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Necessary and sufficient conditions of disconjugacy for the fourth order linear ordinary differential equations

by

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Abstract

We study the disconjugacy of the fourth order linear ordinary differential equation

$$u^{(4)}(t) = p(t)u(t),$$

on the interval $[a,\, b]$. We find necessary and sufficient conditions for the disconjugacy on $[a,\, b]$, which have the comparison theorems character. Our results complete Kondrat'ev's second comparison theorem for the case of the fourth order ODE. The above mentioned conditions significantly improve Coppel's well-known condition which guarantees the disconjugacy of our equation for not necessarily constant sign coefficient p, and generalise some optimal disconjugacy results proved for constant-coefficient equations.

Key Words: Disconjugacy, necessary and sufficient conditions, comparison theorem, 4th order ordinary differential equations.
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1 Introduction

In this paper we study the question of the disconjugacy on the interval $I := [a, b] \subset [0,\ +\infty[$ of the fourth order linear ordinary differential equation

$$u^{(4)}(t) = p(t)u(t), \tag{1.1}$$

where $p : I \to R$ is a Lebesgue integrable function. Also we consider the following two-point boundary conditions

$$u(a) = 0\ ,\quad u^{(i)}(b) = 0\ (i = 0, 1, 2), \tag{1.2$_1$}$$

$$u^{(i)}(a) = 0,\quad u^{(i)}(b) = 0\ \ (i = 0, 1), \tag{1.2$_2$}$$

and

$$u^{(i)}(a) = 0\ (i = 0, 1, 2),\quad u(b) = 0, \tag{1.2$_3$}$$

because it is well known that the question of disconjugacy of equations (1.1) in the interval I is closely related with these boundary conditions.

The study of the fourth order boundary value problems is important since they appear as model equations for a large class of higher order parabolic equations arising, for instance, in statistical mechanics, phase field models, hydrodynamics, suspension bridges models, etc. The study of the disconjugacy property plays an important role in these investigations.

The disconjugacy results obtained in this paper can be also understood as comparison theorems. In this direction, Theorem 6 completes Kondrat'ev's second comparison theorem for $n = 4$.

Theorem 1. *([12], Theorem 2) Let $p_1, p_2 : [a,\, b] \to R$ be continuous functions such that the equations*
$$u^{(4)}(t) = p_1(t)u(t), \;\; u^{(4)}(t) = p_2(t)u(t), \tag{1.3}$$
are disconjugate on I, and $p_1 \le p \le p_2$. Then equation (1.1) is disconjugate too.

Indeed, Theorem 6 shows that if p_1, p_2 are the constant sign functions, then in Theorem 1 the assumption of the disconjugacy of equations (1.3) can be replaced by the more general assumptions that $p_1 \in D_-(I)$, $p_2 \in D_+(I)$ (See Definitions 2 and 3). Theorems 1 and 6 are compared in Remark 6.

On the other hand, Theorem 6 significantly improves Coppel's well known theorem ([6], Theorem 1, p. 86), that the condition $\max\limits_{t\in[a,\, b]} |p(t)| \le \frac{128}{(b-a)^4}$ guarantees the disconjugacy of equation (1.1). This fact is discussed in detail in Remark 7.

Also Theorems 2 and 4 generalize for the Lebesgue integrable coefficient p Theorems 3.1 and 4.1 of Ma et al. [15], which are formulated for the constant-coefficient equations (See Remarks 3, 5).

The following notations are used throughout the paper: $R =]-\infty,\, +\infty[$, $R^+ =]0,\, +\infty[$, $R_0^+ = [0,\, +\infty[$, $R_0^- = R \setminus R^+$, $R^- = R \setminus R_0^+$; $C(I;R)$ is the Banach space of continuous functions $u : I \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$; $\widetilde{C}^3(I;R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their third derivatives; $L(I;R)$ is the Banach space of Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$; For arbitrary $x, y \in L(I;\, R)$, the notation
$$x(t) \preccurlyeq y(t) \;(x(t) \succcurlyeq y(t)) \quad \text{for} \quad t \in I,$$
means that $x \le y$ $(x \ge y)$ and $x \ne y$. Also we use the notations $[x]_\pm = (|x| \pm x)/2$.

By a solution of equation (1.1) we understand a function $u \in \widetilde{C}^3(I;\, R)$, which satisfies equation (1.1) a. e. on I.

Definition 1. *Equation (1.1) is said to be disconjugate (non oscillatory) on I, if every nontrivial solution u has less then four zeros on I, the multiple zeros being counted according to their multiplicity. Otherwise we say that equation (1.1) is oscillatory on I.*

Definition 2. *We will say that $p \in D_+(I)$ if $p \in L(I; R_0^+)$, and problem (1.1), (1.2_2) has a solution u, such that*
$$u(t) > 0 \;\; \text{for} \quad t \in]a,\, b[\, . \tag{1.4}$$

Definition 3. *We will say that $p \in D_-(I)$ if $p \in L(I; R_0^-)$, and problem (1.1), (1.2_3) has a solution u, such that inequality (1.4) holds.*

Remark 1. *Let $p \in L(I;\, R_0^+)\, (p \in L(I;\, R_0^-))$, and consider the equation*
$$u^{(4)}(t) = \lambda^4 p(t)u(t) \quad \text{for} \quad t \in I. \tag{1.5}$$
It follows from Lemma 8 that the set $D_+(I)\,(D_-(I))$ can be interpreted as a set of functions $p : I \to R_0^+\,(R_0^-)$ for which $\lambda = 1$ is the first eigenvalue of problem (1.5), (1.2_2) ((1.5), (1.2_1) and (1.5), (1.2_3)). Also, the fact that $\lambda > 0$ is the first eigenvalue of problem (1.5), (1.2_2) ((1.5), (1.2_1) or (1.5), (1.2_3)) is equivalent to the inclusion $\lambda^4 p \in D_+(I)$ $(\lambda^4 p \in D_-(I))$.

2 Main results

2.1 Disconjugacy of equation (1.1) with non-negative coefficient

We will first consider equation (1.1) when the coefficient p is non-negative. In this case we prove the following.

Theorem 2. *Let $p \in L(I;\, R_0^+)$. Then equation (1.1) is disconjugate on I iff there exists $p^* \in D_+(I)$, such that*

$$p(t) \preccurlyeq p^*(t) \quad \text{for} \quad t \in I. \tag{2.1}$$

Remark 2. *From Theorem 2 it is clear that if $x,\ y \in D_+(I)$, then none of the inequalities $x \preccurlyeq y$ and $y \preccurlyeq x$ holds.*

Corollary 1. *Let $p \in L(I;\, R_0^+)$, $p \not\equiv 0$, and $\lambda_0 > 0$ be the first eigenvalue of problem (1.5), (1.2_2). Then equation (1.1) is disconjugate on I iff $\lambda_0 > 1$.*

Let $\lambda_1 > 0$ be the first eigenvalue of the problem

$$u^{(4)}(t) = \lambda^4 u(t), \;\; u^{(i)}(0) = 0, \;\; u^{(i)}(1) = 0 \; (i = 0, 1), \tag{2.2}$$

then it follows from Remark 1 that $\frac{\lambda_1^4}{(b-a)^4} \in D_+(I)$. Therefore from Theorem 2 and Remark 2 we obtain

Corollary 2. *Equation (1.1) is disconjugate on I if*

$$0 \le p(t) \preccurlyeq \frac{\lambda_1^4}{(b-a)^4} \quad \text{for} \quad t \in I, \tag{2.3}$$

and is oscillatory on I if

$$p(t) \ge \frac{\lambda_1^4}{(b-a)^4} \quad \text{for} \quad t \in I. \tag{2.4}$$

Remark 3. *It is well known that the first eigenvalue λ_1 of problem (2.2) is the first positive root of the equation $\cos\lambda \cdot \cosh\lambda = 1$, and $\lambda_1 \approx 4.73004$ (see [5], [15]). Also in Theorem 3.1 of paper [15] it was proved that the equation $u^{(4)} = \lambda^4 u$ is disconjugate on $[0,\, 1]$ if $0 \le \lambda < \lambda_1$.*

But even if both conditions (2.3) and (2.4) are violated, the question on the disconjugacy of equation (1.1) can be answered by the following theorem

Theorem 3. *Let $p \in L(I;\, R_0^+)$, and there exists $M \in R_0^+$ such that*

$$M\frac{b-a}{2} + \int_a^b [p(s) - M]_+ ds \le \frac{192}{(b-a)^3}. \tag{2.5}$$

Then equation (1.1) is disconjugate on I.

Note that in [3] it is proved that when $p : I \to R_0^+$ is an essentially bounded function and $M = 0$ or $M = \operatorname*{ess\,sup}_{t \in I} p(t)$, then condition (2.5) guarantees the unique solvability of the equation $u^{(4)} = pu + h$ under condition (1.2_2).

The example below shows that: a) There exists such a functions p for which both conditions (2.3) and (2.4) are violated but condition (2.5) holds and therefore equation (1.1) is disconjugate on I; b) For some essentially bounded function p condition (2.5) does not hold if the constant M takes its extremal values 0 and $\operatorname*{ess\,sup}_{t\in I} p(t)$, but there exists $M \in]0,\ \operatorname*{ess\,sup}_{t\in I} p(t)[$ for which condition (2.5) holds.

Example 1. *Let $a = 0$, $b = 1$, and $p(t) = 800t^3$ on $t \in I$. Then conditions* (2.3) *and* (2.4) *are violated, and* (2.5) *does not hold if $M = 0$ or $M = \operatorname*{ess\,sup}_{t\in I} p(t)$. On the other hand it is not difficult to verify that due to the inequality $p(t) \geq 864/5$ on $t \in [3/5,\ 1]$, condition* (2.5) *holds with $M = 864/5$.*

2.2 Disconjugacy of equation (1.1) with non-positive coefficient

Now we will consider equation (1.1) with the non-positive coefficient p, and we prove the following.

Theorem 4. *Let $p \in L(I;\ R_0^-)$. Then equation* (1.1) *is disconjugate on I iff there exists $p_* \in D_-(I)$, such that*

$$p(t) \succcurlyeq p_*(t) \quad \text{for} \quad t \in I. \tag{2.6}$$

Remark 4. *From Theorem 4 it is clear that if $x,\ y \in D_-(I)$, then none of the inequalities $x \preccurlyeq y$ and $y \preccurlyeq x$ holds.*

Corollary 3. *Let $p \in L(I;\ R_0^-)$, $p \not\equiv 0$, and $\lambda_0 > 0$ be the first eigenvalue of problem* (1.5), (1.2_3). *Then equation* (1.1) *is disconjugate on I iff $\lambda_0 > 1$.*

Let $\lambda_2 > 0$ be the first eigenvalue of the problem

$$u^{(4)}(t) = -\lambda^4 u(t), \quad u^{(i)}(0) = 0 \ (i = 0, 1, 2), \quad u(1) = 0, \tag{2.7}$$

then as it follows from Remark 1 the inclusion $-\frac{\lambda_2^4}{(b-a)^4} \in D_-(I)$ holds. Therefore from Theorem 4 and Remark 4 we obtain

Corollary 4. *Equation* (1.1) *is disconjugate on I if*

$$-\frac{\lambda_2^4}{(b-a)^4} \preccurlyeq p(t) \leq 0 \quad \text{for} \quad t \in I, \tag{2.8}$$

and is oscillatory on I if

$$p(t) \leq -\frac{\lambda_2^4}{(b-a)^4} \quad \text{for} \quad t \in I. \tag{2.9}$$

Remark 5. *In Theorem* 4.1 *of [15] (see also [5], Theorems* 3.5 *and* 3.6, *[4] subsection* 4.1*) the following is proved: Let λ_2 be the first positive root of the equation $\tanh \frac{\lambda}{\sqrt{2}} = \tan \frac{\lambda}{\sqrt{2}}$ $(\lambda_2 \approx 5.553)$. Then the equation $u^{(4)} = -\lambda^4 u$ is disconjugate on $[0,\ 1]$ if $0 \leq \lambda < \lambda_2$.*

Even if both conditions (2.8) and (2.9) are violated, the question on the disconjugacy of equation (1.1) can be answered by the following.

Theorem 5. *Let $p \in L(I;\, R_0^-)$, such that there exists $M \in R_0^+$ with*

$$M\frac{495}{1024}(b-a) + \int_a^b [p(s)+M]_- ds \le \frac{110}{(b-a)^3}. \tag{2.10}$$

Then equation (1.1) *is disconjugate on I.*

An example analogous to Example 1 can be also constructed for Theorem 5.

2.3 Disconjugacy of equation (1.1) with not necessarily constant sign coefficient

On the basis of the Theorems 2 and 4 we get the non-improvable results which guarantee the disconjugacy of equation (1.1) on I, when p is a not necessarily constant sign function.

Theorem 6. *Let $p_* \in D_-(I)$ and $p^* \in D_+(I)$. Then for an arbitrary function $p \in L(I;\, R)$, such that*

$$p_*(t) \preccurlyeq -[p(t)]_-, \quad [p(t)]_+ \preccurlyeq p^*(t) \quad \text{for} \quad t \in I, \tag{2.11}$$

equation (1.1) *is disconjugate on I.*

The theorem is optimal in the sense that inequalities (2.11) can not be replaced by the condition $p_* \le p \le p^*$.

Remark 6. *We can see that in the Kondrat'ev's comparison second theorem (see Theorem 1) the permissible coefficients p_1 and p_2 should not necessarily be constant sign functions, while in Theorem 6 for the permissible coefficients p_1 and p_2, equations* (1.3) *should not necessarily be disconjugate. For this reason, if*

$$p(t) = \lambda_1^4[cos(2\pi t/n)]_+ - \lambda_2^4[cos(2\pi t/n)]_-,$$

then it follows from Theorem 6 the disconjugacy of equation (1.1) *on* $[0,\, 1]$ *for all $n \in N$ (see Corollary 6), while this fact does not follow from Kondrat'ev's theorem.*

If $p_* \in D_-(I),\ p^* \in D_+(I)$, and

$$\text{mes}\{\, t \in I \,|\, p_*(t) \cdot p^*(t) \ne 0\} > 0, \tag{2.12}$$

then $p_* \preccurlyeq -[p_* + p^*]_-,\ \ [p_* + p^*]_+ \preccurlyeq p^*$, and then Theorem 6 implies the following.

Corollary 5. *Let $p_* \in D_-(I),\ p^* \in D_+(I)$, and inequality* (2.12) *holds. Then equation* (1.1) *with $p = p_* + p^*$ is disconjugate on I.*

From Theorem 6 with $p_* := -\frac{\lambda_2^4}{(b-a)^4}$ and $p^* := \frac{\lambda_1^4}{(b-a)^4}$ we obtain

Corollary 6. *Let $\lambda_1 > 0$ and $\lambda_2 > 0$ be the first eigenvalues of problems* (2.2) *and* (2.7) *respectively, and the function $p \in L(I;\, R)$ admits the inequalities*

$$-\frac{\lambda_2^4}{(b-a)^4} \preccurlyeq p(t) \preccurlyeq \frac{\lambda_1^4}{(b-a)^4} \quad \text{for} \quad t \in I.$$

Then equation (1.1) *is disconjugate on I.*

Remark 7. *If we take into account that* $\lambda_1^4 \approx 501$ *and* $\lambda_2^4 \approx 951$, *then it is clear that Corollary 6 significantly improves Coppel's well known condition* $\max\limits_{t\in[a,\,b]} |p(t)| \leq \frac{128}{(b-a)^4}$, *proved in [6] (Theorem 1, page 86), which for* $p \in C(I;\,R)$ *guarantees the disconjugacy of equation* (1.1) *on* I.

Also from Theorem 1 by Theorems 3 and 5 we obtain

Corollary 7. *Let the functions* $p_1 \in L(I;\,R_0^-)$, $p_2 \in L(I;\,R_0^+)$, *be such that*

$$\int_a^b |p_1(s)|ds \leq \frac{110}{(b-a)^3}, \qquad \int_a^b p_2(s)ds \leq \frac{192}{(b-a)^3},$$

and the condition $p_1 \leq p \leq p_2$ *holds. Then equation* (1.1) *is disconjugate on* I.

3 Auxiliary propositions

For the equation

$$u^{(4)}(t) = p_1(t)u(t) \quad \text{for} \quad t \in R_0^+, \tag{3.1}$$

where $p_1 \in L_{loc}(R_0^+;\,R)$, and $t_0 \in R_0^+$ is an arbitrary point, we will define the first conjugate point $\eta(t_0,\,p_1)$ to t_0, and the number $\tau(t_0, p_1)$, as in [11] (see Definition 1.5).

Definition 4. *Let* $t_0 \in R_0^+$, *and* $F(t_0,\,p_1)$ *be the set of such* $t_1 > t_0$ *for which some solutions of equation* (3.1) *in the interval* $[t_0,\,t_1]$ *have at least* 4 *zeroes (where zeroes are counted according to their multiplicities). Then we will say that for equation* (3.1), $\eta(t_0,\,p_1) = \inf F(t_0,\,p_1)$ *is the first conjugate point to* t_0 *(*$\eta(t_0,\,p_1) = +\infty$ *if* $F(t_0,\,p_1) = \emptyset$*).*

Definition 5. *Let* $t_0 \in R_0^+$, *and* $E(t_0,\,p_1)$ *be the set of such* $t_1 > t_0$ *for which there exists a solution* u *of equation* (3.1) *such that*

$$u(t_0) = u(t_1) = 0, \;\; u(t) > 0 \quad for \quad t \in]t_0,\,t_1[\,.$$

Then $\tau(t_0, p_1) = \sup E(t_0,\,p_1)$.

For an arbitrary function $x : [a,\ b] \to R$, we introduce the functions $x_-,\,x_+ : R_0^+ \to R$ by the equalities

$$x_{\pm}(t) = \begin{cases} x(t) & \text{for} \;\; t \in I \\ \pm 1 & \text{for} \;\; t \in R_0^+ \setminus I \end{cases}. \tag{3.2}$$

Remark 8. $\eta(a_0, x_{\pm}) < +\infty$ *for an arbitrary* $a_0 \in [a,\,b]$. *(See for example [11], Theorem* 2.15*).*

Now if we assume that

$$p_1(t) \geq 0 \quad \text{for} \quad t \in R_0^+, \tag{3.3}$$

or

$$p_1(t) \leq 0 \quad \text{for} \quad t \in R_0^+, \tag{3.4}$$

then for the numbers $\eta(t_0,\,p_1)$, $\tau(t_0,p_1) \in R^+$ the following results hold.

Lemma 1. *([10], Corollary 5.1) If inequality (3.3) holds and $\eta(x, p_1) < +\infty$ for $x \in [t_*, t^*]$, then $\eta(t_*, p_1) < \eta(t^*, p_1)$.*

Lemma 2. *([13], Theorem 8.6) If inequality (3.4) holds and $0 < t_* < t^*$, then $\eta(t_*, p_1) < \eta(t^*, p_1)$.*

Lemma 3. *([11], Lemma 1.9) Let $\eta(a, p_1) < +\infty$. Then there exist a solution u of equation (3.1) positive on $]a, \eta(a, p_1)[$ and $\ell \in \{1, 2, 3\}$ such, that*

$$u^{(i-1)}(a) = 0 \ (i = \overline{1, \ell}), \quad u^{(i-1)}(\eta(a, p_1)) = 0 \ (i = \overline{1, 4-\ell}). \tag{3.5}$$

Moreover, if condition (3.3) ((3.4)) holds then $\ell = 2$ ($\ell = 1$ or $\ell = 3$).

Lemma 4. *Assume that condition (3.4) holds. Then problem (3.1), (3.5) has a solution positive on $]a, \eta(a, p_1)[$ both for $\ell = 1$ and $\ell = 3$.*

Proof. First we assume that the problem (3.1), (3.5) with $\ell = 3$ has a solutions u_1 positive on $]a, \eta(a, p_1)[$, and u_2 is a solution of equation (3.1) which satisfies the conditions

$$u_2^{(i-1)}(\eta(a, p_1)) = 0 \ (i = 1, 2, 3), \ \ u_2'''(\eta(a, p_1)) < 0. \tag{3.6}$$

Then u_2 also satisfies the conditions (3.5) with $\ell = 1$ and is positive in $]a, \eta(a, p_1)[$. Indeed, if

$$\rho(t) := u_1(t)u_2'''(t) - u_2(t)u_1'''(t) - [u_1'(t)u_2''(t) - u_2'(t)u_1''(t)],$$

due to (3.1) we have $\rho'(t) \equiv 0$, and thus $\rho \equiv Const$. Therefore by (3.5) and (3.6) we get $-u_2(a)u_1'''(a) = \rho(a) = \rho(\eta(a, p_1)) = 0$, and then $u_2(a) = 0$, because it is clear that $u_1'''(a) \neq 0$. I.e., u_2 is a nonzero solution of problem (3.1), (3.5) with $\ell = 1$. If there exists $t_0 \in]a, \eta(a, p_1)[$ such that $u_2(t_0) = 0$, then due to conditions (3.6) we get $\eta(t_0, p_1) \leq \eta(a, p_1)$, which contradicts with Lemma 2. Thus our assumption is invalid and $u_2 > 0$ in $]a, \eta(a, p_1)[$.

Analogously we get that if problem (3.1), (3.5) with $\ell = 1$ has a positive in $]a, \eta(a, p_1)[$ solution u_2, then problem (3.1), (3.5) with $\ell = 3$ has also positive in $]a, \eta(a, p_1)[$ solution u_1. But from Lemma 3 it follows that one of the solutions u_1 and u_2 exists, and the proof is finished. ∎

Lemma 5. *([11], Lemma 2.10) Assume that the inequality (3.3) holds. Then for an arbitrary $t_0 \in R_0^+$ the equality $\tau(t_0, p_1) = \eta(t_0, p_1)$ holds.*

Lemma 6. *([11], Lemma 1.16) Let $p_1(t) \geq p_2(t) \geq 0$ for $t \in R_0^+$. Then for an arbitrary $t_0 \in R_0^+$ the inequality $\tau(t_0, p_1) \leq \tau(t_0, p_2)$ holds.*

Lemma 7. *([13], Theorem 10.1) Let $p_1(t) \leq p_2(t) \leq 0$ for $t \in R_0^+$. Then for an arbitrary $t_0 \in R_0^+$ the inequality $\eta(t_0, p_1) \leq \eta(t_0, p_2)$ holds.*

The following proposition is the special case of the theorems proved for the more general problems than (1.1), (1.2_ℓ) $\ell = 1, 2, 3$ (see [1], [2], [7]-[9], [13]).

Proposition 1. *Let $\ell \in \{1,2,3\}$. Then problem* (1.5), (1.2$_\ell$) *with* $(-1)^{4-\ell}p \succcurlyeq 0$ *has an infinite sequence* $\{\lambda_n\}_{n=1}^{+\infty}$ *of real eigenvalues, where*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lim_{n\to+\infty} \lambda_n = +\infty,$$

and to every eigenvalue λ_n there corresponds the essentially unique eigenfunction u_{λ_n} with exactly $n-1$ zeroes in $]a, b[$.

Then we can prove the following.

Lemma 8. *Let $\lambda > 0$, then the following assertions are equivalent:*
A. $\lambda^4 p \in D_+(I)$ $(\lambda^4 p \in D_-(I))$;
B. $\eta(a,\, \lambda^4 p_+) = b$ $(\eta(a,\, \lambda^4 p_-) = b)$;
C. λ *is the first eigenvalue of problem* (1.5), (1.2$_2$) ((1.5), (1.2$_1$) *and* (1.5), (1.2$_3$)).

Proof. Let $\lambda^4 p \in D_+(I)$, and note that then $p \not\equiv 0$.

From A. and Definitions 4, 5, it is clear that $\eta(a, \lambda^4 p_+) \le b \le \tau(a, \lambda^4 p_+)$, and then from Lemma 5 condition B. follows. Also from B. by Lemma 3 with $\ell = 2$ we obtain A. Equivalency of A and C, and therefore of C and B, immediately follows from Proposition 1 with $\ell = 2$.

Let now $\lambda^4 p \in D_-(I)$, and note that $p \not\equiv 0$.

If condition A. holds, then problem (1.1), (1.2$_3$) has a solution u_0 positive in $]a, b[$, and therefore $\eta(a,\, \lambda^4 p_-) \le b$. Assume that $\eta(a,\, \lambda^4 p_-) < b$. Then due to Lemma 4 there exists positive in $]a,\, \eta(a,\, \lambda^4 p_-)[$ solution u_1 of problems (1.1), (3.5) with $\ell = 3$, and we get the contradiction that $v = u_1 - \frac{u_1'''(a)}{u_0'''(a)} u_0$ is a solution of equation (1.1) and $v^{(i)}(a) = 0$ $(i = 0,1,2,3)$, $v(\eta(a,\, \lambda^4 p_-)) < 0$. Therefore the condition B. holds. Also from B. by Lemma 3 with $\ell = 3$ we obtain A. Equivalency of A. and C., and therefore of C. and B., immediately follows from Proposition 1 with $\ell = 3$, and Lemma 4. □

4 Proof of the main results

Some lemmas from the previous section are formulated for the functions defined on R_0^+. For this reason if in our proofs these lemmas are used, we extend the coefficients of equations to R_0^+ by the equalities (3.2).

Proof of Theorem 2. If $p \equiv 0$, then the validity of our theorem is trivial, therefore assume that $p \not\equiv 0$.

Sufficiency. From condition (2.1) by Lemmas 5 and 6 (with $p_1 = p_+^*$, $p_2 = p_+$) we obtain $\eta(a,\, p_+) \ge \eta(a,\, p_+^*)$, where due to the inclusion $p^* \in D_+(I)$ by Lemma 8 we have $\eta(a,\, p_+^*) = b$, and therefore

$$\eta(a,\, p_+) \ge b. \tag{4.1}$$

But the condition $p^* \in D_+(I)$ implies, that the problem

$$u^{(4)}(t) = p_+^*(t) u(t) \quad \text{for} \quad t \in I, \;\; u^{(i)}(a) = 0, \;\; u^{(i)}(b) = 0 \, (i = 0,1), \tag{4.2}$$

has a solution u positive in $]a,\, b[$. Also, if we assume that $\eta(a,\, p_+) = b$, from Lemma 3 with $\ell = 2$, it follows that the problem

$$v^{(4)}(t) = p_+(t)v(t) \quad \text{for} \quad t \in I, \;\; v^{(i)}(a) = 0, \;\; v^{(i)}(b) = 0\,(i = 0, 1), \tag{4.3}$$

has a solution v positive in $]a,\, b[$. Now if we multiply equations (4.2) and (4.3) respectively by v and $-u$, and integrate their sum from a to b, in view of (3.2) and boundary conditions (4.2) and (4.3), by integration by parts we obtain equality $\int_a^b (p^*(s) - p(s))u(s)v(s)ds = 0$, which contradicts condition (2.1). Thus our assumption is invalid and due to (4.1) we have

$$\eta(a,\, p_+) > b. \tag{4.4}$$

Now assume that equation (1.1) is oscillatory on I, i.e., it has a solution u with at least four zeroes in $[a,\, b]$. Therefore if $t_0 \in [a,\, b[$ is the first zero of u, it is clear that $\eta(t_0,\, p_+) \in]t_0,\, b]$, and then due to (4.4) we get $\eta(t_0,\, p_+) < \eta(a,\, p_+)$. Thus $t_0 \neq a$, and then from Lemma 1 and Remark 8 it follows that $\eta(t_0,\, p_+) > \eta(a,\, p_+)$,, contradicting the previous inequality. Therefore our assumption is invalid and equation (1.1) is disconjugate on I.

Necessity. Let equation (1.1) be disconjugate on I. Due to Proposition 1, problem (1.5), (1.2_2) has the first eigenvalue $\lambda_1 > 0$, and from Lemma 8 with $\ell = 2$ it follows that $p_1 := \lambda_1^4 p \in D_+(I)$, and

$$\eta(a,\, \lambda_1^4 p_+) = b. \tag{4.5}$$

Also from the disconjugacy of (1.1) on I it follows that $\lambda_1 \neq 1$ and $\eta(a,\, p_+) > b$. Now assume that $\lambda_1 < 1$. Then from Lemmas 5, 6, we obtain $\eta(a,\, \lambda_1^4 p_+) \geq \eta(a,\, p_+) > b$, which contradicts (4.5). The obtained contradiction proves that $\lambda_1 > 1$, and therefore condition (2.1) holds, where $p_1 \in D_+(I)$.

□

Proof of Corollary 1. Sufficiency. Let $\lambda_0 > 1$, then $p \preccurlyeq \lambda_0^4 p$, where due to Lemma 8 we have $\lambda_0^4 p \in D_+(I)$, and from Theorem 2 we obtain the disconjugacy of equation (1.1) on I. The proof of the necessity is analogous to the proof of the necessity of Theorem 2. □

Proof of Theorem 3. Assume that equation (1.1) is not disconjugate on I. Then there exists a solution u of equation (3.1) (with $p_1 = p_+$) which has at least four zeroes in $[a,\, b]$, and if $a_0 \in [a,\, b[$ is the first zero of u, then $b_0 := \eta(a_0,\, p_+) \in]a_0,\, b]$. Therefore due to Lemma 3, equation (3.1) (with $p_1 = p_+$) under the boundary conditions $u^{(i)}(a_0) = 0,\; u^{(i)}(b_0) = 0\; (i = 0, 1)$, has a solution u positive in $]a_0,\, b_0[$ and $p_1(t) = p(t)$ on $[a_0,\; b_0]$. Therefore if $\omega_0 := b_0 - a_0$ and

$$v(t) := u(t\omega_0 + a_0), \qquad h(t) := \omega_0^4 p(t\omega_0 + a_0) \quad \text{for} \quad t \in [0,\, 1], \tag{4.6}$$

there exists $t_0 \in]0,\, 1[$ such, that $v(t_0) = ||v||_C$, and then

$$\frac{1}{||v||_C} \int_0^1 G(t_0,\, s)h(s)v(s)ds = 1, \tag{4.7}$$

where

$$G(t,s) = \frac{1}{6}\begin{cases}(1-t)^2 s^2(3t-s-2st) & \text{for } \ 0 \le s \le t \le 1\\ (1-s)^2 t^2(3s-t-2st) & \text{for } \ 0 \le t < s \le 1\end{cases}$$

is the Green's functions of the problem $u^{(4)} = 0,\ u^{(i)}(0) = u^{(i)}(1) = 0\ (i = 0, 1)$. Also it is not difficult to verify that

$$0 \le G(t,\, s) \le \frac{1}{192},\ \ \int_0^1 G(t,\, s)ds = \frac{t^2(1-t)^2}{24} \le \frac{1}{384} \quad \text{for} \quad s, t \in [0,\, 1].$$

Consequently for an arbitrary $M \in R_0^+$ we have the estimation

$$1 < \int_0^1 G(t_0,\, s)h(s)ds \le M\frac{\omega_0^4}{384} + \frac{\omega_0^4}{192}\int_0^1 \Big[\frac{h(s)}{\omega_0^4} - M\Big]_+ ds,$$

which contradicts (2.5), and therefore proves our theorem. ∎

Proof of Theorem 4. If $p \equiv 0$, then the validity of our theorem is trivial, therefore assume that $p \not\equiv 0$.

Sufficiency. From condition (2.6), by Lemma 7 (with $p_1 = p_{*-}, p_2 = p_-$) we obtain $\eta(a,\, p_-) \ge \eta(a,\, p_{*-})$, where due to the inclusion $p_* \in D_-(I)$ by Lemma 8 we have $\eta(a,\, p_{*-}) = b$, and therefore $\eta(a,\, p_-) \ge b$. Now if we assume that $\eta(a,\, p_-) = b$, as in the proof of Theorem 2 by Lemma 3 with $\ell = 1$ (instead of $\ell = 2$) we obtain that

$$\eta(a,\, p_-) > b. \tag{4.8}$$

Now assume that equation (1.1) is oscillatory on I, i.e., it has a solution u with at least four zeroes in $[a,\, b]$. Therefore if $t_0 \in [a,\, b[$ is the first zero of u, it is clear that $\eta(t_0,\, p_-) \in]t_0,\, b]$, and then due to (4.8) we get $\eta(t_0,\, p_-) < \eta(a,\, p_-)$. Thus $t_0 \ne a$, and then from Lemma 2 it follows that $\eta(t_0,\, p_-) > \eta(a,\, p_-)$, which contradicts the previous inequality. Therefore our assumption is invalid and equation (1.1) is disconjugate on I.

Necessity. Let equation (1.1) be disconjugate on I. Due to Proposition 1 problem (1.5), (1.2_3) has the first eigenvalue $\lambda_1 > 0$, and from Lemma 8 with $\ell = 3$ it follows that $p_1 := \lambda_1^4 p \in D_-(I)$, and

$$\eta(a,\, \lambda_1^4 p_-) = b. \tag{4.9}$$

Also from the disconjugacy of (1.1) on I it follows that $\lambda_1 \ne 1$ and $\eta(a,\, p_-) > b$. Now assume that $\lambda_1 < 1$. Then from Lemma 7, we obtain $\eta(a,\, \lambda_1^4 p_-) \ge \eta(a,\, p_-) > b$, which contradicts (4.9). The obtained contradiction proves that $\lambda_1 > 1$, and therefore condition (2.6) holds, where $p_1 \in D_-(I)$.

∎

Proof of Corollary 3. Sufficiency. Let $\lambda_0 > 1$, then $\lambda_0^4 p \preccurlyeq p$, where due to Proposition 1 we have $\lambda_0^4 p \in D_-(I)$. Therefore from Theorem 4 it follows disconjugacy of equation (1.1) on I. Proof of the necessity is analogous to the proof of the necessity of Theorem 4. ∎

Proof of Theorem 5. Assume that equation (1.1) is not disconjugate on I. Then there exists a solution u of equation (3.1) (with $p_1 = p_-$), which has at least four zeroes in $[a, b]$, and if $a_0 \in [a, b[$ is the first zero of u, then $b_0 := \eta(a_0, p_1) \in]a_0, b]$. Therefore due to Lemma 4, equation (3.1) (with $p_1 = p_-$), under the boundary conditions $u^{(i)}(a_0) = 0$ $(i = 0, 1, 2)$, $u(b_0) = 0$, has a solution u positive in $]a_0, b_0[$, and $p_1(t) = p(t)$ on $[a_0, b_0]$. Therefore if $\omega_0 := b_0 - a_0$, and if the functions v and h are defined by the equalities (4.6), then there exists $t_0 \in]0, 1[$ for which $v(t_0) = ||v||_C$ and then equality (4.7) holds where

$$G(t, s) = -\frac{1}{6}\begin{cases} t^3(1-s)^3 - (t-s)^3 & \text{for } \ 0 \le s \le t \le 1 \\ t^3(1-s)^3 & \text{for } \ 0 \le t < s \le 1 \end{cases}$$

is the Green's functions of the problem $u^{(4)} = 0,\ u^{(i)}(0) = 0$ $(i = 0, 1, 2)$, $u(1) = 0$. It is not difficult to verify that

$$-\frac{1}{110} < G(t,\, s) \le 0, \quad \int_0^1 |G(t,\, s)|ds = \frac{t^3(1-t)}{24} \le \frac{9}{2048} \quad \text{for} \quad s, t \in [0,\, 1],$$

and then for an arbitrary $M \in R_0^+$ we have the estimation

$$1 < \int_0^1 |G(t_0,\, s)|(-h(s))ds < M\frac{9\,\omega_0^4}{2048} + \frac{\omega_0^4}{110}\int_0^1 \Big[\frac{h(s)}{\omega_0^4} + M\Big]_- ds,$$

which contradicts (2.10) and therefore proves our theorem. ∎

Proof of Theorem 6. Assume the contrary, i.e., that equation (1.1) has a solution u with at least four zeroes in I. If $a_0 \in [a, b[$ is the first zero of u, it is clear that $b_0 := \eta(a_0, p) \in]a_0, b]$. Therefore due to Lemma 3 there exists $\ell \in \{1, 2, 3\}$ such that the problem

$$u^{(4)}(t) = [p(t)]_+ u(t) - [p(t)]_- u(t) \quad \text{for} \quad t \in [a_0,\ b_0], \tag{4.10}$$

$$u^{(i-1)}(a_0) = 0 \ \ (i = \overline{1, \ell}), \quad u^{(i-1)}(b_0) = 0 \ \ (i = \overline{1, 4-\ell}), \tag{4.11$_\ell$}$$

has a solution u positive on $I_0 :=]a_0, b_0[$. Also due to condition (2.11), from Theorems 2 and 4, we obtain that the equations

$$u^{(4)}(t) = [p(t)]_+ u(t) \quad \text{for} \quad t \in I, \tag{4.12}$$

$$u^{(4)}(t) = -[p(t)]_- u(t) \quad \text{for} \quad t \in I, \tag{4.13}$$

are disconjugate on I, and therefore on $[a_0, b_0]$ too. On the other hand, it follows from Lemma 4.2 of [14] (see also Lemma 2.6 in [4]) that the Green's functions $G_{1\ell}$ and $G_{2\ell}$ of problems (4.12), (4.11$_\ell$) and (4.13), (4.11$_\ell$) respectively, satisfy the conditions

$$(-1)^\ell G_{1\ell}(t,\, s) \ge 0, \quad (-1)^\ell G_{2\ell}(t,\, s) \ge 0 \quad \text{for} \quad t,\, s \in [a_0,\, b_0].$$

Then from (4.10), non-negativity of the functions $[p]_\pm$, and positivity of u on I_0, we have

$$(-1)^{\ell+1}u(t) = \Big|\int_{a_0}^{b_0} G_{1\ell}(t,\, s)[p(s)]_- u(s)ds\Big|, \ \ (-1)^\ell u(t) = \Big|\int_{a_0}^{b_0} G_{2\ell}(t,\, s)[p(s)]_+ u(s)ds\Big|,$$

on I_0, which is a contradiction. Therefore our assumption is invalid and equation (1.1) is diconjugate on I. ∎

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