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Heron means and Pólya inequality for sector matrices

by
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Abstract

We introduce the Heron means and Pólya inequality for sector matrices and give some inequalities involving them. For instance, we show that if $A, B \in S_\alpha$ are two sector matrices and $\nu \in [0,1]$, then

$$0 \le F_\nu(\mathcal{R}A, \mathcal{R}B) \le \mathcal{R}F_\nu(A,B) \le \sec^2 \alpha F_\nu(\mathcal{R}A, \mathcal{R}B)$$

and

$$\cos^3 \alpha \|H_\nu(A,B)\| \le \|F_{\alpha(\nu)}(A,B)\|,$$

where $\alpha(\nu) = 1 - 4(\nu - \nu^2)$. We also present the following inequality for the Pólya inequality

$$\left\| \int_0^1 (A\sharp_\nu B) d\nu \right\| \le \sec^3 \alpha \left\| \frac{2}{3}(A\sharp_\nu B) + \frac{1}{3} A\nabla_\nu B \right\|.$$

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1 Introduction

There are various combinations of means that interpolate between the geometric and the arithmetic mean. The Heinz, Heron and logarithmic means are samples of such means which are defined respectively as follows:

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2},$$
$$F_\nu(a,b) = (1-\nu)\sqrt{ab} + \nu \frac{a+b}{2},$$
$$L(a,b) = \int_0^1 a^\nu b^{1-\nu} d\nu,$$

$0 \le \nu \le 1$. It is obvious that

$$\sqrt{ab} \le H_\nu(a,b) \le \frac{a+b}{2}. \tag{1.1}$$

The second inequality of (1.1) is known as the Heinz inequality for nonnegative real numbers.

Bhatia [3], proved that the Heinz and the Heron means satisfy the following inequality

$$H_\nu(a,b) \leq F_{\alpha(\nu)}(a,b), \tag{1.2}$$

where $\alpha(\nu) = 1 - 4\left(\nu - \nu^2\right)$.

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices. For Hermitian matrices $A, B \in \mathbb{M}_n$, we write that $A \geqslant 0$ if A is positive semidefinite, i.e. if $\langle Ax, x\rangle \geqslant 0$ for all vectors $x \in \mathbb{C}^n$. We also write $A > 0$ if A is positive definite, i.e. if $\langle Ax, x\rangle > 0$ for all vectors $x \in \mathbb{C}^n$, and $A \geqslant B$ if $A - B \geqslant 0$.

A matrix $A \in \mathbb{M}_n$ is called accretive if in its Cartesian (or Toeplitz) decomposition, $A = \mathcal{R}z + i\mathcal{I}z$, $\mathcal{R}z$ is positive definite, where

$$\mathcal{R}z = \frac{A + A^*}{2},\ \mathcal{I}z = \frac{A - A^*}{2}.$$

The numerical range of a matrix $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

A matrix $A \in \mathbb{M}_n$ is said to be sectorial if $W(A) \subseteq S_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$, where S_α denote the sector regions in the complex plane as follows:

$$S_\alpha = \{z \in \mathbb{C} : \mathcal{R}z \geq 0, |\mathcal{I}z| \leq (\mathcal{R}z)\tan\alpha\}.$$

Clearly, A is positive semidefinite if and only if $W(A) \subseteq S_0$, and if $W(A)$, $W(B) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$, then $W(A+B) \subseteq S_\alpha$. Moreover, $W(A) \subseteq S_\alpha$ implies $W(X^*AX) \subseteq S_\alpha$ for any nonzero $n \times m$ matrix X; thus $W(A^{-1}) \subseteq S_\alpha$. The smallest such α is called the sectorial index of A. When $W(A) \subseteq S_\alpha$, we will write $A \in S_\alpha$. The operator mean of two accretive matrices $A, B \in \mathbb{M}_n$ have been defined by Bedrani et al., in [1] as follows

$$A\sigma_f B = \int_0^1 (A!B)d\nu_f(s),$$

where $A!B$ is the harmonic mean of A, B, the function $f : (0,\infty) \longrightarrow (0,\infty)$ is an operator monotone function with $f(1) = 1$ and ν_f is the probability measure characterizing σ_f. Moreover, they also characterize the operator monotone function for an accretive matrix: let $A \in S_\alpha$ and $f : (0,\infty) \longrightarrow (0,\infty)$ be an operator monotone function with $f(1) = 1$. Then

$$f(A) = \int_0^1 ((1-s)I + sA^{-1})^{-1} d\nu_f(s),$$

where ν_f is probability measure satisfying

$$f(x) = \int_0^1 ((1-s)I + sx^{-1})^{-1} d\nu_f(s).$$

Later, Raissouli et. al. [11] defined the following weighted geometric mean of two accretive matrices $A, B \in \mathbb{M}_n$,

$$A\sharp_\nu B = \frac{\sin\nu\pi}{\pi}\int_0^1 t^{\nu-1}(A^{-1} + tB^{-1})^{-1}\frac{dt}{t}$$

Recently, Mao et al [10] defined the Heinz mean for two sector matrices $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subseteq S_\alpha$ as follows

$$H_\nu(A,B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu} B}{2}$$

where $\nu \in [0,1]$.
They derived the following inequalities regarding Heinz mean for sector matrices:

$$H_\nu(\mathcal{R}A, \mathcal{R}B) \le \mathcal{R}H_\nu(A,B) \le \sec^2\alpha H_\nu(\mathcal{R}A, \mathcal{R}B) \tag{1.3}$$

$$\mathcal{R}H_\nu^{-1}(A,B) \le \sec^2\alpha \mathcal{R}H_\nu(A^{-1}, B^{-1}). \tag{1.4}$$

Yang and Lu [13] generalized the results in [10] and proved the following inequalities hold for any unital positive linear map Φ.

$$\mathcal{R}H_\nu^{-1}(\Phi(A), \Phi(B)) \le \sec^2\alpha \mathcal{R}H_\nu(\Phi(A^{-1}), \Phi(B^{-1})).$$

This paper is devoted to the study of the inequalities for the Heron means and Pólya inequality for sector matrices. We show that if $A, B \in S_\alpha$ are two sector matrices and $\nu \in [0,1]$, then

$$\|F_{\alpha(\nu)}(A,B)\| \ge \cos^3\alpha \|H_\nu(A,B)\|,$$

where $\alpha(\nu) = 1 - 4(\nu - \nu^2)$. We also show that the following inequality holds for the Pólya inequality

$$\left\| \int_0^1 (A\sharp_\nu B) d\nu \right\| \le \sec^3\alpha \left\| \frac{2}{3}(A\sharp_\nu B) + \frac{1}{3} A\nabla_\nu B \right\|.$$

2 Heron inequalities for sector matrices

Raissouli et al. in [11] showed that if $A, B \in B(H)$ are accretive and $\nu \in [0,1]$. Then

$$\mathcal{R}A\sharp_\nu \mathcal{R}B \le \mathcal{R}(A\sharp_\nu B) \le \sec^2\alpha((\mathcal{R}A)\sharp_\nu(\mathcal{R}B)). \tag{2.1}$$

Lin [8] proved that if $A \in \mathbb{M}_n$ has a positive definite real part, then

$$\mathcal{R}(A^{-1}) \le \mathcal{R}(A)^{-1} \le \sec^2\alpha \mathcal{R}(A^{-1}), \tag{2.2}$$

and

$$det(\mathcal{R}A) \le |detA| \le \sec^n\alpha det(\mathcal{R}A). \tag{2.3}$$

The operator norm $\|A\|$ of $A \in \mathbb{M}_n$ is defined by

$$\|A\| = \sup\{\langle Ax, y\rangle : x.y \in \mathbb{C}^n, \|x\| = \|y\| = 1\}.$$

Recall that a norm $|||\cdot|||$ on $\mathbb{M}_n$ is unitarily invariant if $|||UAV||| = |||A|||$ for any $A \in \mathbb{M}_n$ and for all unitary matrices $U, V \in \mathbb{M}_n$.

Let $A \in \mathbb{M}_n$. Then

$$\lambda_j(\mathcal{R}A) \le \sigma_j(A) \le \sec^2\alpha \lambda_j(\mathcal{R}A), \quad j = 1, ..., n. \tag{2.4}$$

Consequently,

$$|||\mathcal{R}A||| \leq |||A||| \leq \sec\ \alpha|||\mathcal{R}A||| \tag{2.5}$$

for any unitarily invariant norm $|||\cdot|||$ on $B(H)$, see [5].

Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subseteq S_\alpha$ and let $\nu \in [0,1]$. Liu and Wang [9] showed that the following inequalities hold.

$$\cos^2\alpha\mathcal{R}(A!B) \leq \mathcal{R}(A\sharp B) \leq \sec^2\alpha\mathcal{R}(A\nabla B) \tag{2.6}$$

Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subseteq S_\alpha$. We define the Heron mean of sector matrices (in particular, positive definite matrices) to be as follows:

$$F_\nu(A,B) = \nu(A\nabla B) + (1-\nu)A\sharp B$$

where $\nu \in [0,1]$.

Zhao et al. in [14] gave an inequality for the Heinz-Heron means as follows:

Let A and B be two positive definite operators, then

$$H_\nu(A,B) \leq F_{\alpha(\nu)}(A,B) \tag{2.7}$$

for $\nu \in [0,1]$, where $\alpha(\nu) = 1 - 4(\nu - \nu^2)$.

Theorem 1. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subseteq S_\alpha$ and let $\nu \in [0,1]$. Then*

$$\begin{aligned}
&(a)\ 0 \leq F_\nu(\mathcal{R}A, \mathcal{R}B) \leq \mathcal{R}F_\nu(A,B) \leq \sec^2\alpha F_\nu(\mathcal{R}A,\mathcal{R}B),\\
&(b)\ 0 \leq \cos^{2\nu}\alpha\mathcal{R}A\sharp\mathcal{R}B \leq \cos^{2\nu}\alpha\mathcal{R}(A\sharp B) \leq \mathcal{R}F_\nu(A,B)\\
&\quad \leq \sec^2\alpha(1-\nu\sin^2\alpha)\mathcal{R}(A\nabla B)
\end{aligned}$$

Proof. (a) By (2.1) we have

$$\begin{aligned}
\mathcal{R}F_\nu(A,B) &= (1-\nu)\mathcal{R}(A\sharp B) + \nu\mathcal{R}(A\nabla B)\\
&\geq (1-\nu)(\mathcal{R}(A)\sharp\mathcal{R}(B)) + \nu(\mathcal{R}(A)\nabla\mathcal{R}(B))\\
&= F_\nu(\mathcal{R}A,\mathcal{R}B),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}(F_\nu(A,B)) &= (1-\nu)\mathcal{R}(A\sharp B) + \nu\mathcal{R}(A\nabla B)\\
&\leq (1-\nu)\sec^2\alpha(\mathcal{R}A\sharp\mathcal{R}B) + \nu(\mathcal{R}A\nabla\mathcal{R}B)\\
&\leq \sec^2\alpha F_\nu(\mathcal{R}A,\mathcal{R}B).
\end{aligned}$$

(b) By (2.6), we have

$$\begin{aligned}
\mathcal{R}F_\nu(A,B) &= (1-\nu)\mathcal{R}(A\sharp B) + \nu\mathcal{R}(A\nabla B)\\
&= \mathcal{R}(A\sharp B)\nabla_\nu\mathcal{R}(A\nabla B)\\
&\leq \sec^2\alpha\mathcal{R}(A\nabla B)\nabla_\nu\mathcal{R}(A\nabla B)\\
&= [(1-\nu)\sec^2\alpha + \nu]\mathcal{R}(A\nabla B),
\end{aligned}$$

and by (2.6)

$$\begin{aligned}\mathcal{R}F_\nu(A,B) &= (1-\nu)\mathcal{R}(A\sharp B) + \nu\mathcal{R}(A\nabla B)\\ &= \mathcal{R}(A\sharp B)\nabla_\nu \mathcal{R}(A\nabla B)\\ &\geq \mathcal{R}(A\sharp B)\sharp_\nu \mathcal{R}(A\nabla B)\\ &\geq \mathcal{R}(A\sharp B)\sharp_\nu \cos^2\alpha \mathcal{R}(A\sharp B)\\ &= \cos^{2\nu}\alpha\mathcal{R}(A\sharp B).\end{aligned}$$

□

Lemma 1. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subseteq S_\alpha$ and let $\nu \in (0,1)$ and $\alpha(\nu) = 1 - 4(\nu - \nu^2)$. Then*

$$(a)\ \mathcal{R}F_{\alpha(\nu)}(A,B) \geq \cos^2\alpha\mathcal{R}H_\nu(A,B) \geq 0$$
$$(b)\ \|F_{\alpha(\nu)}(A,B)\| \geq \cos^3\alpha\|H_\nu(A,B)\|.$$

Proof. By Theorem 1, (2.7) and (1.3), we get

$$\begin{aligned}\mathcal{R}F_{\alpha(\nu)}(A,B) &\geq F_{\alpha(\nu)}(\mathcal{R}A, \mathcal{R}B) \geq H_\nu(\mathcal{R}A,\mathcal{R}B)\\ &\geq \cos^2\alpha\mathcal{R}H_\nu(A,B) \geq \cos^2\alpha H_\nu(\mathcal{R}A,\mathcal{R}B) \geq 0.\end{aligned} \tag{2.8}$$

Using (2.5) and (2.8), we obtain

$$\begin{aligned}\|F_{\alpha(\nu)}(A,B)\| &\geq \|\mathcal{R}F_{\alpha(\nu)}(A,B)\| \geq \cos^2\alpha\|\mathcal{R}H_\nu(A,B)\|\\ &\geq \cos^3\alpha\|H_\nu(A,B)\|.\end{aligned}$$

□

Theorem 2. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subseteq S_\alpha$. Then we have*

$$(a)\ |det(F_\nu(A,B))| \leq \sec^{3n}\alpha(1-\nu\sin^2\alpha)^n|detA\nabla B|,$$
$$(b)\ \ |\ det(A\sharp B)\ | \leq \sec^{(2\nu+1)n}\alpha\ |\ detF_\nu(A,B)\ |$$
$$(c)\ |detH_\nu(A,B)| \leq \sec^{3n}\alpha|detF_{\alpha(\nu)}(A,B)|.$$

Proof. (a) By (2.3) and Theorem 1, we obtain

$$\begin{aligned}|det(F_\nu(A,B))| &\leq \sec^n\alpha det\mathcal{R}F_\nu(A,B)\\ &\leq \sec^n\alpha\sec^{2n}\alpha(1-\nu\sin^2\alpha)^n det\mathcal{R}(A\nabla B)\\ &\leq \sec^{3n}\alpha(1-\nu\sin^2\alpha)^n|detA\nabla B|.\end{aligned}$$

(b) By (2.3) and Theorem 1, we have

$$\begin{aligned}|det(A\sharp B)| &\leq \sec^n\alpha\det\mathcal{R}(A\sharp B) && \text{(by (2.3))}\\ &\leq \sec^n\alpha\sec^{2\nu n}\alpha\det\mathcal{R}F_\nu(A,B) && \text{(by Theorem 1)}\\ &\leq \sec^{(2\nu+1)n}\alpha|\det F_\nu(A,B)|. && \text{(by (2.3))}\end{aligned}$$

(c) By (2.3), Lemma 1 and again (2.3), we get

$$\begin{aligned}|detH_\nu(A,B)| &\leq \sec^n\alpha\ det(\mathcal{R}H_\nu(A,B)) \leq \sec^n\alpha\ \sec^{2n}\alpha\ det(\mathcal{R}F_{\alpha(\nu)}(A,B))\\ &\leq \sec^{3n}\alpha\ |det(F_{\alpha(\nu)}(A,B))|.\end{aligned}$$

□

Theorem 3. *Let $A,B\in\mathbb{M}_n$ be such that $W(A),W(B)\subseteq S_\alpha$ and let $\nu\in(0,1)$. Then*

$$\cos^2\alpha\mathcal{R}^{-1}(F_\nu(A,B))\leq\mathcal{R}(F_\nu^{-1}(A,B))\leq\sec^2\alpha\mathcal{R}F_\nu(A^{-1},B^{-1}).$$

Proof. By (2.2), Theorem 1 and operator convexity of the inverse function $f(t)=t^{-1}$ on positive real numbers, we deduce

$$\begin{aligned}
&\cos^2\alpha\mathcal{R}^{-1}(F_\nu(A,B))\\
&\leq\mathcal{R}(F_\nu^{-1}(A,B)) && \text{(by (2.2))}\\
&\leq(\mathcal{R}F_\nu(A,B))^{-1} && \text{(by (2.2))}\\
&\leq(F_\nu(\mathcal{R}A,\mathcal{R}B))^{-1} && \text{(by Theorem 1)}\\
&=(\nu(\mathcal{R}A\nabla\mathcal{R}B)+(1-\nu)(\mathcal{R}A\sharp\mathcal{R}B))^{-1}\\
&\leq\nu(\mathcal{R}A\nabla\mathcal{R}B)^{-1}+(1-\nu)(\mathcal{R}A\sharp\mathcal{R}B)^{-1} && \text{(by operator convexity)}\\
&\leq\nu(\mathcal{R}^{-1}A\nabla\mathcal{R}^{-1}B)+(1-\nu)(\mathcal{R}^{-1}A\sharp\mathcal{R}^{-1}B) && \text{(by operator convexity)}\\
&\leq\sec^2\alpha[\nu(\mathcal{R}A^{-1}\nabla\mathcal{R}B^{-1})+(1-\nu)(\mathcal{R}A^{-1}\sharp\mathcal{R}B^{-1})] && \text{(by (2.2))}\\
&=\sec^2\alpha F_\nu(\mathcal{R}A^{-1},\mathcal{R}B^{-1})\\
&\leq\sec^2\alpha\mathcal{R}F_\nu(A^{-1},B^{-1}). && \text{(by Theorem 1)}
\end{aligned}$$

□

3 Numerical range of sector matrices

The numerical radius $\omega(A)$ of $A\in\mathbb{M}_n$ is defined by

$$\omega(A)=\sup\{\langle Ax,x\rangle : x\in\mathbb{C}^n,\|x\|=1\}.$$

Kittaneh et al. [7] proved that

$$\omega(\mathcal{R}A)\leq\omega(A)\leq\sec^2\alpha\omega(\mathcal{R}A). \tag{3.1}$$

Bedrani et al. [2] showed that if $A,B\in S_\alpha$ and $\nu\in[0,1]$, then

$$\cos^3\alpha\omega^{-1}(A)\leq\omega(A^{-1}), \tag{3.2}$$

$$\cos\ \alpha\parallel A\parallel\leq\omega(A)\leq\parallel A\parallel, \tag{3.3}$$

and

$$\omega(A\sharp_\nu B) \leq \sec^3\alpha\omega^{1-\nu}(A)\omega^\nu(B). \tag{3.4}$$

Weyl's monotonicity theorem [4, p. 63] implies that if $0 \leq A \leq B$, then

$$\lambda_j(A) \leq \lambda_j(B) \text{ for all } 1 \leq j \leq n. \tag{3.5}$$

Theorem 4. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subseteq S_\alpha$ and $\nu \in (0,1)$. Then*

$$a)\ \sigma_j(F_\nu(A,B)) \leq \sec^4\alpha(1-\nu\sin^2\alpha)\sigma_j(A\nabla B),$$
$$b)\ \sigma_j(A\sharp B) \leq \sec^{2\nu+2}\alpha\sigma_j(F_\nu(A,B)).$$

Proof. a) By (2.4), Theorem 1 and (3.5), we obtain

$$\begin{aligned}
&\sigma_j(F_\nu(A,B)) \leq \sec^2\alpha\lambda_j(\mathcal{R}F_\nu(A,B)) && \text{(by (2.4))}\\
&\leq \sec^2\alpha\sec^2\alpha(1-\nu\sin^2\alpha)\lambda_j(\mathcal{R}(A\nabla B)) && \text{(by Theorem 1 and (3.5))}\\
&= \sec^4\alpha(1-\nu\sin^2\alpha)\lambda_j(\mathcal{R}(A\nabla B))\\
&\leq \sec^4\alpha(1-\nu\sin^2\alpha)\sigma_j(A\nabla B). && \text{(by (2.4))}
\end{aligned}$$

b) By (2.4) and Theorem 1 and (3.5), we have

$$\begin{aligned}
&\sigma_j(A\sharp B) \leq \sec^2\alpha\lambda_j(\mathcal{R}(A\sharp B)) && \text{(by (2.4))}\\
&\leq \sec^2\alpha\sec^{2\nu}\alpha\lambda_j(\mathcal{R}F_\nu(A,B)) && \text{(by Theorem 1 and (3.5))}\\
&\leq \sec^{2\nu+2}\alpha\sigma_j(F_\nu(A,B)). && \text{(by (2.4))}
\end{aligned}$$

□

Theorem 5. *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subseteq S_\alpha$ and let $\nu \in (0,1)$. Then for any unitarily invariant norm $\|\cdot\|$, the following inequalities hold*

$$\cos^{2\nu+1}\alpha \parallel A\sharp B \parallel \leq \parallel F_\nu(A,B) \parallel \leq \sec^3\alpha(1-\nu\sin^2\alpha) \parallel A\nabla B \parallel .$$

Proof. By (2.5) and Theorem 1, we get

$$\begin{aligned}
\parallel F_\nu(A,B) \parallel &\leq \sec\ \alpha \parallel \mathcal{R}F_\nu(A,B) \parallel && \text{(by (2.5))}\\
&\leq \sec\alpha\sec^2\alpha(1-\nu\sin^2\alpha) \parallel \mathcal{R}(A\nabla B) \parallel && \text{(by Theorem 1)}\\
&\leq \sec^3\alpha(1-\nu\sin^2\alpha) \parallel A\nabla B \parallel, && \text{(by (2.5))}
\end{aligned}$$

and

$$\begin{aligned}
\parallel A\sharp B \parallel &\leq \sec\ \alpha \parallel \mathcal{R}(A\sharp B) \parallel && \text{(by (2.5))}\\
&\leq \sec^{2\nu+1}\alpha \parallel \mathcal{R}(F_\nu(A,B)) \parallel && \text{(by Theorem 1)}\\
&\leq \sec^{2\nu+1}\alpha \parallel F_\nu(A,B) \parallel . && \text{(by (2.5))}
\end{aligned}$$

□

Theorem 6. *Let $A, B \in S_\alpha$. Then, for $\nu \in (0,1)$,*

$$\cos^{2\nu+2} \alpha\omega(A\sharp B) \leq \omega(F_\nu(A,B)) \leq \sec^4 \alpha(1-\nu\sin^2\alpha)\omega(A\nabla B).$$

Proof. By Theorem 5 and (3.3) we have

$$\begin{aligned}\omega(A\sharp B) &\leq \| A\sharp B \| \\ &\leq \sec^{2\nu+1} \alpha \parallel F_\nu(A,B) \parallel \\ &\leq \sec^{2\nu+2} \alpha\omega(F_\nu(A,B)),\end{aligned}$$

and

$$\begin{aligned}\omega(F_\nu(A,B)) &\leq \| F_\nu(A,B) \| \\ &\leq \sec^3 \alpha(1-\nu\sin^2\alpha) \parallel \frac{A+B}{2} \parallel \\ &= \sec^4 \alpha(1-\nu\sin^2\alpha)\omega(A\nabla B).\end{aligned}$$

∎

Remark 1. *Let $A, B \in S_\alpha$ and $\nu \in (0,1)$, then by (1.4) and (3.1) we have*

$$\begin{aligned}(a)\ \omega(H_\nu^{-1}(A,B)) &\leq \sec^2\alpha\ \omega(\mathcal{R}H_\nu^{-1}(A,B)) \\ &\leq \sec^4\alpha\ \omega(\mathcal{R}H_\nu(A^{-1},B^{-1})) \\ &\leq \sec^4\alpha\ \omega(H_\nu(A^{-1},B^{-1})),\end{aligned}$$

and by Theorem 3 and (3.1) we have

$$\begin{aligned}(b)\ \omega(F_\nu^{-1}(A,B)) &\leq \sec^2\alpha\ \omega(\mathcal{R}F_\nu^{-1}(A,B)) \\ &\leq \sec^4\alpha\ \omega(\mathcal{R}F_\nu(A^{-1},B^{-1})) \\ &\leq \sec^4\alpha\ \omega(F_\nu(A^{-1},B^{-1})).\end{aligned}$$

By (3.2) and (3.4) we obtain

$$\begin{aligned}(c)\ \omega^{-1}(H_\nu(A,B)) &\leq \sec^3\alpha\omega(H_\nu^{-1}(A,B)) \\ &= \sec^3\alpha\omega\left(\frac{A\sharp_\nu B + A\sharp_{1-\nu}B}{2}\right)^{-1} \\ &\leq \sec^3\alpha\omega\left(\frac{(A\sharp_\nu B)^{-1} + (A\sharp_{1-\nu}B)^{-1}}{2}\right) \\ &= \sec^3\alpha\omega\left(\frac{A^{-1}\sharp_\nu B^{-1} + A^{-1}\sharp_{1-\nu}B^{-1}}{2}\right) \\ &= \sec^3\alpha\omega(H_\nu(A^{-1},B^{-1}))\end{aligned}$$

and

$$
\begin{aligned}
(d)\ \omega^{-1}(F_\nu(A,B)) &\le \sec^3\alpha\omega(F_\nu^{-1}(A,B))\\
&= \sec^3\alpha\omega((1-\nu)A\sharp B+\nu A\nabla B)^{-1}\\
&\le \sec^3\alpha\omega((1-\nu)(A\sharp B)^{-1}+\nu(A\nabla B)^{-1})\\
&= \sec^3\alpha\omega((1-\nu)(A^{-1}\sharp B^{-1})+\nu(A\nabla B)^{-1})\\
&\le \sec^3\alpha\omega((1-\nu)(A^{-1}\sharp B^{-1})+\nu(A^{-1}\nabla B^{-1}))\\
&= \sec^3\alpha\omega(F_\nu(A^{-1},B^{-1})).
\end{aligned}
$$

4 The Pólya inequality for sector matrices

The classical Pólya inequality asserts that if $a, b \ge 0$, then

$$\int_0^1 a^\nu b^{1-\nu}d\nu \le \frac{1}{3}\left(2\sqrt{ab}+\frac{a+b}{2}\right). \tag{4.1}$$

Zou [15], obtained a matrix version of (4.1) for all positive definite matrices $A, B \in \mathbb{M}_n$, as follows:

$$\int_0^1 A\sharp_\nu B d\nu \le \frac{1}{3}\left(2A\sharp B + A\nabla B\right). \tag{4.2}$$

Meanwhile, this author also presented the following norm inequality of Pólya type for matrices:

$$\left\|\int_0^1 A^\nu X B^{1-\nu}d\nu\right\|_2 \le \frac{1}{3}\left\|2A^{1/2}XB^{1/2}+\frac{AX+XB}{2}\right\|_2, \tag{4.3}$$

where $A, B, X \in \mathbb{M}_n$ such that A and B are positive semidefinite.

Theorem 7. *Let $A, B \in S_\alpha$. Then, for $\nu \in [0,1]$,*

$$\left\|\int_0^1 (A\sharp_\nu B)d\nu\right\| \le \sec^3\alpha\left\|\frac{2}{3}(A\sharp_\nu B)+\frac{1}{3}A\nabla_\nu B\right\|.$$

Proof. By Lemma 1 of [6], we have $(\int_0^1(A\sharp_\nu B)d\nu)^* = \int_0^1(A\sharp_\nu B)^* d\nu$.
Therefore$\mathcal{R}\int_0^1(A\sharp_\nu B)d\nu = \int_0^1\mathcal{R}(A\sharp_\nu B)d\nu$. By Theorem 3 of [12],

$$
\begin{aligned}
0 \le \int_0^1(\mathcal{R}A\sharp_\nu\mathcal{R}B)d\nu \le \mathcal{R}\int_0^1(A\sharp_\nu B)d\nu &= \int_0^1\mathcal{R}(A\sharp_\nu B)d\nu\\
&\le \sec^2\alpha\int_0^1(\mathcal{R}A\sharp_\nu\mathcal{R}B)d\nu\\
&\le \sec^2\alpha\left(\frac{2}{3}\mathcal{R}A\sharp_\nu\mathcal{R}B+\frac{1}{3}\mathcal{R}A\nabla_\nu\mathcal{R}B\right)\\
&\le \sec^2\alpha\mathcal{R}\left(\frac{2}{3}(A\sharp_\nu B)+\frac{1}{3}A\nabla_\nu B\right).
\end{aligned} \tag{4.4}
$$

Using (2.5) and (4.4), we get

$$\begin{aligned}\left\|\int_0^1 (A\sharp_\nu B)d\nu\right\| &\leq \sec\ \alpha \left\|\mathcal{R}\int_0^1 (A\sharp_\nu B)d\nu\right\| \\ &\leq \sec^3\alpha \left\|\mathcal{R}\left(\frac{2}{3}(A\sharp_\nu B)+\frac{1}{3}A\nabla_\nu B\right)\right\| \\ &\leq \sec^3\alpha \left\|\frac{2}{3}(A\sharp_\nu B)+\frac{1}{3}A\nabla_\nu B\right\|.\end{aligned}$$

□

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