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Some new relations between sums of squares and triangular numbers

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Abstract

Let $T(a_1, a_2, a_3, a_4; n)$ denote the number of representations of n as $a_1\frac{x_1(x_1+1)}{2} + a_2\frac{x_2(x_2+1)}{2} + a_3\frac{x_3(x_3+1)}{2} + a_4\frac{x_4(x_4+1)}{2}$, where a_1, a_2, a_3, a_4 are positive integers, n, x_1, x_2, x_3, x_4 are arbitrary nonnegative integers, and let $N(a_1, a_2, a_3, a_4; n)$ denote the number of representations of n as $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$, where this time x_1, x_2, x_3, x_4 are integers. In a recent paper, Sun not only discovered many relations between $T(a_1, a_2, a_3, a_4; n)$ and $N(a_1, a_2, a_3, a_4; n)$, but also posed a number of conjectures on the relations between $T(a_1, a_2, a_3, a_4; n)$ and $N(a_1, a_2, a_3, a_4; n)$. In this paper, we confirm some of Sun's conjectures by using Ramanujan's theta function identities and (p, k)-parametrization of theta functions.

Key Words: Ramanujan's theta function identities, sum of squares, sum of triangular numbers.
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1 Introduction

In this paper, we always let $\mathbb{N}^+$, $\mathbb{N}$ and $\mathbb{Z}$ denote the set of positive integers, the set of nonnegative integers and the set of integers, respectively. The aim of this paper is to confirm a number of conjectures on the relations between $T(a_1, a_2, a_3, a_4; n)$ and $N(a_1, a_2, a_3, a_4; n)$ by utilizing Ramanujan's theta function identities and (p, k)-parametrization of theta functions, where $a_1, a_2, a_3, a_4 \in \mathbb{N}^+$, $n \in \mathbb{N}$ and $T(a_1, a_2, a_3, a_4; n)$ and $N(a_1, a_2, a_3, a_4; n)$ are defined by

$$\begin{aligned} &T(a_1, a_2, a_3, a_4; n) \\ :=&\# \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{N}^4 | a_1\frac{x_1^2 + x_1}{2} + a_2\frac{x_2^2 + x_2}{2} + a_3\frac{x_3^2 + x_3}{2} + a_4\frac{x_4^2 + x_4}{2} = n \right\} \end{aligned} \tag{1.1}$$

and

$$N(a_1, a_2, a_3, a_4; n) := \# \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 | a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = n \right\}. \tag{1.2}$$

As usual, set $N(a_1, a_2, a_3, a_4; 0) = T(a_1, a_2, a_3, a_4; 0) = 1$.

In recent years, the relations between $N(a_1, a_2, a_3, a_4; n)$ and $T(a_1, a_2, a_3, a_4; n)$ have received a lot of attention. Recently, Wang and Sun [9] revealed new connections between $T(a_1, a_2, a_3, a_4; n)$ and $N(a_1, a_2, a_3, a_4; 8n + a_1 + a_2 + a_3 + a_4)$. Sun [5] also posed 23 conjectures on the relations between $N(a_1, a_2, a_3, a_4; n)$ and $T(a_1, a_2, a_3, a_4; n)$. Some of these conjectures were confirmed by Xia and Zhong [11]. Very recently, Sun [8] not only proved a number of formulas between $N(a_1, a_2, a_3, a_4; n)$ and $T(a_1, a_2, a_3, a_4; n)$, but also posed many conjectures. For more details, see for example [2, 6, 7, 10].

In this paper, employing Ramanujan's theta function identities and (p, k)-parametrization of theta functions, we will confirm some of the conjectures due to Sun [8]. The following formulas were conjectured by Sun [8].

Theorem 1. *For $n \in \mathbb{N}$,*

$$13T(1,3,9,27;8n+7) = N(1,3,9,27;16n+24), \tag{1.3}$$

$$12T(1,2,3,18;4n+4) = N(1,2,3,18;8n+14), \tag{1.4}$$

$$36T(1,2,3,18;4n+4) = N(1,2,3,18;32n+56), \tag{1.5}$$

$$6T(1,2,3,18;8n+3) = N(1,2,3,18;16n+12), \tag{1.6}$$

$$4T(1,2,3,18;12n+6) = N(1,2,3,18;24n+18), \tag{1.7}$$

$$10T(1,2,3,18;24n+15) = N(1,2,3,18;48n+36), \tag{1.8}$$

$$12T(1,3,18,18;4n+2) = N(1,3,18,18;8n+14), \tag{1.9}$$

$$36T(1,3,18,18;4n+2) = N(1,3,18,18;32n+56), \tag{1.10}$$

$$6T(1,3,18,18;8n+1) = N(1,3,18,18;16n+12), \tag{1.11}$$

$$30T(1,3,18,18;8n+1) = N(1,3,18,18;64n+48), \tag{1.12}$$

$$9T(1,3,18,18;16n+7) = N(1,3,18,18;32n+24), \tag{1.13}$$

$$33T(1,3,18,18;16n+7) = N(1,3,18,18;128n+96), \tag{1.14}$$

$$4T(1,3,18,18;12n+4) = N(1,3,18,18;24n+18), \tag{1.15}$$

$$28T(1,3,18,18;12n+4) = N(1,3,18,18;96n+72), \tag{1.16}$$

$$10T(1,3,18,18;24n+13) = N(1,3,18,18;48n+36), \tag{1.17}$$

$$34T(1,3,18,18;24n+13) = N(1,3,18,18;192n+144), \tag{1.18}$$

$$12T(2,3,9,18;4n+3) = N(2,3,9,18;8n+14), \tag{1.19}$$

$$36T(2,3,9,18;4n+3) = N(2,3,9,18;32n+56), \tag{1.20}$$

$$6T(2,3,9,18;8n+2) = N(2,3,9,18;16n+12), \tag{1.21}$$

$$9T(2,3,9,18;16n+8) = N(2,3,9,18;32n+24), \tag{1.22}$$

$$15T(2,3,9,18;32n+20) = 2N(2,3,9,18;64n+48), \tag{1.23}$$

$$10T(2,3,9,18;24n+14) = N(2,3,9,18;48n+36), \tag{1.24}$$

$$4T(2,3,9,18;12n+5)=N(2,3,9,18;24n+18), \tag{1.25}$$

$$8T(1,6,9,12;4n+2)=N(1,6,9,12;8n+11), \tag{1.26}$$

$$8T(1,6,9,12;4n+3)=N(1,6,9,12;8n+13), \tag{1.27}$$

$$4T(1,6,8,9;8n+3)=N(1,6,8,9;16n+12), \tag{1.28}$$

$$6T(2,2,3,9;8n+4)=N(2,2,3,9;16n+12), \tag{1.29}$$

$$12T(1,6,9,24;8n+7)=N(1,6,9,24;16n+24). \tag{1.30}$$

2 Preliminaries

Recall that the well-known Jacobi triple product identity is

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-qz;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty, \qquad z\neq 0, \tag{2.1}$$

where

$$(a;q)_\infty=\prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

By (2.1), the Ramanujan theta functions $\varphi(q)$ and $\psi(q)$ can be represented as products

$$\varphi(q):=\sum_{n=-\infty}^{\infty} q^{n^2}=\frac{f_2^5}{f_1^2f_4^2}, \tag{2.2}$$

$$\psi(q):=\sum_{n=0}^{\infty} q^{n(n+1)/2}=\frac{f_2^2}{f_1}, \tag{2.3}$$

where here and throughout this paper, for any positive integer k,

$$f_k=(q^k;q^k)_\infty.$$

The generating function for $T(a_1,a_2,a_3,a_4;n)$ and $N(a_1,a_2,a_3,a_4;n)$ are

$$\sum_{n=0}^{\infty} T(a_1,a_2,a_3,a_4;n)q^n=\psi(q^{a_1})\psi(q^{a_2})\psi(q^{a_3})\psi(q^{a_4}), \tag{2.4}$$

and

$$\sum_{n=0}^{\infty} N(a_1,a_2,a_3,a_4;n)q^n=\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}). \tag{2.5}$$

In order to prove the main results of this paper, we require some Ramanujan's theta function identities.

Lemma 1. *The following 2-dissection formulas hold:*

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{2.6}$$

$$\varphi(q)^2 = \varphi(q^2)^2 + 4q\psi(q^4)^2, \tag{2.7}$$

$$\psi(q)^2 = \psi(q^2)\varphi(q^4) + 2q\psi(q^2)\psi(q^8), \tag{2.8}$$

$$\psi(q)\psi(q^3) = \psi(q^4)\varphi(q^6) + q\varphi(q^2)\psi(q^{12}), \tag{2.9}$$

$$\varphi(q)\varphi(q^3) = \varphi(q^4)\varphi(q^{12}) + 2q\psi(q^2)\psi(q^6) + 4q^4\psi(q^8)\psi(q^{24}), \tag{2.10}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.11}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.12}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{2.13}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{2.14}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{2.15}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \tag{2.16}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{2.17}$$

Proof. Identities (2.6) and (2.7) follow from (1.9.4) and (1.10.1) on page 14 in Hirschhorn's book [4]. By (2.2) and (2.3),

$$\psi(q)^2 = \varphi(q)\psi(q^2). \tag{2.18}$$

Identity (2.8) follows from (2.6) and (2.18). Identity (2.9) follows from [4, p.219, (25.2.1)]. Identity (2.10) follows from (25.2.2) and (25.2.4) on page 219 in Hirschhorn's book [4]. Identity (2.11) is equivalent to (2.6) and identity (2.12) is equivalent to (2.7). Replacing q by $-q$ in [4, (22.7.3)] yields (2.13). Identity (2.14) is equivalent to (2.9) and identity (2.15) follows from [4, p.196, (22.7.5)]. Identities (2.16) and (2.17) follow from (30.10.3) and (30.10.4) in [4]. ∎

Lemma 2. *The following 3-dissection formulas are true:*

$$\psi(q) = \frac{f_6 f_9^2}{f_3 f_{18}} + q\psi(q^9), \tag{2.19}$$

$$\varphi(q) = \varphi(q^9) + 2q\frac{f_6^2 f_9 f_{36}}{f_3 f_{12} f_{18}}. \tag{2.20}$$

Identities (2.19) and (2.20) follow from Corollary (i) and (ii) on page 49 in Berndt's book [3].

At last, we need the following (p,k)-parametrization of theta functions.

Lemma 3. *We have*

$$q^{\frac{1}{24}} f_1 = 2^{-\frac{1}{6}} p^{\frac{1}{24}} (1-p)^{\frac{1}{2}} (1+p)^{\frac{1}{6}} (1+2p)^{\frac{1}{8}} (2+p)^{\frac{1}{8}} k^{\frac{1}{2}}, \tag{2.21}$$

$$q^{\frac{1}{12}} f_2 = 2^{-\frac{1}{3}} p^{\frac{1}{12}} (1-p)^{\frac{1}{4}} (1+p)^{\frac{1}{12}} (1+2p)^{\frac{1}{4}} (2+p)^{\frac{1}{4}} k^{\frac{1}{2}}, \tag{2.22}$$

$$q^{\frac{1}{8}} f_3 = 2^{-\frac{1}{6}} p^{\frac{1}{8}} (1-p)^{\frac{1}{6}} (1+p)^{\frac{1}{2}} (1+2p)^{\frac{1}{24}} (2+p)^{\frac{1}{24}} k^{\frac{1}{2}}, \tag{2.23}$$

$$q^{\frac{1}{6}} f_4 = 2^{-2/3} p^{\frac{1}{6}} (1-p)^{\frac{1}{8}} (1+p)^{\frac{1}{24}} (1+2p)^{\frac{1}{8}} (2+p)^{\frac{1}{2}} k^{\frac{1}{2}}, \tag{2.24}$$

$$q^{\frac{1}{4}} f_6 = 2^{-\frac{1}{3}} p^{\frac{1}{4}} (1-p)^{\frac{1}{12}} (1+p)^{\frac{1}{4}} (1+2p)^{\frac{1}{12}} (2+p)^{\frac{1}{12}} k^{\frac{1}{2}}, \tag{2.25}$$

$$q^{\frac{1}{2}} f_{12} = 2^{-2/3} p^{\frac{1}{2}} (1-p)^{\frac{1}{24}} (1+p)^{\frac{1}{8}} (1+2p)^{\frac{1}{24}} (2+p)^{\frac{1}{6}} k^{\frac{1}{2}}, \tag{2.26}$$

where

$$p := p(q) = \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \qquad k := k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}.$$

Identities (2.21)–(2.26) follow from (2.14)–(2.19) in [1].

3 Proof of Theorem 1

In this section, we are ready to prove Theorem 1. We only present a proof of (1.30). The rest can be proved similarly by using Lemmas 1–3, so we omit the details. In order to prove (1.30), it suffices to prove (1.30) holds when $n \equiv 0, 1, 2 \pmod 3$, namely,

$$12T(1,6,9,24;24n+7) = N(1,6,9,24;48n+24), \tag{3.1}$$

$$12T(1,6,9,24;24n+15) = N(1,6,9,24;48n+40), \tag{3.2}$$

$$12T(1,6,9,24;24n+23) = N(1,6,9,24;48n+56). \tag{3.3}$$

In order to prove (3.1)–(3.3), we first define an operator. For integers $0 \le b < a$ and a power series $\sum_{n=-\infty}^{\infty} s(n)q^n$, define an operator $U_{a,b}$ as

$$U_{a,b}\left(\sum_{n=-\infty}^{\infty} s(n)q^n\right) = \sum_{n=-\infty}^{\infty} s(an+b)q^n.$$

By (2.5),

$$\sum_{n=0}^{\infty} N(1,6,9,24;n)q^n = \varphi(q)\varphi(q^6)\varphi(q^9)\varphi(q^{24}). \tag{3.4}$$

It follows from (2.6) and (3.4) that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,6,9,24;2n)q^n \\
&= U_{2,0}\left(\sum_{n=0}^{\infty} N(1,6,9,24;n)q^n\right) = U_{2,0}\left(\varphi(q)\varphi(q^6)\varphi(q^9)\varphi(q^{24})\right) \\
&= \varphi(q^3)\varphi(q^{12})U_{2,0}\left(\left(\varphi(q^4)+2q\psi(q^8)\right)\left(\varphi(q^{36})+2q^9\psi(q^{72})\right)\right) \\
&= \varphi(q^3)\varphi(q^{12})U_{2,0}\left(\varphi(q^4)\varphi(q^{36})+2q\psi(q^8)\varphi(q^{36})+2q^9\varphi(q^4)\psi(q^{72})+4q^{10}\psi(q^8)\psi(q^{72})\right) \\
&= 4q^5\varphi(q^3)\psi(q^4)\varphi(q^{12})\psi(q^{36})+\varphi(q^2)\varphi(q^3)\varphi(q^{12})\varphi(q^{18}). \qquad (3.5)
\end{aligned}$$

In view of (2.6) and (3.5),

$$\begin{aligned}
\sum_{n=0}^{\infty} N(1,6,9,24;4n)q^n &= U_{2,0}\left(\sum_{n=0}^{\infty} N(1,6,9,24;2n)q^n\right) \\
&= U_{2,0}\left(4q^5\varphi(q^3)\psi(q^4)\varphi(q^{12})\psi(q^{36})+\varphi(q^2)\varphi(q^3)\varphi(q^{12})\varphi(q^{18})\right) \\
&= 4\psi(q^2)\varphi(q^6)\psi(q^{18})U_{2,0}\left(q^5\varphi(q^3)\right)+\varphi(q)\varphi(q^6)\varphi(q^9)U_{2,0}\left(\varphi(q^3)\right) \\
&= 4\psi(q^2)\varphi(q^6)\psi(q^{18})U_{2,0}\left(q^5\varphi(q^{12})+2q^8\psi(q^{24})\right) \\
&\quad +\varphi(q)\varphi(q^6)\varphi(q^9)U_{2,0}\left(\varphi(q^{12})+2q^3\psi(q^{24})\right) \\
&= 8q^4\psi(q^2)\varphi(q^6)\psi(q^{12})\psi(q^{18})+\varphi(q)\varphi(q^6)^2\varphi(q^9). \qquad (3.6)
\end{aligned}$$

Thanks to (2.6) and (3.6),

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,6,9,24;8n)q^n = U_{2,0}\left(\sum_{n=0}^{\infty} N(1,6,9,24;4n)q^n\right) \\
&= U_{2,0}\left(8q^4\psi(q^2)\varphi(q^6)\psi(q^{12})\psi(q^{18})+\varphi(q)\varphi(q^6)^2\varphi(q^9)\right) \\
&= 8q^2\psi(q)\varphi(q^3)\psi(q^6)\psi(q^9)+\varphi(q^3)^2U_{2,0}\left(\varphi(q)\varphi(q^9)\right) \\
&= 8q^2\psi(q)\varphi(q^3)\psi(q^6)\psi(q^9) \\
&\quad +\varphi(q^3)^2U_{2,0}\left(\varphi(q^4)\varphi(q^{36})+2q\psi(q^8)\varphi(q^{36})+2q^9\varphi(q^4)\psi(q^{72})+4q^{10}\psi(q^8)\psi(q^{72})\right) \\
&= 4q^5\varphi(q^3)^2\psi(q^4)\psi(q^{36})+8q^2\psi(q)\varphi(q^3)\psi(q^6)\psi(q^9)+\varphi(q^2)\varphi(q^3)^2\varphi(q^{18}). \qquad (3.7)
\end{aligned}$$

Based on (2.18),

$$\psi(q)\varphi(q^3)\psi(q^6)\psi(q^9)=\psi(q)\psi(q^3)^2\psi(q^9). \qquad (3.8)$$

Combining (3.7) and (3.8) yields

$$\sum_{n=0}^{\infty} N(1,6,9,24;8n)q^n = 4q^5\varphi(q^3)^2\psi(q^4)\psi(q^{36}) + 8q^2\psi(q)\psi(q^3)^2\psi(q^9) + \varphi(q^2)\varphi(q^3)^2\varphi(q^{18}). \tag{3.9}$$

In light of (2.7), (2.9) and (3.9),

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,6,9,24;16n+8)q^n = U_{2,1}\left(\sum_{n=0}^{\infty} N(1,6,9,24;8n)q^n\right) \\
&= 4q^2\psi(q^2)\psi(q^{18})U_{2,0}\left(\varphi(q^3)^2\right) + 8qU_{2,1}\left(\psi(q)\psi(q^3)^2\psi(q^9)\right) + \varphi(q)\varphi(q^9)U_{2,1}\left(\varphi(q^3)^2\right) \\
&= 4q^2\psi(q^2)\psi(q^{18})U_{2,0}\left(\varphi(q^6)^2 + 4q^3\psi(q^{12})^2\right) \\
&\quad + 8qU_{2,1}\left(\left(\psi(q^4)\varphi(q^6) + q\varphi(q^2)\psi(q^{12})\right)\left(\psi(q^{12})\varphi(q^{18}) + q^3\varphi(q^6)\psi(q^{36})\right)\right) \\
&\quad + \varphi(q)\varphi(q^9)U_{2,1}\left(\varphi(q^6)^2 + 4q^3\psi(q^{12})^2\right) \\
&= 12q^2\psi(q^2)\varphi(q^3)^2\psi(q^{18}) + 12q\varphi(q)\psi(q^6)^2\varphi(q^9).
\end{aligned} \tag{3.10}$$

Substituting (2.19) and (2.20) in (3.10) yields

$$\begin{aligned}
\sum_{n=0}^{\infty} N(1,6,9,24;16n+8)q^n =& 12q^2\varphi(q^3)^2\psi(q^{18})\left(\frac{f_{12}f_{18}^2}{f_6f_{36}} + q^2\psi(q^{18})\right) \\
&+ 12q\psi(q^6)^2\varphi(q^9)\left(\varphi(q^9) + 2q\frac{f_6^2f_9f_{36}}{f_3f_{12}f_{18}}\right) \\
=& 12q^4\varphi(q^3)^2\psi(q^{18})^2 + 24q^2\psi(q^6)^2\varphi(q^9)\frac{f_6^2f_9f_{36}}{f_3f_{12}f_{18}} \\
&+ 12q^2\varphi(q^3)^2\psi(q^{18})\frac{f_{12}f_{18}^2}{f_6f_{36}} + 12q\psi(q^6)^2\varphi(q^9)^2,
\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} N(1,6,9,24;48n+8)q^n = 0, \tag{3.11}$$

$$\sum_{n=0}^{\infty} N(1,6,9,24;48n+24)q^n = 12q\varphi(q)^2\psi(q^6)^2 + 12\psi(q^2)^2\varphi(q^3)^2, \tag{3.12}$$

$$\sum_{n=0}^{\infty} N(1,6,9,24;48n+40)q^n = 24\psi(q^2)^2\varphi(q^3)\frac{f_2^2f_3f_{12}}{f_1f_4f_6} + 12\varphi(q)^2\psi(q^6)\frac{f_4f_6^2}{f_2f_{12}}. \tag{3.13}$$

In view of (2.4),

$$\sum_{n=0}^{\infty} T(1,6,9,24;n)q^n = \psi(q)\psi(q^6)\psi(q^9)\psi(q^{24}). \tag{3.14}$$

Substituting (2.19) into (3.14), we arrive at

$$\sum_{n=0}^{\infty} T(1,6,9,24;n)q^n = \psi(q^6)\psi(q^9)\psi(q^{24})\left(\frac{f_6 f_9^2}{f_3 f_{18}} + q\psi(q^9)\right)$$
$$= \psi(q^6)\psi(q^9)\psi(q^{24})\frac{f_6 f_9^2}{f_3 f_{18}} + q\psi(q^6)\psi(q^9)^2\psi(q^{24}),$$

which implies

$$\sum_{n=0}^{\infty} T(1,6,9,24;3n)q^n = \frac{f_2 f_3^2}{f_1 f_6}\psi(q^2)\psi(q^3)\psi(q^8), \tag{3.15}$$

$$\sum_{n=0}^{\infty} T(1,6,9,24;3n+1)q^n = \psi(q^2)\psi(q^3)^2\psi(q^8), \tag{3.16}$$

$$\sum_{n=0}^{\infty} T(1,6,9,24;3n+2)q^n = 0. \tag{3.17}$$

Identity (3.3) follows from (3.11) and (3.17).

By (2.8) and (3.16),

$$\sum_{n=0}^{\infty} T(1,6,9,24;6n+1)q^n = U_{2,0}\left(T(1,6,9,24;3n+1)q^n\right) = U_{2,0}\left(\psi(q^2)\psi(q^3)^2\psi(q^8)\right)$$
$$= \psi(q)\psi(q^4)U_{2,0}\left(\psi(q^3)^2\right) = \psi(q)\psi(q^3)\psi(q^4)\varphi(q^6). \tag{3.18}$$

Thanks to (2.9) and (3.18),

$$\sum_{n=0}^{\infty} T(1,6,9,24;12n+7)q^n = U_{2,1}\left(T(1,6,9,24;6n+1)q^n\right) = U_{2,1}\left(\psi(q)\psi(q^3)\psi(q^4)\varphi(q^6)\right)$$
$$= \psi(q^2)\varphi(q^3)U_{2,1}\left(\psi(q)\psi(q^3)\right) = \varphi(q)\psi(q^2)\varphi(q^3)\psi(q^6). \tag{3.19}$$

In view of (2.10) and (3.19),

$$\sum_{n=0}^{\infty} T(1,6,9,24;24n+7)q^n = U_{2,0}\left(T(1,6,9,24;12n+7)q^n\right) = U_{2,0}\left(\varphi(q)\psi(q^2)\varphi(q^3)\psi(q^6)\right)$$
$$= \psi(q)\psi(q^3)U_{2,0}\left(\varphi(q)\varphi(q^3)\right)$$
$$= \psi(q)\varphi(q^2)\psi(q^3)\varphi(q^6) + 4q^2\psi(q)\psi(q^3)\psi(q^4)\psi(q^{12}). \tag{3.20}$$

In light of (2.2), (2.3), (3.12) and (3.20),

$$12\sum_{n=0}^{\infty} T(1,6,9,24;24n+7)q^n - \sum_{n=0}^{\infty} N(1,6,9,24;48n+24)q^n$$

$$=48q^2\psi(q)\psi(q^3)\psi(q^4)\psi(q^{12})-12q\varphi(q)^2\psi(q^6)^2+12\psi(q)\varphi(q^2)\psi(q^3)\varphi(q^6)-12\psi(q^2)^2\varphi(q^3)^2$$

$$=12\frac{F_1(q)}{f_1f_3}, \tag{3.21}$$

where

$$F_1(q)=4q^2\frac{f_2^2f_6^2f_8^2f_{24}^2}{f_4f_{12}}-q\frac{f_3}{f_1^3}\cdot\frac{f_2^{10}f_{12}^4}{f_4^4f_6^2}+\frac{f_4^5f_{12}^5}{f_8^2f_{24}^2}-\frac{f_1}{f_3^3}\cdot\frac{f_4^4f_6^{10}}{f_2^2f_{12}^4}. \tag{3.22}$$

Replacing q by $-q$ in (2.15) yields

$$\frac{f_1}{f_3^3}=\frac{f_2f_4^2f_{12}^2}{f_6^7}-q\frac{f_2^3f_{12}^6}{f_4^2f_6^9}. \tag{3.23}$$

Substituting (2.13) and (3.23) into (3.22), we deduce that

$$F_1(q)=F_2(q^2), \tag{3.24}$$

where

$$F_2(q)=4q\frac{f_1^2f_3^2f_4^2f_{12}^2}{f_2f_6}-3q\frac{f_1^3f_6^6}{f_2^2f_3}+\frac{f_2^5f_6^5}{f_4^2f_{12}^2}-\frac{f_2^6f_3^3}{f_1f_6^2}. \tag{3.25}$$

Substituting (2.21)–(2.26) into (3.25), we arrive at

$$F_2(q)=0$$

and therefore,

$$F_1(q)=0. \tag{3.26}$$

By (3.21) and (3.26),

$$12\sum_{n=0}^{\infty}T(1,6,9,24;24n+7)q^n=\sum_{n=0}^{\infty}N(1,6,9,24;48n+24)q^n, \tag{3.27}$$

which yields (3.1).

At last, we turn to prove (3.2).

By (2.3), we rewrite (3.15) as

$$\sum_{n=0}^{\infty}T(1,6,9,24;3n)q^n=\frac{f_3f_4^2f_6f_{16}^2}{f_1f_8}. \tag{3.28}$$

In view of (2.16) and (3.28),

$$\sum_{n=0}^{\infty}T(1,6,9,24;6n+3)q^n=U_{2,1}\left(\sum_{n=0}^{\infty}T(1,6,9,24;3n)q^n\right)=U_{2,1}\left(\frac{f_3f_4^2f_6f_{16}^2}{f_1f_8}\right)$$

$$=\frac{f_2^2f_3f_8^2}{f_4}U_{2,1}\left(\frac{f_3}{f_1}\right)=\frac{f_2^2f_3^2f_4f_8f_{24}}{f_1^2f_{12}}. \tag{3.29}$$

It follows from (2.17) and (3.29) that

$$\begin{aligned}\sum_{n=0}^{\infty} T(1,6,9,24;12n+3)q^n &= U_{2,0}\left(\sum_{n=0}^{\infty} T(1,6,9,24;6n+3)q^n\right) = U_{2,0}\left(\frac{f_2^2 f_3^2 f_4 f_8 f_{24}}{f_1^2 f_{12}}\right)\\ &= \frac{f_1^2 f_2 f_4 f_{12}}{f_6} U_{2,0}\left(\frac{f_3^2}{f_1^2}\right) = \frac{f_2^5 f_3 f_6}{f_1^3}. \qquad (3.30)\end{aligned}$$

Based on (2.13) and (3.30),

$$\begin{aligned}\sum_{n=0}^{\infty} T(1,6,9,24;24n+15)q^n &= U_{2,1}\left(\sum_{n=0}^{\infty} T(1,6,9,24;12n+3)q^n\right) = U_{2,1}\left(\frac{f_2^5 f_3 f_6}{f_1^3}\right)\\ &= f_1^5 f_3 U_{2,1}\left(\frac{f_3}{f_1^3}\right) = 3\frac{f_2^2 f_3^2 f_6^2}{f_1^2}. \qquad (3.31)\end{aligned}$$

Combining (2.2), (2.3), (3.13) and (3.31) yields

$$\begin{aligned}&12\sum_{n=0}^{\infty} T(1,6,9,24;24n+15)q^n - \sum_{n=0}^{\infty} N(1,6,9,24;48n+40)q^n\\ =&36\frac{f_2^2 f_3^2 f_6^2}{f_1^2} - 24\psi(q^2)^2\varphi(q^3)\frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6} - 12\varphi(q)^2\psi(q^6)\frac{f_4 f_6^2}{f_2 f_{12}} = 12F_3(q), \qquad (3.32)\end{aligned}$$

where

$$F_3(q) = 3\frac{f_2^2 f_3^2 f_6^2}{f_1^2} - 2\frac{f_4^3 f_6^4}{f_1 f_3 f_{12}} - \frac{f_2^9 f_6 f_{12}}{f_1^4 f_4^3}. \qquad (3.33)$$

Substituting (2.21)–(2.26) into (3.33) yields

$$F_3(q) = 0. \qquad (3.34)$$

Therefore,

$$12\sum_{n=0}^{\infty} T(1,6,9,24;24n+15)q^n = \sum_{n=0}^{\infty} N(1,6,9,24;48n+40)q^n. \qquad (3.35)$$

Identity (3.2) follows from (3.35). This completes the proof. □

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