $F_2 = 0$, $F_{hk} = \frac{c}{2}\delta_{hk} = \frac{F_1}{2}\delta_{hk}$, for any $h, k \in \{1, \ldots, s\}$.
ii) if $n = 1$, then M^{2+s} is conformally flat if and only if $F_{hk} = \frac{c}{2}\delta_{hk} = \frac{F_1 + 3F_2}{2}\delta_{hk}$, for any $h, k \in \{1, \ldots, s\}$.

Proof: In both cases, i) and ii), the necessary condition follows from Lemma 1 and Proposition 4.
Vice versa, (2.1) becomes (3.6), taking account of Proposition 1 when $n = 1$.
Then, by direct calculation we obtain (3.2) and conclude as in Theorem 2. □

Proposition 5. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$, $s > 2$, be a conformally flat generalized f.pk-space-form. If the φ-sectional curvature is constant, then M^{2n+s} is flat.*

Proof: Assume that c is constant and, arguing by contradiction, suppose $c \neq 0$. Applying (3.5), τ is constant, so that the Ricci tensor is a Codazzi tensor, [3]. By (3.4) the Ricci operator has two distinct eigenfunctions $\lambda_1 = (2n + \frac{s}{2} - 1)c$ and $\lambda_2 = nc$. The corresponding eigendistributions are $\mathcal{D}, \mathcal{D}^\perp$ of rank $2n \geq 2$ and $s > 2$, respectively, and, since λ_1, λ_2 are constant, they are orthogonal and totally geodesic. It follows that M^{2n+s} is locally a Riemannian product $N^{2n} \times N^s$, N^{2n} being a leaf of $\mathcal{D}$ and N^s a leaf of $\mathcal{D}^\perp$. Hence, given $k \in \{1, \ldots, s\}$ and $X \in \mathcal{D}$, for the sectional curvature we have $K(X, \xi_k) = 0$. On the other hand $K(X, \xi_k) = F_{kk} = \frac{c}{2}$, so obtaining a contradiction. Therefore, $c = 0$ and then $R = 0$. □

4   A local description in the case $c \neq 0$

We are going to describe locally the conformally flat generalized $f.pk$-space-forms $M^{2n+s}(F_1, F_2, \mathcal{F})$, $s > 2$, for which the φ-sectional curvature does not vanish and then, by Proposition 5, it is a non-constant function. Firstly, we recall a result given in [2] as Theorem 4.6, p. 129.

Proposition 6. *Let $\mathcal{F}$ be a totally geodesic foliation on a Riemannian manifold (M, g). If all the mixed sectional curvatures in a point $x_0 \in M$ are positive, then the transverse distribution is not integrable.*

Lemma 3. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$, $s > 2$, be a generalized f.pk-space-form with φ-sectional curvature c. For any $X \in \mathcal{D}$ and $\xi \in \mathcal{D}^\perp$, for the sectional curvature one has $K(X, \xi) = \frac{c}{2}$.*

Proof: Considering $X \in \mathcal{D}$ and $\xi \in \mathcal{D}^\perp$, $\|X\| = 1$, $\|\xi\| = 1$, using (3.2) one obtains $K(X, \xi) = g(R(X, \xi, \xi), X) = c - \frac{c}{2} \sum_{i=1}^{s} \eta^i(\xi)\eta^i(\xi) = \frac{c}{2}$. $\square$

Theorem 4. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$, $s > 2$, be a conformally flat generalized f.pk-space-form with f.pk-structure $(\varphi, \xi_i, \eta^i, g)$ and φ-sectional curvature $c \neq 0$. Then, for any point p such that $c(p) \neq 0$, there exists an open set W where $c < 0$ and (W, g) is a warped product $N_1 \times_f N_2$, with warping function $f = \frac{1}{|c|}$, (N_1, g_0) and (N_2, g_2) being leaves of $\mathcal{D}^\perp$ and $\mathcal{D}$, respectively. Moreover (N_1, g_0) is a flat manifold.*

Proof: Since c is a smooth function the set $M_c = \{p \in M \mid c(p) \neq 0\}$ is an open subset. Moreover, being M^{2n+s} conformally flat, using (3.4) and (3.5) the Schouten tensor acts as $L(X) = \frac{c}{2}\varphi^2 X$, so, at any $p \in M_c$, it has two eigenvalues $\lambda_1(p) = -\frac{c(p)}{2}$ and $\lambda_2(p) = 0$. Following [3, 16.10, p. 436], M_c is dense in M^{2n+s}, and, fixing a point $p \in M_c$, we consider its connected component in M_c, W, which is an open subset of M^{2n+s}, and $c_{|W} \neq 0$ everywhere. Hence $L_{|W}$ has two distinct eigenfunctions $\lambda_1 = -\frac{c}{2}$ and $\lambda_2 = 0$ and the corresponding eigendistributions are $V_1 = \mathcal{D}_{|W}$ and $V_2 = \mathcal{D}^\perp_{|W}$, of rank $2n \geq 2$ and $s > 2$, respectively. We obtain that V_1, V_2 are integrable and totally umbilical distributions and $X(\lambda_1) = 0$ for any $X \in V_1$, [3, 7, 21]. Hence the function c is constant on the leaves of V_1. We recall that if λ, μ are distinct eigenfunctions, for any $Y \in V_\lambda$, $X, Z \in V_\mu$ one has

$$Y(\mu)g(X, Z) = (\lambda - \mu)g(\nabla_X Y, Z). \tag{4.1}$$

Thus, for any $Y \in \mathcal{D}_{|W}$, $\xi, \xi' \in \mathcal{D}^\perp_{|W}$ we get $cg(\nabla_\xi Y, \xi') = 0$ so that $g(\nabla_\xi \xi', Y) = 0$ and the distribution $\mathcal{D}^\perp_{|W}$ is totally geodesic. By Proposition 6 and Lemma 3 we have $c_{|W} < 0$. It follows that $(W, g_{|W})$ is locally isometric to a twisted product manifold $N_1 \times_f N_2 = (N_1 \times N_2, g_0 + f^2 g_2)$, where N_1 is a leaf of $\mathcal{D}^\perp$, N_2 a leaf of $\mathcal{D}$, f a smooth positive function such that $H = -grad\,log f$ is the mean curvature vector field of N_2, [22]. We prove that f is constant on N_2 and $f = \frac{1}{|c|}$. In fact, applying (4.1), for any $X, Z \in \mathcal{D}$ and $i \in \{1, \ldots, s\}$, one has $-\frac{1}{2}\xi_i(c)g(X, Z) = \frac{c}{2}g(\nabla_X \xi_i, Z)$ so that

$$g(\nabla_X Z, \xi_i) = \xi_i(log|c|)g(X, Z). \tag{4.2}$$

It follows that $H = \sum_{i=1}^{s} \xi_i(log|c|)\xi_i = grad\,log|c|$. Moreover, for any X tangent to N_2, we have $[H, X] = 0$ so $\nabla_X H = \nabla_H X$ is tangent to N_2, $\mathcal{D}^\perp_{|W}$ being totally geodesic. It follows $\nabla_X^\perp H = 0$. Hence $\mathcal{D}_{|W}$ is a spherical foliation, $N_1 \times_f N_2$ a warped product manifold, that is f is constant on N_2. Finally, by the relation

$H = -grad\,log\,f = grad\,log|c|$ we get $|c|f = \alpha$, α being a constant and choosing $\alpha = 1$ we have $f = \frac{1}{|c|}$. Finally, the Riemannian manifold (N_1, g_0) is flat. In fact it is a leaf of $\mathcal{D}^\perp_{|W}$ which is totally geodesic and flat. $\square$

In order to obtain more information on the Riemannian structure of the manifolds considered in Theorem 4, we firstly recall a result due to Yau, [24].

Theorem 5. *Let M^m, $m \geq 3$, be a conformally flat Riemannian manifold. Suppose that M is a non trivial Riemannian product $M_1 \times M_2$. Then both M_1, M_2 have constant curvature and if both M_1, M_2 have dimension ≥ 2, then the curvatures of M_1, M_2 differ by a sign.*

Now, we consider an *f.pk*-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ as in Theorem 4, the open set W such that $c_{|W} < 0$ and the metric $\bar{g} = c^2 g_{|W}$. Then, both the distributions $\mathcal{D}_{|W}$, $\mathcal{D}^\perp_{|W}$, which are $\bar{g}$-orthogonal, are totally geodesic, [3]. This can be checked by direct calculation, observing that the Levi-Civita connections $\overline{\nabla}$, ∇ of $\bar{g}, g$ are related by

$$\overline{\nabla}_X Y = \nabla_X Y + g(H,X)Y + g(H,Y)X - g(X,Y)H, \qquad (4.3)$$

where $H = grad\,log|c| \in \mathcal{D}^\perp$ is the mean curvature vector field of the foliation $\mathcal{D}$ on $(W, g_{|W})$.

Hence $(W, \bar{g})$ is locally isometric to the Riemannian product $(N_1, \bar{g}_1) \times (N_2, \bar{g}_2)$, $(N_1, \bar{g}_1)$ being a leaf of $\mathcal{D}^\perp$, $(N_2, \bar{g}_2)$ a leaf of $\mathcal{D}$. Note that $(W, \bar{g})$ is conformally flat, also. By Theorem 5, either $\bar{g}$ is a flat metric, so that both the metrics $\bar{g}_1, \bar{g}_2$ are flat, or $(N_1, \bar{g}_1)$ and $(N_2, \bar{g}_2)$ have non zero constant curvature, one the opposite of the other. Now we evaluate the curvature of $(N_2, \bar{g}_2)$. Firstly we remark that the curvature $\overline{R}$ of $\bar{g}$, for every vector fields X, Y, Z, acts as:

$$\begin{aligned}
\overline{R}(X,Y,Z) = \ & R(X,Y,Z) + g(X,Z)(\nabla_Y H - g(Y,H)H) \\
& -g(Y,Z)(\nabla_X H - g(X,H)H) \\
& -(g(\nabla_Y H, Z) - g(Y,H)g(Z,H) + \|H\|^2 g(Y,Z))X \\
& +(g(\nabla_X H, Z) - g(X,H)g(Z,H) + \|H\|^2 g(X,Z))Y
\end{aligned} \qquad (4.4)$$

where $\|H\|^2 = g(H,H)$.

Moreover, for any $X \in \mathcal{D}$, since $\overline{\nabla}_X H = 0$, by (4.3) we have

$$\nabla_X H = -\|H\|^2 X \qquad (4.5)$$

In particular $\|H\|$ is constant on any leaf of $\mathcal{D}$.

Let $X, Y \in \mathcal{D}$ be $\bar{g}$-orthonormal. By (4.4), (4.5) and (3.2) we get

$$\overline{R}(X,Y,Y) = R(X,Y,Y) + \|H\|^2 g(Y,Y)X = \frac{c + \|H\|^2}{c^2} X.$$

Then, for the sectional curvature we have $\overline{K}(X,Y) = \frac{c+\|H\|^2}{c^2}$ and it follows that $(N_2, \bar{g}_2)$ has constant curvature $k = \frac{c+\|H\|^2}{c^2}$.

Obviously $-k$ is the constant sectional curvature of the leaves of $\mathcal{D}^{\perp}$.
Finally, we point out that the condition $\overline{R}(X, \xi, Z) = 0$ for any $X, Z \in \mathcal{D}$ and $\xi \in \mathcal{D}^{\perp}$ gives

$$\nabla_{\xi} H = g(\xi, H)H + \frac{c}{2}\xi. \tag{4.6}$$

Summing up, the sign of $k = \frac{c + \|H\|^2}{c^2}$ determines the model spaces of the manifold $(W, \bar{g})$. More precisely, these spaces are $\mathbb{H}^s(-k) \times S^{2n}(k)$, $\mathbb{R}^s \times \mathbb{R}^{2n} = \mathbb{R}^{2n+s}$, $S^s(-k) \times \mathbb{H}^{2n}(k)$ according the cases $k > 0$, $k = 0$, $k < 0$, respectively. As usual, $\mathbb{H}^m(k)$, $S^m(k)$ denote, respectively, the hyperbolic space and the sphere endowed with the metric of curvature k, as well as $\mathbb{R}^m$ is endowed with the Euclidean metric.

The next proposition characterizes the conformal flatness of the local models $N_1 \times_f N_2$ of the manifold $(W, g_{|W})$ considered in Theorem 4.

Proposition 7. *Let* (M_1, g_0) *be a flat Riemannian manifold with* $\dim M_1 \geq 2$, $c : M_1 \to \mathbb{R}$ *a non-constant smooth function such that* $c \neq 0$ *everywhere and put* $H = grad \, log|c|$. *Given a Riemannian manifold* (M_2, g_2) *with constant sectional curvature* k, *the following conditions are equivalent:*
i) the warped product manifold $M_1 \times_{\frac{1}{|c|}} M_2$ *is conformally flat*

ii) $k = \frac{c + \|H\|^2}{c^2}$ *and* $\nabla^0_{\xi} H = g_0(\xi, H)H + \frac{c}{2}\xi$, *for any* $\xi \in TM_1$, ∇^0 *denoting the Levi-Civita connection of* (M_1, g_0).

Proof: The statement follows by Theorem 3.3 in [6], which characterizes the conformal flatness of multiply warped product spaces. $\qquad\square$

We recall the following result coming from Theorem 3.4 in [6].

Proposition 8. *Let* $M = U^s \times_f F$ *be a warped product space where* $U^s \subset \mathbb{R}^s$, $s \geq 2$ *and* $\dim F \geq 2$. *Then,* M *is conformally flat if and only if the warping function satisfies* $f(x) = a\|x\|^2 + g_0(b, x) + d$ *for all* $x \in U$, *where* $a > 0, d \in \mathbb{R}$ *and* $b \in \mathbb{R}^s$.
Moreover, the sectional curvature of F *is given by* $K = \|b\|^2 - 4ad$.

Finally, for any $s \in \mathbb{N}$, $s > 2$, we give explicit examples of conformally flat generalized *f.pk*-space-forms with negative φ-sectional curvature.

Example 1. *Given* $s \in \mathbb{N}$, $s > 2$, *and* $k \in \mathbb{R}$, $k > 0$, *we consider the open set* $U = \{t = (t^1, \ldots, t^s) \in \mathbb{R}^s \mid \|t\|^2 > 4k\}$ *and the function* $c : U \to \mathbb{R}$ *defined by* $c(t) = \frac{4}{4k - \|t\|^2}$. *Put* $f = \frac{1}{|c|}$. *On the warped product* $M^{2n+s} = U \times_f S^{2n}$ *we want to define an f.pk-structure* (φ, ξ_i, η^i) *compatible with the metric* $g = g_0 + \frac{1}{c^2}g_2$, g_2 *being the metric of constant curvature* k *on* S^{2n}. *To this aim one needs to require that* S^{2n} *admits an almost Hermitian structure* (J, g_2) *and then* $n = 1$ *or* $n = 3$, *[18]. By direct calculation from Proposition 7 or by Proposition 8, choosing* $a = \frac{1}{4}$, $b = 0$, $d = -k$, *we have that* M^{2n+s}, $n = 1, 3$, *is conformally*

flat. For any $i \in \{1, \ldots, s\}$ we put $\xi_i = \frac{\partial}{\partial t^i}$, $\eta^i = dt^i$ and define φ by $\varphi(\xi_i) = 0$ and $\varphi(X) = J(X)$ for any $X \in TS^{2n}$, $n = 1, 3$.
Then it is easy to check that $(\varphi, \xi_i, \eta^i, g_0 + \frac{1}{c^2}g_2)$ is a metric f.pk-structure on M^{2n+s}, $n = 1, 3$. Finally, a direct calculation, using the curvature formulas of a warped product, allows to show that (3.2) holds and M^{2n+s}, $n = 1, 3$, is a generalized f.pk-space-form with φ-sectional curvature c.

Example 2. *Let c be the smooth function defined on $((\mathbb{R}^s)^*, g_0)$ by $c(t) = -\frac{4}{\|t\|^2}$. For any $n \geq 1$ we consider the canonical Hermitian structure (J_0, g_0') on $\mathbb{R}^{2n}$. We put $f = \frac{1}{|c|}$ and consider the warped product $M^{2n+s} = (\mathbb{R}^s)^* \times_f \mathbb{R}^{2n}$. By direct calculation from Proposition 7 or by Theorem 8, choosing $a = \frac{1}{4}$, $b = 0$, $d = 0$, we have that M^{2n+s} is conformally flat. For any $i \in \{1, \ldots, s\}$ we put $\xi_i = \frac{\partial}{\partial t^i}$, $\eta^i = dt^i$ and define φ by $\varphi(\xi_i) = 0$ and $\varphi(X) = J(X)$ for any $X \in T\mathbb{R}^{2n}$.*
Then it is easy to check that $(\varphi, \xi_i, \eta^i, g)$, $g = g_0 + \frac{1}{c^2}g_0'$, is a metric f.pk-structure on M^{2n+s}. Finally, computing the curvature tensor field, as in Example 1, we obtain (3.2) so that the manifold is a generalized f.pk-space-form with φ-sectional curvature c.

Example 3. *Given $s > 2$ and $k \in \mathbb{R}$, $k < 0$, we define on $\mathbb{R}^s$ the smooth function c by $c(t) = \frac{4}{4k - \|t\|^2}$ and we put $f = \frac{1}{|c|}$. We consider (D_r^{2n}, g_r), where $r = \frac{1}{\sqrt{-k}}$, $D_r^{2n} = \{x \in \mathbb{R}^{2n} | \|x\|^2 < r^2\}$, $g_r = (\frac{2r^2}{r^2 - \|x\|^2})^2 g_0'$. Hence g_r is compatible with the almost complex structure J induced on D_r^{2n} by the canonical complex structure on $\mathbb{R}^{2n}$. Then, by direct calculation from Proposition 7 or by Theorem 8, choosing $a = \frac{1}{4}$, $b = 0$, $d = -k$, we have that the warped product $M^{2n+s} = \mathbb{R}^s \times_f D_r^{2n}$ is conformally flat. For any $i \in \{1, \ldots, s\}$ we put $\xi_i = \frac{\partial}{\partial t^i}$, $\eta^i = dt^i$ and define φ by $\varphi(\xi_i) = 0$ and $\varphi(X) = J(X)$ for any $X \in T(D_r^{2n})$.*
It is easy to check that $(\varphi, \xi_i, \eta^i, g)$, when $g = g_0 + \frac{1}{c^2}g_r$, is a metric f.pk-structure on M^{2n+s}. As in the above examples, a direct calculation gives (3.2) showing that M^{2n+s} is a generalized f.pk-space-form with φ-sectional curvature c.

5 Compatibility with underlying geometric structures

Several subclasses of metric f.pk-manifolds have been studied from different points of view (see [4, 5, 8, 9, 10, 11] and references therein). In this section we discuss the existence of conformally flat generalized f.pk-space-forms M^{2n+s}, $n \geq 1, s > 2$, whose underlying f.pk-structure is of some special type.

We begin considering $\mathcal{K}$-structures. As in [4], a metric f.pk-manifold is said to be:

i) a $\mathcal{K}$-manifold if it is normal and the fundamental 2-form Φ is closed. In such a manifold the vector fields ξ_i, $i \in \{1, \ldots, s\}$, are Killing,

ii) an $\mathcal{S}$-manifold if it is a $\mathcal{K}$-manifold such that $d\eta^i = \Phi$, $i \in \{1, \ldots, s\}$,

iii) a $\mathcal{C}$-manifold if it is a $\mathcal{K}$-manifold such that $d\eta^i = 0$, $i \in \{1, \ldots, s\}$.

We shall prove that a conformally flat generalized $f.pk$-space-form M^{2n+s}, $n \geq 1, s > 2$, with underlying $\mathcal{K}$-structure, is a flat $\mathcal{C}$-manifold. Then, M^{2n+s} is locally $\mathbb{R}^s \times \mathbb{C}^n$, according to the well-known result stating that a $\mathcal{C}$-manifold is locally a product of a flat s-dimensional manifold and a $2n$-dimensional Kähler manifold.

Namely, let us suppose $c \neq 0$ and recall that, owing to the normality, a $\mathcal{K}$-manifold satisfies $[\xi_i, \xi_j] = 0$ and $\mathcal{L}_{\xi_j}\eta^i = 0$ for any $i, j \in \{1, \ldots, s\}$, [4, 14]. Hence $d\eta^i(X, \xi_j) = 0$ for any $X \in \Gamma(TM^{2n+s})$. By the results of the previous section, the conformal flatness, when $c \neq 0$, implies that the distribution $\mathcal{D}$ should be integrable so involutive and $2d\eta^i(X, Y) = -\eta^i([X, Y]) = 0$, for any $X, Y \in \mathcal{D}$ and $i \in \{1, \ldots, s\}$. Thus, $d\eta^i = 0$ for every $i \in \{1, \ldots, s\}$. That forces the structure to be a $\mathcal{C}$-structure and by Remark 2 in [12] we know that if $M^{2n+s}(F_1, F_2, \mathcal{F})$ is a generalized $f.pk$-space-form with an underlying C-structure, then $F_{ij} = 0$ for any $i, j \in \{1, \ldots, s\}$ and $F_1 = F_2$. Hence, the conformal flatness implies, for $n \geq 2$, $F_2 = 0, c = F_1 = 0$ and, for $n = 1$, $F_1 + 3F_2 = c$, $0 = F_{hh} = \frac{c}{2}$ and $c = 0$.

Therefore, the case to be discussed is that of flat $\mathcal{K}$-manifolds. Obviously, $\mathcal{S}$-manifolds are excluded since they cannot have constant curvature [19].

Now, in a $\mathcal{K}$-manifold, the Killing condition on the ξ_i's and $[\xi_i, \xi_j] = 0$ imply $\nabla_{\xi_i}\xi_j = 0$ and $\nabla_X\xi_i \in \mathcal{D}$, $\nabla_{\xi_i}X \in \mathcal{D}$, for any $X \in \mathcal{D}$. Then, evaluating the mixed sectional curvatures, we obtain $K(X, \xi_i) = \|\nabla_X\xi_i\|^2$ and the flatness implies $\nabla_X\xi_i = 0$. Thus $\nabla\xi_i = 0$ and the manifold is a $\mathcal{C}$-manifold.

We recall that the condition $d\eta^i = 0$ for any $i \in \{1, \ldots, s\}$ holds for the classes of almost $\mathcal{C}$-manifolds and of (almost) Kenmotsu $f.pk$-manifolds, [10].

Firstly, we consider a conformally flat generalized $f.pk$-space-form with underlying almost $\mathcal{C}$-structure, with φ-sectional curvature c, and we prove that such a structure turns out to be a $\mathcal{C}$-structure, so we fall in the above described situation. In fact, by definition, we have $d\Phi = 0$, $d\eta^i = 0$, $1 \leq i \leq s$, and the leaves of the foliation defined by the distribution $\mathcal{D}$, called the canonical foliation, are almost Kähler and minimal, [15]. Now, assume that $c \neq 0$. Being $H = 0$, directly by (4.6), we get $c_{|W} = 0$ on each W provided by Theorem 4. It follows that $M_c = \emptyset$ and c vanishes everywhere on M^{2n+s}, obtaining a contradiction.

Hence we get $c = 0$ and then $R = 0$. As proved in [20] in any almost $\mathcal{C}$-manifold one has $\tau - \tau^* - \sum_{i=1}^s Ric(\xi_i, \xi_i) + \frac{1}{2}\|\nabla\varphi\|^2 = 0$, where τ denotes the scalar curvature and τ^* the *-scalar curvature. Hence, we get $\nabla\varphi = 0$, and the metric $f.pk$-manifold is a $\mathcal{C}$-manifold, [4].

Now we are going to discuss Kenmotsu $f.pk$-manifolds.

By definition, a metric $f.pk$-manifold M^{2n+s}, with $f.pk$-structure $(\varphi, \xi_i, \eta^i, g)$, is said to be a Kenmotsu $f.pk$-manifold if it is normal, the 1-forms η^i's are all closed and $d\Phi = 2\eta^j \wedge \Phi$ for some and then only for one 1-form, that, up to

a rearrangement, we fix as η^1. We recall that in a Kenmotsu $f.pk$-manifold we have $\nabla \xi_1 = -\varphi^2$, $\nabla \xi_i = 0$, $i \in \{2, \ldots, s\}$. In [12] we proved that, if $M^{2n+s}(F_1, F_2, \mathcal{F})$ is a generalized $f.pk$-space-form with structure of Kenmotsu type and p.c. φ-sectional curvature c, then $F_{11} = -1$ and $F_{ij} = 0$ for any $(i, j) \neq (1, 1)$. Hence we get:

Proposition 9. *Let $M^{2n+s}(F_1, F_2, \mathcal{F})$, $s > 2$, be a generalized $f.pk$-space-form with underlying structure of Kenmotsu type, and p.c. φ-sectional curvature c. Then M^{2n+s} cannot be conformally flat.*

Proof: Assuming that M^{2n+s} is conformally flat, by Theorem 3 we have $F_{hh} = \frac{c}{2}$ for any $h \in \{1, \ldots, s\}$, so $F_{22} = 0$ implies $c = 0$ and then $0 = F_{11} = -1$, a contradiction.   □

Finally, we consider almost Kenmotsu $f.pk$-manifolds.

Definition 1. *A metric $f.pk$-manifold M of dimension $2n + s$, $s \geq 1$, with $f.pk$-structure $(\varphi, \xi_i, \eta^i, g)$, is said to be an almost Kenmotsu $f.pk$-manifold if the 1-forms η^i's are closed and $d\Phi = 2\eta^1 \wedge \Phi$.*

In the next Lemma we summarize some properties stated in [11]. As usual, N denotes the normality tensor field, $N = [\varphi, \varphi] + 2d\eta^i \otimes \xi_i$.

Lemma 4. *Let M^{2n+s} be an almost Kenmotsu $f.pk$-manifold. Then, considering $h_i = \frac{1}{2}\mathcal{L}_{\xi_i}\varphi$, for each $i \in \{1, \ldots, s\}$, one has:*
1) h_i is a symmetric operator and $h_i \circ \varphi + \varphi \circ h_i = 0$,
2) $\nabla_X \xi_i = -\varphi h_i X$, for any $i \in \{2, \ldots, s\}$, for any $X \in \Gamma(TM^{2n+s})$,
3) $\nabla_X \xi_1 = -\varphi^2(X) - \varphi h_1 X$, for any $X \in \Gamma(TM^{2n+s})$,
4) $N(Y, \xi_i) = 2\varphi h_i(Y)$, for any $i \in \{1, \ldots, s\}$, for any $Y \in \Gamma(TM^{2n+s})$.

Proposition 10. *Let M^{2n+s}, $s > 2$, be a generalized $f.pk$-space-form with underlying almost Kenmotsu $f.pk$-structure. Then M^{2n+s} cannot be conformally flat.*

Proof: Suppose that M^{2n+s}, $s > 2$, is a conformally flat generalized $f.pk$-space-form with underlying almost Kenmotsu $f.pk$-structure and $c \neq 0$. In each W provided by Theorem 4, the leaves of $\mathcal{D}$ are totally umbilical with mean curvature vector field $H = -\xi_1$ and applying (4.6) for $\xi = \xi_i$, $i \geq 2$, we get $0 = \frac{c}{2}\xi_i$ and so $c = 0$, a contradiction. We conclude that $c = 0$ and $R = 0$.
Firstly, we prove that any integral manifold N^{2n} of $\mathcal{D}$ is Kähler. Denoting by ∇' the induced connection on the almost Kähler N^{2n} and putting $J = \varphi_{|N^{2n}}$, using the Gauss equation, we compute the scalar curvature τ' and the *-scalar curvature τ'^*. We get $\tau' = 2n(2n-1) - \sum_{j=1}^{s} tr(h_j^2)$ and $\tau'^* = 2n - \sum_{j=1}^{s} tr(h_j^2)$. Hence $\tau' - \tau'^* = 4n(n-1)$ and using the well-known formula $\tau' - \tau'^* = -\frac{1}{2}\|\nabla' J\|^2$ we obtain $\nabla' J = 0$ and $n = 1$. Thus the case $n \geq 2$ is excluded. Assuming $n = 1$, we consider a φ-basis $(X, \varphi X, \xi_1, \ldots, \xi_s)$ and, by direct calculation, using

Lemma 4, for each $i \geq 2$, we get $\rho(\xi_i, \xi_i) = \|h_i X\|^2 = 0$. Hence $h_i X = 0$ and $h_i(\varphi X) = -\varphi(h_i X) = 0$ imply $h_i = 0$ and $\nabla \xi_i = 0$, $i \geq 2$.

Now, we consider the splitting $(\mathcal{D} \oplus < \xi_1 >) \oplus < \xi_2, \ldots, \xi_s >$ and observe that the integrable distributions $\mathcal{D} \oplus < \xi_1 >$ and $< \xi_2, \ldots, \xi_s >$ are both totally geodesic. Namely, for the second fundamental form of a leaf of $\mathcal{D} \oplus < \xi_1 >$, we have

$$\alpha(X, Y) = \sum_{i=2}^{s} g(\nabla_X Y, \xi_i)\xi_i = -\sum_{i=2}^{s} g(Y, \nabla_X \xi_i)\xi_i = 0.$$

It follows that M^{2+s} is locally a Riemannian product $N^3 \times \mathbb{R}^{s-1}$ and both the factors are flat. On the other hand, N^3 inherits an almost Kenmotsu structure. As proved in [8], an almost Kenmotsu manifold of constant curvature K is a Kenmotsu manifold and $K = -1$. So, in our case we should have $0 = -1$. $\qquad \square$

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