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Simple singularities of parametrized space curves in positive characteristic

by

Nimra Javed(1), Gerhard Pfister(2)

Abstract

Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. The aim of the article is to give a classification of simple parametrized space curve singularities over $\mathbb{K}$. The idea is to give explicitly a class of families of singularities which are not simple such that almost all singularities deform to one of those and show that remaining singularities are simple.

Key Words: Simple singularities, space curves, parametrized curves, characteristic $p \geq 0$.
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1 Introduction

The study and classification of singularities have a long history. Very important contributions go back to Zariski [21] and Arnold [2]. Most of the results were obtained over the complex numbers. Greuel and his students started a classification for hypersurface singularities in characteristic $p > 0$ ([3],[9],[10]). Bruce and Gaffney [5] classified the simple[1] parameterized plane curve singularities over the complex numbers. Parametrization of space curve singularities were studied by Gibson and Hobbs over the complex numbers [8]. We recall their classification in Theorem 1. Their way of proving cannot be adapted to positive characteristics. The reason is that in characteristic zero heavily the results of Mather [16] are used. Mather uses so called complete transversals in the orbit space (of the group associated to $\mathcal{A}$- equivalence). There is no such theory for positive characteristic.
Mehmood and the second author [17] classified the simple plane curve parametrizations in characteristic p. The aim of this paper is to give the classification of parametrized irreducible curve singularities in 3-space in characteristic $p > 0$. The classification is given in Theorem 3. The classification depends very much on the characteristic. We found parametrizations which do not occur in characteristic 0. On the other hand not all simple parametrizations from characteristic 0 are simple in characteristic $p > 0$. To give an example, the parametrizations $(t^4, t^5, 0)$ and $(t^4, t^5 + t^6, 0)$ are $\mathcal{A}$-equivalent in characteristic $p \neq 5$ but not in characteristic 5. The parametrization $(t^4, t^6 + t^k, 0)$ is simple in characteristic 0 for all k but not simple in characteristic 17 if $k > 9$. In characteristic 0 the following holds. Let $(x(t), y(t))$ define a plane curve and consider the space curve $(x(t), y(t), z(t))$. The classification of Gibson and Hobbs in characteristic zero ([8], table 1 and table 2) shows

[1] See Definition 4.

that $(x(t), y(t), z(t))$ being simple implies that $(x(t), y(t))$ is simple[2]. This is not true in positive characteristic. In characteristic 2 the curve $(t^3, t^{10}, 0)$ is not simple, but the curve (t^3, t^{10}, t^{11}) is simple (cf. Theorem 3).
Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. This field is fixed now during this paper. A parametrized space curve singularity is given by a map $f : \mathbb{K}[[x, y, z]] \to \mathbb{K}[[t]]$. If $f(x) = x(t)$, $f(y) = y(t)$ and $f(z) = z(t)$ then we write shortly $f = (x(t), y(t), z(t))$. The image of f is the subalgebra $\mathbb{K}[[x(t), y(t), z(t)]] \subseteq \mathbb{K}[[t]]$ and we will always assume that the δ-invariant of the parametrization is finite

$$\delta(f) := dim_{\mathbb{K}}\mathbb{K}[[t]]/\mathbb{K}[[x(t), y(t), z(t)]] < \infty.$$

The finiteness condition implies that there exist a minimal c such that the conductor ideal satisfies $t^c\mathbb{K}[[t]] \subseteq \mathbb{K}[[x(t), y(t), z(t)]]$. Two parametrized space curve singularities $f = (x(t), y(t), z(t))$ and $g = (\hat{x(t)}, \hat{y(t)}, \hat{z(t)})$ are called $\mathcal{A}$- equivalent, $f \sim g$, if there exist automorphisms,

$$\psi : \mathbb{K}[[t]] \to \mathbb{K}[[t]]$$

$$\varphi : \mathbb{K}[[x, y, z]] \to \mathbb{K}[[x, y, z]]$$

such that the following diagram commutes:

$$\begin{tikzcd}
K[[x,y,z]] \arrow[r, "f"] \arrow[d, "\varphi"'] & K[[x(t),y(t),z(t)]] \subseteq K[[t]] \arrow[d, "\psi"] \\
K[[x,y,z]] \arrow[r, "g"] & K[[\hat{x}(t),\hat{y}(t),\hat{z}(t)]] \subseteq K[[t]]
\end{tikzcd}$$

i.e.

$$(x(\psi(t)), y(\psi(t)), z(\psi(t))) = (\varphi_1(\hat{x}(t), \hat{y}(t), \hat{z}(t)), \varphi_2(\hat{x}(t), \hat{y}(t), \hat{z}(t)), \varphi_3(\hat{x}(t), \hat{y}(t), \hat{z}(t))).$$

Definition 1. *Given a parametrization $f = (x(t), y(t), z(t))$, we define the semigroup as*

$$\Gamma = \Gamma_f = \{ord_t(h) | h \in \mathbb{K}[[x(t), y(t), z(t)]] \setminus \{0\}\}.$$

If $t^c\mathbb{K}[[t]]$ is the conductor ideal then $c - 1 \notin \Gamma$ and $l \in \Gamma$ if $l \geq c$. The integer $c = c(\Gamma)$ is called conductor of Γ. The cardinality of the set $\mathbb{Z}_{\geq 0} \setminus \Gamma$ is called $\delta = \delta(\Gamma)$. The semigroup Γ has a unique minimal system of generators $\{\beta_1, \ldots, \beta_k\}$ and we write $\Gamma = \langle \beta_1, \ldots, \beta_k \rangle$. We will always assume that the minimal generators of a semigroup are given in an increasing way.

Definition 2. *Given a parametrization $f = (x(t), y(t), z(t))$. A set $\{w_1(t), \ldots, w_k(t)\} \subset K[[x(t), y(t), z(t)]] \setminus \{0\}$ is called sagbi basis of $K[[x(t), y(t), z(t)]]$ if*[3] *$ord_t(w_1), \ldots, ord_t(w_k)$ generate Γ.*

[2]The other implication is always true by definition.

[3]In general the definition of sagbi bases (the analogon of Gröbner bases for subalgebras) is more involved. There are efficient algorithms to compute sagbi bases. For more details see [12] or [13].

Definition 3. *Let* $f = (x(t), y(t), z(t)) \in t\mathbb{K}[[t]]^3$ *define a parametrized space curve singularity. A deformation of* f *over the affine ring* $A = \mathbb{K}[x_1, \ldots, x_n]/I$ *is a pair* $(F, \mathfrak{m})$, $F \in tA[[t]]^3$ *and* $\mathfrak{m} \subseteq A$ *a maximal ideal , such that* F *mod* $\mathfrak{m}A[[t]]^3 = f$. *Since the field* $\mathbb{K}$ *is algebraically closed a closed point* $p \in V(I) \subseteq \mathbb{K}^n$ *corresponds to a maximal ideal* $\mathfrak{m}_p \subseteq A$ *and we will write* $F(p, t) \in t\mathbb{K}[[t]]^3$ *for* F *mod* $\mathfrak{m}_p A[[t]]^3$. *We will only consider closed points in this paper. We will denote the point corresponding to* $\mathfrak{m}$ *by* o.

Definition 4. *Let* $f = (x(t), y(t), z(t)) \in t\mathbb{K}[[t]]^3$ *define a parametrized space curve singularity.* f *is called simple if for any deformation* $(F, \mathfrak{m})$ *of* f, $F \in tA[[t]]^3$, $A = \mathbb{K}[x_1, ..., x_n]/I$, *there exsist a Zariski open subset* U *of* $V(I) \subset \mathbb{K}^n$ *containing* o *such that the set* $\{F(p, t) \in \mathbb{K}[[t]]^3 | p \in U\}$ *contains only finitely many* $\mathcal{A}$*-equivalence classes.*

Remark 1. *Let* $f = (x(t), y(t), z(t)) \in t\mathbb{K}[[t]]^3$ *define a parametrized space curve singularity and* c *the conductor of the semigroup. We will see in Corollary 1 that* f *is finitely determined, i.e.* f *is determined by its coefficients up to degree* c. *Given a deformation* $(F, \mathfrak{m})$ *of* $f = (x(t), y(t), z(t)) \in \mathbb{K}[[t]]^3$, *we will always choose an open set* $U \subset Specmax(A)$ *such that for* $\mathfrak{m}_p \in U$ *all monomials of* f *of degree* $\leq c$ *occur in* $F(p, t)$, *i.e. we do not allow cancellation of these terms in the family. Especially, if* $F = (X(t), Y(t), Z(t))$ *we have* $ord_t(x(t)) \geq ord_t(X(p, t))$, $ord_t(y(t)) \geq ord_t(Y(p, t))$ *and* $ord_t(z(t)) \geq ord_t(Z(p, t))$.

Remark 2. *Let* $f = (x(t), y(t), z(t)) \in t\mathbb{K}[[t]]^3$ *define a parametrized space curve singularity and* $(F, \mathfrak{m})$ *a deformation of* f, $F \in tA[[t]]^3$. *Let* $U \subset Specmax(A)$ *be a non-empty open set such that* o *is in the closure of* U *and for all* $p \in U$ *the parametrization* $F(p, t)$ *is not simple, then* f *is not simple.*

We will give the idea of the classification in case of characteristic ≥ 17. The case of smaller characteristic is similar. We first prove that parametrizations with semigroups greater[4] or equal to $\langle 5, 6, 7 \rangle$ or $\langle 4, 9, 10 \rangle$ are not simple. Parametrizations with smaller semigroup are discussed case by case and turn out to be simple.

We now recall the results of the classification of Gibson and Hobbs over the complex numbers [8].

Theorem 1. *Let* $\mathbb{C}$ *be the field of complex numbers. Let* $f \in t\mathbb{C}[[t]]^3$ *be a simple parametrized space curve singularity, then* f *is* $\mathcal{A}$*- equivalent to a parametrized space curve singularity in the following table:*

[4] See Definition 6.

Characteristic $p=0$	
Γ	Normal Form
$\langle 1\rangle$	$(t,0,0)$
$\langle 2,k\rangle$	$(t^2,t^k,0)$, k >2 odd
$\langle 3,k,r\rangle$ $k\cdot r\equiv 2 \bmod 3$ or $r=\infty$	(t^3,t^k+t^l,t^r) $l=\infty$ or $k<l\leq 2k-6$ and $k\cdot l\equiv 2 \bmod 3$ r $=\infty$ or $k<r<2k-2$
$\langle 4,5,6\rangle$	(t^4,t^5,t^6)
$\langle 4,5,7\rangle$	(t^4,t^5,t^7)
$\langle 4,5,11\rangle$	(t^4,t^5,t^{11}) (t^4,t^5+t^7,t^{11})
$\langle 4,5\rangle$	$(t^4,t^5,0)$ $(t^4,t^5+t^7,0)$
$\langle 4,6,k+6,r\rangle$ $r\in\{k-2,k+2,k+4,k+8,\infty\}$ $k\geq 7$ odd	(t^4,t^6+t^k,t^{k-2}), $k\geq 9$ (t^4,t^6+t^k,t^{k+2}) (t^4,t^6+t^k,t^{k+8}) (t^4,t^6+t^k,t^{k+4}) $(t^4,t^6+t^k,0)$
$\langle 4,6,k\rangle$	(t^4,t^6,t^k), k ≥ 7, k odd
$\langle 4,7,9\rangle$	(t^4,t^7,t^9), (t^4,t^7,t^9+t^{10}),
$\langle 4,7,10\rangle$	(t^4,t^7,t^{10}), (t^4,t^7+t^9,t^{10}),
$\langle 4,7,13\rangle$	(t^4,t^7,t^{13}), (t^4,t^7+t^9,t^{13}),
$\langle 4,7,17\rangle$	(t^4,t^7,t^{17}), (t^4,t^7+t^9,t^{17}), (t^4,t^7+t^{13},t^{17}),
$\langle 4,7\rangle$	$(t^4,t^7,0)$, $(t^4,t^7+t^9,0)$, $(t^4,t^7+t^{13},0)$,

An important basis for the classification is the following theorem of Zariski ([21], Chapter III, Proposition 1.2) generalized to space curves.

Theorem 2. *Given a parametrization*

$$(t^l + \Sigma_{i>l} a_i t^i, t^m + \Sigma_{i>m} b_i t^i, t^n + \Sigma_{i>n} c_i t^i)$$

$2 < l < m < n$, $l \nmid m$ *and* $n \notin \langle l, m \rangle$ *if* $n < \infty$, *with semigroup* Γ *and conductor* c. *Let* $k \in \Gamma$ *then there exists an* $\mathcal{A}$*-equivalent parametrization*

$$(t^l + \Sigma_{i>l} \hat{a}_i t^i, t^m + \Sigma_{i>m} \hat{b}_i t^i, t^n + \Sigma_{i>n} \hat{c}_i t^i)$$

with $a_i = \hat{a}_i, b_i = \hat{b}_i, c_i = \hat{c}_i$ *if* $i < k$ *and* $\hat{a}_k = \hat{b}_k = \hat{c}_k = 0$, $\hat{a}_s = \hat{b}_s = \hat{c}_s = 0$, *for all* $s \geq c$.

Corollary 1. *Given a parametrization*

$$(x(t), y(t), z(t)) \text{ with semigroup } \Gamma,$$

there exists an $\mathcal{A}$*-equivalent parametrization of the form*

$$(t^l + \Sigma_{i>l, i\notin\Gamma} a_i t^i, t^m + \Sigma_{i>m, i\notin\Gamma} b_i t^i, t^n + \Sigma_{i>n, i\notin\Gamma} c_i t^i),$$

$l < m < n$ *(*$n = \infty$ *included),* $l \nmid m$ *and* $n \notin \langle l, m \rangle$ *if* $n < \infty$.

Definition 5. *Let* $(x(t), y(t), z(t))$ *be a parametrization. A parametrization with the properties of Corollary 1 is called a weak normal form*[5] *of* $(x(t), y(t), z(t))$.

The main result of this paper is the following theorem:

Theorem 3. *Let* $(x(t), y(t), z(t))$ *be a parametrization of a simple space curve*[6] *then it is* $\mathcal{A}$*-equivalent to a parametrization in the following table:*

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[5] also called short parametrization cf. [21].

[6] The classification includes also the classification of simple plane curves (last component 0).

Characteristic $p > 2$	
Γ	Normal Form
$\langle 1 \rangle$	$(t, 0, 0)$
$\langle 2, k \rangle$	$(t^2, t^k, 0)$, k > 2 odd
$\langle 3, k, r \rangle$, $p \neq 3$ $k \cdot r \equiv 2 \bmod 3$ or $r = \infty$	$(t^3, t^k + t^l, t^r)$ $l = \infty$ or $k < l \leq 2k - 6$ and $k \cdot l \equiv 2 \bmod 3$ r $= \infty$ or $k < r < 2k - 2$ $k < p + 9$ or $2p + 9 > k \geq p + 9$ and $l < k + p$ or $r \leq k + p$.
$\langle 3, k, r \rangle, p = 3$	$(t^3, t^5, 0)$ $(t^3 + t^4, t^5, 0)$ (t^3, t^5, t^7) $(t^3 + t^4, t^5, t^7)$ (t^3, t^7, t^8) $(t^3 + t^4, t^7, t^8)$ $(t^3 + t^5, t^7, t^8)$
$\langle 4, 5 \rangle$	$(t^4, t^5, 0)$ $(t^4, t^5 + t^7, 0)$ and additionally if p =5 $(t^4, t^5 + t^6, 0)$
$\langle 4, 5, 6 \rangle$	(t^4, t^5, t^6) and additionally if p =3 $(t^4, t^5, t^6 + t^7)$
$\langle 4, 5, 7 \rangle$	(t^4, t^5, t^7) and additionally if p =5 $(t^4, t^5 + t^6, t^7)$
$\langle 4, 5, 11 \rangle$	(t^4, t^5, t^{11}) $(t^4, t^5 + t^7, t^{11})$ and additionally if p =5, $(t^4, t^5 + t^6, t^{11})$
$\langle 4, 6, k + 6, r \rangle$ $r \in \{k - 2, k + 2, k + 4, k + 8, \infty\}$ k odd $p \neq$ 3,13	$(t^4, t^6 + t^k, t^{k-2})$ $k \geq$9 $(t^4, t^6 + t^k, t^{k+2})$ if $p \nmid k + 2$. $(t^4, t^6 + t^k, t^{k+8})$ $(t^4, t^6 + t^k, t^{k+4})$ $(t^4, t^6 + t^k, 0)$ $7 \leq k \leq p - 8$ if $p \geq 17$, $7 \leq k \leq 8$ if p =11, $7 \leq k \leq 13$ if p =7, k =7 if p =5

$\langle 4,6,r\rangle$ r odd, $p\neq 3,13$	(t^4,t^6,t^r) (t^4,t^6,t^r+t^{r+2}) if $p\mid r$ $7\leq r\leq p+8$ if $p\geq$17, $7\leq r\leq 29$ if $p=$11, $7\leq r\leq 15$ if $p=$7, $7\leq r\leq 11$ if $p=$5
	if $p=$3 (t^4,t^6,t^7) (t^4,t^6+t^9,t^7)
	if $p=$13 (t^4,t^6,t^7) (t^4,t^6,t^9) (t^4,t^6+t^7,t^9)

Assume that $p\neq 3,7$.

$\langle 4,7\rangle$	$(t^4,t^7,0)$, $(t^4,t^7+t^9,0)$, $(t^4,t^7+t^{13},0)$,
$\langle 4,7,9\rangle$	(t^4,t^7,t^9), (t^4,t^7,t^9+t^{10}),
	let $p=5$ (t^4,t^7,t^9)
	let $p=$13 (t^4,t^7,t^9) (t^4,t^7,t^9+t^{10}).
$\langle 4,7,10\rangle$	(t^4,t^7,t^{10}), (t^4,t^7+t^9,t^{10}),
$\langle 4,7,13\rangle$	(t^4,t^7,t^{13}), (t^4,t^7+t^9,t^{13}),
$\langle 4,7,17\rangle$	(t^4,t^7,t^{17}), (t^4,t^7+t^9,t^{17}), (t^4,t^7+t^{13},t^{17}),

Characteristic $p = 2$	
$\langle 1\rangle$	$(t, 0, 0)$
$\langle 2, k\rangle$	$(t^2, t^k, 0), k \geq 3$ odd $(t^2 + t^m, t^k, 0)$, $0 < m < k$, k, m odd
$\langle 3, 4\rangle$	$(t^3, t^4, 0)$ $(t^3, t^4 + t^5, 0)$
$\langle 3, 4, 5\rangle$	(t^3, t^4, t^5)
$\langle 3, 5\rangle$	$(t^3, t^5, 0)$
$\langle 3, 5, 7\rangle$	(t^3, t^5, t^7)
$\langle 3, 7\rangle$	$(t^3, t^7, 0)$ $(t^3, t^7 + t^8, 0)$
$\langle 3, 7, 8\rangle$	(t^3, t^7, t^8)
$\langle 3, 7, 11\rangle$	(t^3, t^7, t^{11}) $(t^3, t^7 + t^8, t^{11})$
$\langle 3, 8, 10\rangle$	(t^3, t^8, t^{10})
$\langle 3, 8, 13\rangle$	(t^3, t^8, t^{13}) $(t^3, t^8 + t^{10}, t^{13})$
$\langle 3, 10, 11\rangle$	(t^3, t^{10}, t^{11})
$\langle 3, 10, 14\rangle$	(t^3, t^{10}, t^{14}) $(t^3, t^{10} + t^{11}, t^{14})$.

Remark 3. *The list does not include all normal forms of Gibson and Hobbs since some of them depend on the characteristic of the field. If the characteristic is tending to infinity we obtain in the limit the list of Gibson and Hobbs. Especially our methods for the classification would give the corresponding classification over algebraically closed fields of characteristic 0 with the same normal forms obtained by Gibson and Hobbs over the complex numbers.*

2 Semigroups and deformations

We start by collecting some useful properties of numerical semigroups.

Lemma 1. *Let* $\Gamma = \langle g_1, ..., g_m\rangle$ *be a semigroup given by a minimal set of generators. Then*

1. $m \leq g_1$,
2. *if* $m = g_1$ *then* $a + b - g_1 \in \Gamma$ *for* $a, b \in \Gamma, a, b \neq 0$,
3. $g_i \leq c(\Gamma) + g_1 - 1$,
4. $\delta(\Gamma) \leq c(\Gamma) \leq 2\delta(\Gamma)$.

Proof. A proof of these properties can be found in [6] (pages 234 and 235) and [19] (page 316). ∎

For the classification of parametrizations we need the following results about semigroups.

Definition 6. *Let $\Gamma = \langle a_1, ..., a_l \rangle$, $\bar{\Gamma} = \langle b_1, ..., b_s \rangle$ be semigroups given by minimal sets of generators. If $l < s$ (resp. $l > s$) then we extend the set of generators $a_1, ..., a_l$ to $a_1, ..., a_l$, $\infty, ..., \infty$ (resp. $b_1, ..., b_s$ to $b_1, ..., b_s, \infty, ..., \infty$). We define $\Gamma < \bar{\Gamma}$ if $\Gamma \neq \bar{\Gamma}$ and there exist i such that $a_j = b_j$ for $j < i$ and $a_i < b_i$.*

Example 1. $\langle g_1, g_2 \rangle > \langle g_1, g_2, g_3 \rangle$ *if* $g_3 \neq \infty$.

Lemma 2. *Let $\bar{f} = (\bar{x}(t), \bar{y}(t), \bar{z}(t))$ and $f = (x(t), y(t), z(t))$ be parametrizations with semigroup $\bar{\Gamma}$ resp. Γ. If $\Gamma \subseteq \bar{\Gamma}$ then $\bar{\Gamma} \leq \Gamma$.*

Proof. Let $\bar{\Gamma} = \langle \bar{\beta}_1, ..., \bar{\beta}_s \rangle$ resp. $\Gamma = \langle \beta_1, ..., \beta_t \rangle$ be given by their minimal system of generators. Consider $\beta_1 \in \Gamma \subseteq \bar{\Gamma}$. It implies $\beta_1 = \sum c_i \bar{\beta}_i$ with $c_i \in \mathbb{Z}$, $c_i \geq 0$. This implies $\beta_1 \geq \bar{\beta}_1$. If $\beta_1 > \bar{\beta}_1$ then $\bar{\Gamma} < \Gamma$. Assume we have found i such that $\beta_1 = \bar{\beta}_1, ..., \beta_{i-1} = \bar{\beta}_{i-1}$. Since $\beta_i \in \Gamma \subseteq \bar{\Gamma}$ and $\beta_i \notin \langle \beta_1, ..., \beta_{i-1} \rangle = \langle \bar{\beta}_1, ..., \bar{\beta}_{i-1} \rangle$. We have $\beta_i = \sum c_i \bar{\beta}_i$, $c_i \geq 0$ and $c_k \neq 0$ for some $k \geq i$. This implies that $\beta_i \geq \bar{\beta}_i$. If $\beta_i > \bar{\beta}_i$ then $\bar{\Gamma} < \Gamma$. Using induction we obtain $\bar{\Gamma} \leq \Gamma$. ∎

Lemma 3. *Assume that the characteristic $p \neq 2$. Let $(x(t), y(t), z(t))$ be a parametrization in weak normal form such that $ord_t(x(t)) = 4$, $ord_t(y(t)) = 6$ and the semigroup Γ is minimally generated by 4 elements. Then $(x(t), y(t))$ defines a plane curve with semigroup $\Gamma_0 = \langle 4, 6, k \rangle$ and $\Gamma = \langle 4, 6, s, k \rangle$ with $s \in \{k-8, k-4, k-2\}$ or $\Gamma = \langle 4, 6, k, k+2 \rangle$.*

Proof. We may assume that $x(t) = t^4$ since $p \neq 2$. If $y(t) \in K[[t^2]]$ then $\Gamma = \langle 4, 6, \mathrm{ord}_t(z(t)) \rangle$. This is a contradiction to our assumption. This implies that $(x(t), y(t))$ defines a plane curve with semigroup $\Gamma_0 = \langle 4, 6, k \rangle$ for a suitable odd[7] k. The conductor of this semigroup is $k+3$. Since the curve $(x(t), y(t), z(t))$ is in weak normal form we have $s := \mathrm{ord}_t(z(t)) \notin \Gamma_0$. If $s \leq k-10$ or $s = k-6$ then $k-6 \in \langle 4, 6, s \rangle$ this is a contradiction to the assumption that $(x(t), y(t), z(t))$ is in weak normal form. The remaining possibilities for s are $k-8, k-4, k-2$ and $k+2$. ∎

Proposition 1. *Let Γ be the semigroup of the parametrization $(x(t), y(t), z(t))$ and assume that $\Gamma \leq \langle 4, 7 \rangle$. If the parametrization has multiplicity 4 assume additionally that the characteristic $p > 2$. Let $(X(t), Y(t), Z(t)) \in A[[t]]^3$ be a deformation and for $\mathfrak{m} \in Specmax(A)$ let $\Gamma_{\mathfrak{m}}$ be the corresponding semigroup. Then there exists an open neighborhood U of o such that for all $\mathfrak{m} \in U$ we have $\Gamma_{\mathfrak{m}} \leq \Gamma$.*

[7] k is minimal such that t^{k-6} occurs as a monomial in $y(t)$.

Proof. We may assume that $(x(t), y(t), z(t))$ is in weak normal form. If $z(t) = 0$ then the result follows from the corresponding proposition for plane curves ([17], Lemma 14). We choose an open neighborhood U of o such that $ord_t X(u,t) \leq ord_t x(t)$, $ord_t Y(u,t) \leq ord_t y(t)$ and $ord_t Z(u,t) \leq ord_t z(t)$. If $z(t) \neq 0$ and Γ has as minimal generators 3 elements then $\{x(t), y(t), z(t)\}$ form a sagbi basis of the algebra $\mathbb{K}[[x(t), y(t), z(t)]]$ ([12], Proposition 3.1.), $\Gamma = \langle ord_t x(t), ord_t y(t), ord_t z(t)\rangle$ and we are in one of the following cases

1. $\Gamma = \langle 3, k, s\rangle$
2. $\Gamma = \langle 4, 5, s\rangle$
3. $\Gamma = \langle 4, 6, s\rangle$
4. $\Gamma = \langle 4, 7, s\rangle$

If the deformation decreases the order we have $\Gamma_u < \Gamma$. If the order is constant then we may assume that $X(u,t) = x(t) \mod t^{ord_t x(t)+1}$ and $ord_t Y(u,t) > ord_t X(u,t)$ and $ord_t Z(u,t) > ord_t X(u,t)$ and both orders are not divisible by $ord_t X(u,t)$. If one of the two orders is smaller than $ord_t y(t)$ then $\Gamma_u < \Gamma$. Now we may assume additionally that $Y(u,t) = y(t) \mod t^{ord_t y(t)+1}$ and $ord_t Z(u,t) > ord_t Y(u,t)$ and $ord_t Z(u,t) \notin \langle ord_t x(t), ord_t y(t)\rangle$. If $ord_t Z(u,t) < ord_t z(t)$ we have $\Gamma_u < \Gamma$. If $ord_t Z(u,t) = ord_t z(t)$ it is possible that the deformation is no more a sagbi basis. But this would enlarge the set of generators $ord_t x(t), ord_t y(t), ord_t z(t)$ and again by definition we have $\Gamma_u \leq \Gamma$.
Finally we have to consider the case that Γ is generated by 4 elements. We may assume that the parametrization is in weak normal form. We apply Lemma 3 and obtain that $(x(t), y(t))$ defines a plane curve with semigroup $\Gamma_0 = \langle 4, 6, k\rangle$. Again we apply the corresponding Proposition for the plane curve to $(x(t), y(t))$. Let $\Gamma_{0,u}$ be the semigroup corresponding to $(X(u,t), Y(u,t))$. Then $\Gamma_{0,u} \leq \Gamma_0$. If $\Gamma_{0,u} < \Gamma_0$, we are done. If $\Gamma_{0,u} = \Gamma_0 = \langle 4, 6, k\rangle$. We obtain $\Gamma_u \leq \Gamma$ since $ord_t Z(u,t) \leq ord_t z(t)$. ∎

Lemma 4. *Given a parametrization $f = (x(t), y(t), z(t))$ with the semigroup $\langle 4, 6, k, s\rangle$, $k > 6$ odd ($s = \infty$ included), and $F(u,t)$ a deformation of f with multiplicity 3 and semigroup Γ_u for $u \neq 0$. Then $\Gamma_u \leq \langle 3, 7\rangle$.*

Proof. Let $f = (x(t), y(t), z(t))$ be a parametrization with semigroup $\langle 4, 6, k, s\rangle$. We may assume that
$$f = (t^4, t^6 + t^{k-6}, t^s) \text{ or } f = (t^4, t^6, t^s).$$
We give the proof for the first case. Consider any deformation for this parametrization

$$F = (\sum_{i\geq 3} \alpha_i t^i, \sum_{i\geq 3} \beta_i t^i, \sum_{i\geq 3} \gamma_i t^i),$$

by definition of deformation we have the following conditions $\alpha_i(0) = 0$, $i \neq 4$, $\alpha_4(0) = 1$, $\beta_i(0) = 0$, $i \neq 6, k-6$, $\beta_6(0) = 1$, $\beta_{k-6}(0) = 1$, $\gamma_i(0) = 0$, $i \neq s$, $\gamma_s(0) = 1$.
If $\alpha_3 = 0$, we obtain $\Gamma_u = \langle 3, 4\rangle$ or $\langle 3, 4, 5\rangle$.
Now assume $\alpha_3 \neq 0$, consider $\beta_3 \sum_{i\geq 3} \alpha_i t^i - \alpha_3 \sum_{i\geq 3} \beta_i t^i$
$= (\beta_3\alpha_4 - \alpha_3\beta_4)t^4 + (\beta_3\alpha_5 - \alpha_3\beta_5)t^5 + (\beta_3\alpha_6 - \alpha_3\beta_6)t^6 + (\beta_3\alpha_7 - \alpha_3\beta_7)t^7 + \ldots.$
If $(\beta_3\alpha_4 - \alpha_3\beta_4) \neq 0$, we obtain $\Gamma_u = \langle 3, 4\rangle$ or $\langle 3, 4, 5\rangle$.

If $(\beta_3\alpha_4 - \alpha_3\beta_4) = 0$, and $(\beta_3\alpha_5 - \alpha_3\beta_5) \neq 0$ then $3, 5 \in \Gamma$. This implies that $\Gamma_u \leq \langle 3, 5\rangle$.
If $(\beta_3\alpha_4 - \alpha_3\beta_4) = 0$, and $(\beta_3\alpha_5 - \alpha_3\beta_5) = 0$, and $(\beta_3\alpha_6 - \alpha_3\beta_6) \neq 0$.
then $\alpha_3^2(\beta_3\sum_{i\geq 3}\alpha_i t^i - \alpha_3\sum_{i\geq 3}\beta_i t^i) - (\beta_3\alpha_6 - \alpha_3\beta_6)(\sum_{i\geq 3}\alpha_i t^i)^2$
$= [\alpha_3^2(\beta_3\alpha_7 - \alpha_3\beta_7) - 2(\beta_3\alpha_6 - \alpha_3\beta_6)\alpha_3\alpha_4]t^7 + ...$
$= \alpha_3^2[(\beta_3\alpha_7 - \alpha_3\beta_7) - 2(\beta_4\alpha_6 - \alpha_4\beta_6)]t^7 + ...$
Since $\beta_6(0)\alpha_4(0) = 1$, we obtain that the coefficient of t^7 is different from zero. If $\beta_3\alpha_6 - \alpha_3\beta_6 = 0$, we obtain multiplying with α_4 and using $\beta_3\alpha_4 = \alpha_3\beta_4$ that $\alpha_3(\beta_4\alpha_6 - \alpha_4\beta_6) = 0$. But $\alpha_3 \neq 0$ and $\alpha_4\beta_6$ is a unit. This implies $\beta_4\alpha_6 - \alpha_4\beta_6 \neq 0$. This is a contradiction. This implies that $7 \in \Gamma_u$ and therefore $\Gamma_u \leq \langle 3, 7\rangle$ in this case. □

3 Minimal non-simple curves

The idea is to prove the Theorem 3 for almost all characteristics is the following: We prove for a given parameterized space curve singularities $f = (x(t), y(t), z(t))$ with $ord_t x(t) = 5$ or $ord_t x(t) = 4$ and $ord_t y(t) \geq 9$ and $ord_t z(t) \geq 10$ that f is not simple. For the other cases, we give normal forms not depending on parameters. The property $ord_t x(t) \leq 4$, $ord_t y(t) \leq 7$ is kept under deformation.

Lemma 5. *The following parametrizations are not simple:*

1. $(t^5, t^6, 0)$ *and* (t^5, t^6, t^7).
2. $(t^4, t^9, 0)$ *and* (t^4, t^9, t^{10}).

Proof. We will prove that

$$(t^5, t^6 + t^8 + at^9, t^7) \sim (t^5, t^6 + t^8 + bt^9, t^7)$$

implies $a = b$ or $a = -b$.
This will prove the lemma since for different a modulo sign, the parametrizations are in different classes. This gives infinitely many different classes since the field is algebraically closed.
The case $(t^4, t^9 + t^{11}, t^{10} + at^{11})$ can be treated similarly.
Set

$$\psi(t) = a_1 t + \Sigma_{i>1} a_i t^i$$

with $a_1 \neq 0$, and let

$$\varphi(x, y, z) = (\varphi_1, \varphi_2, \varphi_3)\ ;\ \varphi_j = \Sigma {b^j}_{k,l,m} x^k y^l z^m$$

be an automorphism of $\mathbb{K}[[x, y, z]]$. Assume that

$$\psi^5 = \varphi_1(t^5, t^6 + t^8 + at^9, t^7)$$

$$\psi^6 + \psi^8 + b\psi^9 = \varphi_2(t^5, t^6 + t^8 + at^9, t^7)$$

$$\psi^7 = \varphi_3(t^5, t^6 + t^8 + at^9, t^7).$$

This is the condition for $(t^5, t^6 + t^8 + at^9, t^7) \sim (t^5, t^6 + t^8 + bt^9, t^7)$ according to the definition of $\mathcal{A}$-equivalence. Writing down this explicitly we see if $p \neq 5$ and $p \neq 7$ then $a_2 = ... = a_5 = 0$, $a_1^2 = 1$ and

$$\psi^5 = \varphi_1(t^5, t^6 + t^8 + at^9, t^7) \bmod t^{10}$$

$$\psi^6 + \psi^8 + b\psi^9 = \varphi_2(t^5, t^6 + t^8 + at^9, t^7) + (b - a)t^9 \bmod t^{10} \ (*)$$

$$\psi^7 = \varphi_3(t^5, t^6 + t^8 + at^9, t^7) \bmod t^{10}$$

This implies $a = a_1^2 b$.
The computation can be done using Singular. The computations of all possible cases took less than a minute per case. The corresponding code for $(t^5, t^6 + t^8 + at^9, t^7)$ is as follows: We define the ring

$$R = \mathbb{Q}(a_1, \ldots, a_{10}, u_1, \ldots, u_{10}, v_1, \ldots, v_{10}, w_1, \ldots, w_{10}, a, b)[[x, y, z, t]]$$

and the map ψ as above. The map ϕ is given by

$$\phi(x) = \phi_1 = Hx = u_1x + u_2y + u_3z + \ldots,\ \phi(y) = \phi_2 = Hy = v_1x + v_2y + v_3z + \ldots, \text{ and}$$
$$\phi(z) = \phi_3 = Hz = w_1x + w_2y + w_3z + \ldots.$$

The relations (*) are given by the polynomials W, X, Y. Their coefficients are collected in the ideal I. We compute a Gröbner basis of I with respect to the lexicographical ordering to obtain the relations between a, b, a_1.

```
ring R=(0,a1,a2,a3,a4,a5,a6,a7,a8,a9,a10,u1,u2,u3,u4,u5,u6,u7,u8,u9,
u10,v1,v2,v3,v4,v5,v6,v7,v8,v9,v10,w1,w2,w3,w4,w5,w6,w7,w8,w9,w10,a,b)
,(x,y,z,t) ,ds;
poly psi=a1*t+a2*t2+a3*t3+a4*t4+a5*t5+a6*t6+a7*t7+a8*t8+a9*t9+a10*t10;
poly Hx=u1*x+u2*y+u3*z+u4*x2+u5*xy+u6*xz+u7*y2+u8*yz+u9*z2+u10*x3;
poly Hy=v1*x+v2*y+v3*z+v4*x2+v5*xy+v6*xz+v7*y2+v8*yz+v9*z2+v10*x3;
poly Hz=w1*x+w2*y+w3*z+w4*x2+w5*xy+w6*xz+w7*y2+w8*yz+w9*z2+w10*x3;
poly W=jet(psi^5-subst(Hx,x,t5,y,t^6+t8+b*t9,z,t7) ,9);
poly X=jet(psi^6+phi^8+a*phi^9-subst(Hy,x,t5,y,t^6+t8+b*t9,z,t7) ,9);
poly Y=jet(psi^7-subst(Hz,x,t5,y,t^6+t8+b*t9,z,t7) ,9);
matrix M1=coef(W,t);matrix M2=coef(X,t);matrix M3=coef(Y,t);
ideal I;int ii;
for(ii=1;ii<=ncols(M1);ii++){I[size(I)+1]=M1[2,ii];}
for(ii=1;ii<=ncols(M2);ii++){I[size(I)+1]=M2[2,ii];}
for(ii=1;ii<=ncols(M3);ii++){I[size(I)+1]=M3[2,ii];}
ring S=integer,(a2,a3,a4,a5,a6,a7,a8,a9,a10,u1,u2,u3,u4,u5,u6,u7,u8,u9,u10,
v1,v2,v3,v4,v5,v6,v7,v8,v9,v10,w1,w2,w3,w4,w5,w6,w7,w8,w9,w10,
a,b,a1),lp;
ideal I=imap(R,I);std(I);
//==The first 2 polynomials of the standard basis of I are
_[1]=7*a1^10-7*a1^8
_[2]=35*a*a1^8-35*b*a1^9
```

Since a_1 is not zero and the characteristic is different from 5 and 7 we obtain $a_1^2 = 1$ and $a = a_1 b$. This proves our assertion.
It remains to discuss the cases $p = 5$ and $p = 7$. Let us start with $p = 5$. We will show that the family $(t^5 + t^8 + at^9, t^6, t^7)$ contains infinitely many different equivalence classes. The computation can be done using Singular as follows:

```
int ch=5;
ring R=(ch,a1,a2,a3,a4,a5,a6,a7,a8,a9,a10,u1,u2,u3,u4,u5,u6,u7,u8,u9,
u10,v1,v2,v3,v4,v5,v6,v7,v8,v9,v10,w1,w2,w3,w4,w5,w6,w7,w8,w9,w10,a,b)
,(x,y,z,t) ,ds;
poly psi=a1*t+a2*t2+a3*t3+a4*t4+a5*t5+a6*t6+a7*t7+a8*t8+a9*t9+a10*t10;
poly Hx=u1*x+u2*y+u3*z+u4*x2+u5*xy+u6*xz+u7*y2+u8*yz+u9*z2+u10*x3;
poly Hy=v1*x+v2*y+v3*z+v4*x2+v5*xy+v6*xz+v7*y2+v8*yz+v9*z2+v10*x3;
poly Hz=w1*x+w2*y+w3*z+w4*x2+w5*xy+w6*xz+w7*y2+w8*yz+w9*z2+w10*x3;
poly W=jet(psi^5+psi^8+a*psi^9-subst(Hx,x,t5+t8+b*t9,y,t^6,z,t7) ,9);
poly X=jet(psi^6-subst(Hy,x,t5+t8+b*t9,y,t^6,z,t7) ,9);
poly Y=jet(psi^7-subst(Hz,x,t5+t8+b*t9,y,t^6,z,t7) ,9);
matrix M1=coef(W,t);matrix M2=coef(X,t);matrix M3=coef(Y,t);
ideal I;int ii;
for(ii=1;ii<=ncols(M1);ii++){I[size(I)+1]=M1[2,ii];}
for(ii=1;ii<=ncols(M2);ii++){I[size(I)+1]=M2[2,ii];}
for(ii=1;ii<=ncols(M3);ii++){I[size(I)+1]=M3[2,ii];}
ring S=ch,(a2,a3,a4,a5,a6,a7,a8,a9,a10,u1,u2,u3,u4,u5,u6,u7,u8,u9,u10,
v1,v2,v3,v4,v5,v6,v7,v8,v9,v10,w1,w2,w3,w4,w5,w6,w7,w8,w9,w10,
a,b,a1),lp;
ideal I=imap(R,I);std(I);
//==The first 2 polynomials in the standard basis are
_[1]=a1^8-a1^5
_[2]=a*a1^5-b*a1^7
```

Since a_1 is not zero we obtain $a_1^3 = 1$ and $a = a_1^2 b$.
Now we consider the case $p = 7$. We will show that the family $(t^5, t^6, t^7 + t^8 + at^9)$ contains infinitely many different equivalence classes. The computation can be done using Singular as follows:
Since the code is the same as above and only the definition of the integer ch and the polynomials W, X, Y change we will only give those data and the result.

```
int ch=7;
poly W=jet(psi^5-subst(Hx,x,t5,y,t^6,z,t7+t8+b*t9) ,9);
poly X=jet(psi^6-subst(Hy,x,t5,y,t^6,z,t7+t8+b*t9) ,9);
poly Y=jet(psi^7+psi^8+a*psi^9-subst(Hz,x,t5,y,t^6,z,t7+t8+b*t9) ,9);
//==The first 2 polynomials in the standard basis are
_[1]=a1^8-a1^7
_[2]=a^2*a1^7+a*b*a1^9-a*b*a1^7-2*a*a1^9-b^2*a1^7+2*b*a1^7
```

Since a_1 is not zero we obtain $a_1 = 1$ and $a = b$ or $a = 2 - b$. □

Lemma 6. *Let $\mathbb{K}$ be a field of characteristic 3 then the parametrizations with the semigroup $\langle 3, 7, 11\rangle$ are not simple.*

Proof. To prove this we just need to show that

$$(t^3 + \sum\nolimits_{i\geq 4} \alpha_i t^i, t^7 + \sum\nolimits_{i\geq 8} \gamma_i t^i, t^{11}) \sim (t^3 + \sum\nolimits_{i\geq 4} \beta_i t^i, t^7 + \sum\nolimits_{i\geq 8} \delta_i t^i, t^{11}).$$

with $\alpha_4 = \beta_4 = 1$ implies[8] $\alpha_5 = \beta_5$.
The computation can be done using SINGULAR:
Since the code is the same as above and only the definition of the integer ch and the polynomials W, X, Y change we will only give those data and the result.

```
int ch=3;
poly W=jet(psi^3+psi^4+a*psi^5-subst(Hx,x,t3+t4+b*t5,y,t7,z,t11) ,11);
poly X=jet(psi^7-subst(Hy,x,t3+t4+b*t5,y,t7,z,t11) ,11);
poly Y=jet(psi^11-subst(Hz,x,t3+t4+b*t5,y,t7,z,t11) ,11);
//==The first 2 polynomials in the standard basis are
_[1]=a1^4-a1^3
_[2]=a*a1^3-b*a1^3
```

Since a_1 is not zero we obtain $a_1 = 1$ and $a = b$. ∎

Corollary 2. *Let $\mathbb{K}$ be a field of characteristic* 3 *then the parametrizations with the semigroup $\langle 4, 7, 9\rangle$ and $\langle 4, 6, k, s\rangle$, $k \geq 13$, $s \geq 9$ (the case $k = \infty$ included), are not simple.*

Proof. These parametrizations deform into parametrizations with the semigroup $\langle 3, 7, 11\rangle$ which are not simple.
To see this let us consider the case $s = 9$. Using Corollary 1 (Zariski's Theorem) we obtain that the corresponding parametrization is equivalent to $(t^4, t^6 + at^7 + bt^{11}, t^9 + ct^{11})$. As a deformation we consider $(\alpha t^3 + t^4, t^6 + at^7 + bt^{11}, t^9 + ct^{11})$. For $\alpha \neq 0$ and we obtain that this parametrization is equivalent to

$$(\alpha t^3 + t^4, (\alpha^2 a - 2\alpha)t^7 - t^8 + \alpha^2 bt^{11}, \alpha^3 ct^{11} - t^{12})$$

having semigroup $\langle 3, 7, 11\rangle$.
In case of the semigroup $\langle 4, 7, 9\rangle$, we consider the parametrization $(t^4, t^7 + at^{10}, t^9 + bt^{10})$ obtained from the generic one using Corollary 1 (Zariski's Theorem). As a deformation we consider $(\alpha t^3 + t^4, t^7 + at^{10}, t^9 + bt^{10}) \sim (\alpha t^3 + t^4, t^7, 0)$ having semigroup $\langle 3, 7\rangle > \langle 3, 7, 11\rangle$.
∎

Lemma 7. *Let $\mathbb{K}$ be a field of characteristic 5 then the parametrizations with the semigroup $\langle 4, 7, 10\rangle$ are not simple.*

Proof. We will show that the family $(t^4, t^7 + t^9, 1^{10} + at^{13})$ contains infinitely many different equivalence classes. The computation can be done using SINGULAR as follows:
Since the code is the same as above and only the definition of the integer ch and the polynomials W, X, Y change we will only give those data and the result.

[8] If α_4 and β_4 are different from 0 we can always obtain after applying a suitable automorphism of $\mathbb{K}[[t]]$ that they are 1.

```
int ch=5;
poly W=jet(psi^4-subst(Hx,x,t4,y,t7+t9,z,t10+b*t13) ,13);
poly X=jet(psi^7+psi^9-subst(Hy,x,t4,y,t7+t9,z,t10+b*t13) ,13);
poly Y=jet(psi^10+a*psi^13-subst(Hz,x,t4,y,t7+t9,z,t10+b*t13) ,13);
```

We obtain $a_1^2 = 1$ and $a = a_1 b$. □

Lemma 8. *Let $\mathbb{K}$ be a field of characteristic 7 then the parametrizations with the semigroup $\langle 4, 7, 13\rangle$ are not simple.*

Proof. We will show that the family $(t^4, t^7 + t^9 + at^{10}, t^{13})$ contains infinitely many different equivalence classes. The computation can be done using SINGULAR as follows:
Since the code is the same as above and only the definition of the integer *ch* and the polynomials W, X, Y change we will only give those data and the result.

```
int ch=7;
poly W=jet(psi^4-subst(Hx,x,t4,y,t7+t9+b*t10,z,t13) ,13);
poly X=jet(psi^7+psi^9+a*psi^10-subst(Hy,x,t4,y,t7+t9+b*t10,z,t13) ,13);
poly Y=jet(psi^11-subst(Hz,x,t4,y,t7+t9+b*t10,z,t13) ,13);
//==The first 2 polynomials in the standard basis are
_[1]=a1^9-a1^7
_[2]=a*a1^7-b*a1^8
```

Since a_1 is different from zero we obtain $a_1^2 = 1$ and $a = a_1 b$. □

Lemma 9. *Let $\mathbb{K}$ be a field of characteristic 13 then a parametrization with the semigroup $\langle 4, 6, 11, 13\rangle$ is not simple.*

Proof. We will show that the family $(t^4, t^6 + t^7 + at^9, t^{11})$ whose semigroup is $\langle 4, 6, 11, 13\rangle$ contains infinitely many different equivalence classes. The computation can be done using SINGULAR as follows:
Since the code is the same as above and only the definition of the integer *ch* and the polynomials W, X, Y change we will only give those data and the result.

```
int ch=13;
poly W=jet(psi^4-subst(Hx,x,t4,y,t6+t7+b*t9,z,t11) ,10);
poly X=jet(psi^6+psi^7+a*psi^9-subst(Hy,x,t4,y,t6+t7+b*t9,z,t11) ,10);
poly Y=jet(psi^11-subst(Hz,x,t4,y,t6+t7+b*t9,z,t11) ,10);
//==The first 2 polynomials in the standard basis are
_[1]=a1^7-a1^6
_[2]=a*a1^6-b*a1^7
```

Since a_1 is different from zero we obtain $a_1 = 1$ and $a = b$. □

Corollary 3. *Let $\mathbb{K}$ be a field of characteristic* 13 *then a parametrization with the semigroup $\langle 4, 7, 10\rangle$ are not simple.*

Proof. The parametrization (t^4, t^7, t^{10}) can be deformed to $(t^4, \alpha t^6 + t^7, t^{10}) \sim (t^4, \alpha t^6 + t^7, t^{11})$ with semigroup $\langle 4, 6, 11, 13 \rangle$ which is not simple. □

Proposition 2. *Let $\mathbb{K}$ be a field of characteristic 2 Let $f = (x(t), y(t), z(t))$ be a space curve singularity with the semigroup $\langle 4, 5, 6 \rangle$, $\langle 3, 10, 17 \rangle$ or $\langle 3, 8 \rangle$. Then f is not simple.*

Proof. We will first show that the family $(t^4 + at^7, t^5, t^6 + t^7)$ contains infinitely many different equivalence classes. The computation can be done using Singular as follows:
Since the code is the same as above and only the definition of the integer *ch* and the polynomials W, X, Y change we will only give those data and the result.

```
int ch=2;
poly W=jet(psi^4+a*psi^7-subst(Hx,x,t4+b*t7,y,t5,z,t6+t7) ,7);
poly X=jet(psi^5-subst(Hy,x,t4+b*t7,y,t5,z,t6+t7) ,7);
poly Y=jet(psi^6+psi^7-subst(Hz,x,t4+b*t7,y,t5,z,t6+t7) ,7);
//==The first 2 polynomials in the standard basis are
_[1]=a1^7+a1^6
_[2]=a*a1^5+b*a1^4
```

Since a_1 is different from zero and $p = 2$ we obtain $a_1 = 1$ and $a = b$.
Now we will show that the family $(t^3, t^{10} + t^{11} + at^{14}, t^{17})$ contains infinitely many different equivalence classes. The computation can be done using Singular as follows:
Since the code is the same as above and only the definition of the integer *ch* and the polynomials W, X, Y change we will only give those data and the result.

```
int ch=2;
poly W=jet(psi^3-subst(Hx,x,t3,y,t10+t11+b*t14,z,t17) ,14);
poly X
=jet(psi^10+ psi^11+a*psi^14-subst(Hy,x,t3,y,t10+t11+b*t14,z,t17) ,14);
poly Y=jet(psi^17-subst(Hz,x,t3,y,t10+t11+b*t14,z,t17) ,14);
//==The first 2 polynomials in the standard basis are
_[1]=a1^11+a1^10
_[2]=a^3*a1^10+a^2*b*a1^10+a*b^2*a1^10+b^3*a1^10
```

Since a_1 is different from zero and $p = 2$ we obtain $a^3 a_1^{10} + a^2 b a_1^{10} + a b^2 a_1^{10} + b^3 a_1^{10} = a_1^{10}(a + b)^3$ and therefore $a_1 = 1$ and $a = b$.
The case of the semigroup $\langle 3, 8 \rangle$ is proved in [17]. □

4 Curves of multiplicity 2

In this section we assume that the characteristic $p > 2$.

Proposition 3. *Let $(x(t), y(t), z(t))$ be a parametrized space curve singularity and ord $x(t) = 2$. Then for a suitable odd k, $(x(t), y(t), z(t)) \sim (t^2, t^k, 0)$.*

Proof. Since the characteristic $p > 2$, we may assume that $x(t) = t^2$. If $y(t) \in \mathbb{K}[[t^2]]$ then $(x(t), y(t), z(t))$ is equivalent to $(t^2, z(t), 0)$.
If $y(t) \notin \mathbb{K}[[t^2]]$. We may assume $y = \sum_{i \geq k} b_i t^i$, k odd, $b_k \neq 0$. We obtain

$$(t^2, y(t), z(t)) \sim (t^2, \sum_{i \geq k} b_i t^i, \sum_{i > k} c_i t^i).$$

Since the conductor of the semigroup is equal to $k-1$, we obtain using Zariski's Theorem (Corollary 1) that $(t^2, \sum_{i \geq k} b_i t^i, \sum_{i > k} c_i t^i) \sim (t^2, t^k, 0)$. ∎

5 Curves of multiplicity 3

In this section we assume that the characteristic $p > 2$. First we recall the results of Lemma 4,5,6 and 7 of [17] and join them to the following Proposition.

Proposition 4. *Consider the plane curve* $f = (t^3, t^k + t^l + \sum_{i > l} a_i t^i)$ *with* [9] $k < l$ *and* $k \cdot l \equiv 2$ *mod 3.*

1. *If* $l \geq 2k - 2$ *then* $f \sim (t^3, t^k)$.
2. *If* $l \geq 2k - 8$ *and* $p \neq 3$ *then* $f \sim (t^3, t^k + t^l)$.
3. *If* $p \nmid l - k$ *then* $f \sim (t^3, t^k + t^l)$.
4. *If* $p | l - k$ *and* $l \leq 2k - 9$ *then* f *is not simple*[10].

Proposition 5. *Let* $(x(t), y(t), z(t))$ *be a parametrization of a simple space curve singularity of multiplicity 3.*
If $p \neq 3$ *then* $(x(t), y(t), z(t)) \sim (t^3, t^k + t^l, t^r)$ *and*

1. $3 \nmid k$
2. $l = \infty$ *or* $k < l \leq 2k - 6$ *and* $k \cdot l \equiv 2$ *mod* 3.
3. $r = \infty$ *or* $k < r < 2k - 2$ *and* $k \cdot r \equiv 2$ *mod* 3.
4. $k < p + 9$ *or* $2p + 9 > k \geq p + 9$ *and* $l < k + p$ *or* $r \leq k + p$.
5. $\Gamma = \langle 3, k, r \rangle$ *with the conductor*[11] *min*$\{r - 2, 2k - 2\}$.

If $p = 3$ *then* $(x(t), y(t), z(t))$ *is equivalent to one of the following parametrizations:*

6. $(t^3, t^5, 0)$
7. $(t^3 + t^4, t^5, 0)$
8. (t^3, t^5, t^7)

[9] Note that $k \cdot l \equiv 2$ mod 3 and $2k - 9 < l < 2k - 2$ implies $l = 2a - 3$ or $l = 2a - 6$.
[10] Note that $p | l - k$ and $l \leq 2k - 9$ implies $k \geq p + 9$. Especially the curve $(t^3, t^{p+9} + t^{2p+9})$ is not simple.
[11] If $r < \infty$ then the conductor is $r - 2$.

9. (t^3+t^4, t^5, t^7)
10. (t^3, t^7, t^8)
11. (t^3+t^4, t^7, t^8)
12. (t^3+t^5, t^7, t^8).

Proof. If $p=3$, then the simple plane parametrizations with multiplicity 3 are equivalent to (t^3,t^5) or (t^3+t^4,t^5). Since the conductor of $\langle 3,5\rangle$ is 8 we obtain the curves (6), (7), (8) or (9).
Lemma 6 implies that the parametrization (t^3,t^7,t^{11}) is not simple. It remains to prove that parametrizations with semigroup $\langle 3,7,8\rangle$ are simple. Corollary 1 (Zariski's Theorem) implies that such a parametrization is equivalent to $(t^3+at^4+bt^5,t^7,t^8)$. If $a=0$ (respectively $b=0$ we obtain using the $\mathbb{K}^*-$ action (t^3+t^5,t^7,t^8) respectively (t^3+t^4,t^7,t^8). If $a=b=0$, we obtain (t^3,t^7,t^8).
If $a\neq 0$, we use the $\mathbb{K}[[t]]-$ automorphism defined by $t\to t-\frac{b}{a}t^2$ and the $\mathbb{K}^*$- action to obtain (t^3+t^4,t^7,t^8). Since in a deformation of a parametrization with semigroup $\langle 3,7,8\rangle$ we may only have the semigroups $\Gamma\leq\langle 3,7,8\rangle$, we obtain the cases $(6)-(12)$ or the curves of multiplicity 2. This implies that parametrizations with semigroup $\langle 3,7,8\rangle$ are simple.
Now assume that $p>3$ and our parametrization is in weak normal form, i.e. $x(t)=t^3$, $y(t)=t^k+\sum_{i>k,i\notin\Gamma}a_it^i$, $z(t)=0$ or $z(t)=t^r+\sum_{i>r,i\notin\Gamma}b_it^i$ and $3\nmid k$, $r\notin\langle 3,k\rangle$.
We apply Proposition 4 and obtain that the plane curve $(x(t),y(t))$ is simple if and only if

$$(x(t),y(t))\sim(t^3,t^k+t^l)$$

with the following properties:

1. $3\nmid k$
2. $l=\infty$ or $k<l\leq 2k-6$ and $k\cdot l\equiv 2\ mod\ 3$.
3. $k<p+9$ or $2p+9>k\geq p+9$ and $l<k+p$ or $r\leq k+p$.

Now assume that $z(t)\neq 0$. Since the conductor of the semigroup $\langle 3,k\rangle$ is $2k-2$, we know that $k<r<2k-2$ and $k\cdot r\equiv 2\ mod\ 3$. The conductor of the semigroup $\langle 3,k,r\rangle$ is $r-2$. This implies that

$$(x(t),y(t),z(t))\sim(t^3,t^k+t^l,t^r).$$

If $l\in\langle 3,k,r\rangle$ then

$$(x(t),y(t),z(t))\sim(t^3,t^k,t^r).$$

(t^3,t^k+t^l,t^r) is simple if $k<p+9$ or $2p+9>k\geq p+9$ and $l<p+k$ since the plane curve (t^3,t^k+t^l) is simple and r can only decrease[12] in a deformation.
If $2p+9>k\geq p+9$ and $l\geq p+k$ but $r\leq k+p$ we can add a suitable multiple of t^r to t^k+t^l to obtain a simple parametrization. ∎

[12]In a deformation the term t^r survives.

6 Curves of multiplicity 4

In this section we assume that the characteristic $p > 2$.

Proposition 6. *Assume that the characteristic $p > 3$ and $p \neq 13$. Let $(x(t), y(t), z(t))$ be a parametrized simple space curve singularity of multiplicity 4 with the semigroup Γ. Assume $5 \notin \Gamma$ and $6 \in \Gamma$. Then $(x(t), y(t), z(t))$ is equivalent to one of the following parametrization. Let k be odd and $7 \leq k \leq p-8$ if $p \geq 17$, $7 \leq k \leq 8$ if $p = 11$, $7 \leq k \leq 13$ if $p = 7$, $k = 7$ if $p = 5$.*

1. $(t^4, t^6 + t^k, t^{k-2})$, $k \geq 9$.
2. $(t^4, t^6 + t^k, t^{k+2})$ *if* $p \nmid k+2$.
3. $(t^4, t^6 + t^k, t^{k+4})$
4. $(t^4, t^6 + t^k, t^{k+8})$
5. $(t^4, t^6 + t^k, 0)$

Let r be odd and $7 \leq r \leq p+8$ if $p \geq 17$, $7 \leq r \leq 29$ if $p = 11$, $7 \leq r \leq 15$ if $p = 7$ and $7 \leq r \leq 11$ if $p = 5$.

6. (t^4, t^6, t^r)
7. $(t^4, t^6, t^r + t^{r+2})$ *if* $p|r$.

Proof. We may assume that the parametrization is in weak normal form, i.e. $x(t) = t^4$, $y(t) = t^6 + \sum_{i>6, i\notin\Gamma} a_i t^i$ and $z(t) = 0$ or $z(t) = t^r + \sum_{i>r, i\notin\Gamma} b_i t^i$, $r > 6$ and odd.
If $y(t) \in \mathbb{K}[[t^2]]$ then the weak normal form implies $y(t) = t^6$. In this case the conductor of the semigroup of $(x(t), y(t), z(t))$ is $r+3$ and we have as weak normal form $(t^4, t^6, t^r + b_{r+2}t^{r+2})$.
If $p \nmid r$ then $(t^4, t^6, t^r + b_{r+2}t^{r+2}) \sim (t^4, t^6, t^r)$. If $p|r$ and $b_{r+2} \neq 0$ then $(t^4, t^6, t^r + b_{r+2}t^{r+2}) \sim (t^4, t^6, t^r + t^{r+2})$. This are the cases (6) and (7) of the proposition.
Now assume that $y(t) \notin \mathbb{K}[[t^2]]$. Then we apply the plane curve classification (cf.[17]) to $(x(t), y(t))$ and obtain

$$(x(t), y(t)) \sim (t^4, t^6 + t^k) \text{ , } k \geq 7, \text{ odd.}$$

This parametrization is simple if $p \neq 13$, $k \leq p-8$ if $p \geq 17$, $k \leq 25$, if $p = 11$, $k \leq 13$, if $p = 7$ and $k = 7$, if $p = 3$ or $p = 5$. If $z(t) = 0$, we obtain (5).
If $z(t) \neq 0$ then $r \notin \langle 4, 6, k+6\rangle$. Since $k \notin \Gamma$, we have $r > k$ or $r = k-2$ since the semigroup $\langle 4, 6, r\rangle$ has $r+3$ as conductor. On the other hand $r < k+9$ since $k+9$ is the conductor of $\langle 4, 6, k+6\rangle$. This implies $r \in \{k-2, k+2, k+4, k+8\}$ and the conductor of Γ is $k+5$ if $r = k+8$ or $k+2$, $k+3$ if $r = k+4$ and $k+1$ if $r = k-2$. We obtain the cases (1) to (4) of the proposition and it remains to prove that they are simple.
Let $(X(u,t), Y(u,t), Z(u,t))$ be a deformation of $(t^4, t^6 + \alpha t^k, \beta t^r + \gamma t^{r+2})$, $\alpha, \beta, \gamma \in \{0, 1\}$. If the deformation has constant multiplicity 4 then k and r can only decrease, i.e. for a given fixed u

$$(X(u,t), Y(u,t), Z(u,t)) \sim (t^4, t^6 + \bar{\alpha} t^{\bar{k}}, \bar{\beta} t^{\bar{r}} + \bar{\gamma} t^{\bar{r}+2})$$

with $\bar{k} \leq k$ and $\bar{r} \leq r$. This is an immediate consequence of proposition 3.2.3. This implies that there are only finitely many different equivalence classes in this deformation.
If $(X(u,t), Y(u,t), Z(u,t))$ is a deformation with multiplicity 3. Lemma 4 implies that the associated semigroup Γ_u satisfies $\Gamma_u \leq \langle 3, 7\rangle$. This implies that for fixed $u \neq 0$, the parametrization $(X(u,t), Y(u,t), Z(u,t))$ is simple since the characteristic $p > 3$. If $(X(u,t), Y(u,t), Z(u,t))$ is a deformation with multiplicity 2 then $\Gamma_u \leq \langle 2, min\{r,k\}\rangle$. This implies that there are again finitely many different equivalence classes. All together we obtain that the parametrization in the proposition are simple. ∎

Proposition 7. *Assume that the characteristic $p = 3$. Let $(x(t), y(t), z(t))$ be a parametrized simple space curve singularity of multiplicity 4 with the semigroup Γ. Assume $5 \notin \Gamma$ and $6 \in \Gamma$. Then $(x(t), y(t), z(t))$ is equivalent to one of the following parametrizations.*

1. (t^4, t^6, t^7)
2. $(t^4, t^6 + t^9, t^7)$.

Proof. We know that parametrizations with semigroup $\langle 4, 6, k, s\rangle$ are not simple if $s \geq 9$, $k \geq 13$ (Corollary 2). This implies that the parametrizations with semigroup $\langle 4, 6, 7\rangle$ are the only candidates for simple singularities. It is not difficult to see that a parametrization

$$(t^4, t^6 + \sum_{i>6} a_i t^i, t^7 + \sum_{i>7} b_i t^i)$$

is equivalent to (1) or (2).
In a deformation with multiplicity ≤ 3 only semigroups Γ with $\Gamma \leq \langle 3, 7, 8\rangle$ are possible. These parametrizations are simple. ∎

Proposition 8. *Assume that the characteristic $p = 13$. Let $(x(t), y(t), z(t))$ be a parametrized simple space curve singularity of multiplicity 4 with the semigroup Γ. Assume $5 \notin \Gamma$ and $6 \in \Gamma$. Then $(x(t), y(t), z(t))$ is equivalent to one of the following parametrizations.*

1. (t^4, t^6, t^7)
2. (t^4, t^6, t^9)
3. $(t^4, t^6 + t^7, t^9)$.

Proof. Lemma 3.3.6 implies that parametrizations with semigroup $\langle 4, 6, 11, 13\rangle$ are not simple. This implies that simple parametrizations with semigroup Γ with $4, 6 \in \Gamma$ must satisfy $\Gamma < \langle 4, 6, 11, 13\rangle$. We obtain the following semigroups with this property $\langle 4, 6, 7\rangle$ and $\langle 4, 6, 9\rangle$. We only prove the case with the semigroup $\langle 4, 6, 9\rangle$, the other case is similiar. It is clear that a parametrization with the semigroup $\langle 4, 6, 9\rangle = \{0, 4, 6, 8, 9, 10, 12, ...\}$ is of the form

$$(t^4 + \sum\nolimits_{i \geq 5} \alpha_i t^i, t^6 + at^7 + \sum\nolimits_{i \geq 9} \hat{\alpha}_i t^i, t^9 + \sum\nolimits_{i \geq 10} \bar{\beta}_i t^i).$$

By Corollary 1 (Zariski's Theorem) it is equivalent to $(t^4, t^6 + a_7 t^7 + a_{11} t^{11}, t^9 + b_{11} t^{11})$. We map t to $t - \frac{b_{11}}{9} t^3$, we obtain that our parametrization is equivalent to $(t^4, t^6 + \bar{a}_7 t^7 +$

$\bar{a}_{11}t^{11}, t^9)$. We map t to $t - \frac{\bar{a}_{11}}{6}t^6$, we obtain that our parametrization is equivalent to $(t^4, t^6 + \bar{\bar{a}}_7 t^7, t^9)$. If $\bar{\bar{a}}_7 = 0$ then we have (t^4, t^6, t^9). If $\bar{\bar{a}}_7 \neq 0$ then using the $\mathbb{K}^*-$ action, we obtain $(t^4, t^6 + t^7, t^9)$.

A parametrization with semigroup $\langle 4, 6, 9\rangle$ is simple since in a deformation the semigroup cannot increase. For parametrizations with multiplicity 4 we have only the possibilities $\langle 4, 6, 7\rangle$ and $\langle 4, 6, 9\rangle$. As in the proof of proposition 3.5.2, it follows that semigroup Γ_u corresponding to a parametrization of multiplicity 3 in a deformation must satisfy $\Gamma_u \leq \langle 3, 7\rangle$ and the parametrization is simple.

The same holds for parametrizations of multiplicity 2 in a deformation, since in this case $\Gamma_u \leq \langle 2, 9\rangle$. □

Proposition 9. *Let $(x(t), y(t), z(t))$ be a parametrization of a simple space curve singularity of multiplicity 4 with semigroup Γ. Assume that $5 \in \Gamma$. Then $(x(t), y(t), z(t))$ is equivalent to one of the following parametrizations:*

1. $(t^4, t^5, 0)$
2. $(t^4, t^5 + t^7, 0)$
3. (t^4, t^5, t^6)
4. (t^4, t^5, t^7)
5. (t^4, t^5, t^{11})
6. $(t^4, t^5 + t^7, t^{11})$

If $p = 5$, then we have additionally

7. $(t^4, t^5 + t^6, 0)$
8. $(t^4, t^5 + t^6, t^7)$
9. $(t^4, t^5 + t^6, t^{11})$

If $p = 3$ then we have additionally

10. $(t^4, t^5, t^6 + t^7)$.

Proof. We may assume that the parametrization is in weak normal form, i.e. $x(t) = t^4$, $y(t) = t^5 + \sum_{i>5, i\notin\Gamma} a_i t^i$ and $z(t) = 0$ or $z(t) = t^r + \sum_{i>r, i\notin\Gamma} d_i t^i$, $r \notin \langle 4, 5\rangle$, i.e. $r \in \{6, 7, 11\}$.

We apply the plane curve classification (cf.[17]) to $(x(t), y(t))$ and obtain that $(x(t), y(t))$ is equivalent to one of the following parametrization:

(i) (t^4, t^5)

(ii) $(t^4, t^5 + t^7)$

(iii) additionally $(t^4, t^5 + t^6)$ if $p = 5$.

If $z(t) = 0$, we obtain (1),(2) or (7). If $z(t) \neq 0$ then the conductor of the semigroup $\langle 4, 5, r\rangle$ is smaller or equal to 8.
If $p \neq 5$ then $(t^4, t^5 + t^6) \sim (t^4, t^5 + t^7)$. This implies that we obtain one of the cases (3)-(6). The cases $p = 3$ and $p = 5$ can be treated similarly.
It remains to prove that the parametrizations above are simple. Obviously deformation with the multiplicity 4 leads again to one of the cases (1)-(10), i.e. finitely many equivalence classes.
A deformation with multiplicity 2 has a semigroup smaller or equal to $\langle 2, 7\rangle$, i.e. again finitely many equivalence classes. A deformation of multiplicity 3 leads to a semigroup smaller or equal to $\langle 3, 5\rangle$ belonging to simple singularities. ∎

Proposition 10. *Assume $p > 7$. Let $(x(t), y(t), z(t))$ be a parametrization of a simple space curve singularity of multiplicity 4 with semigroup Γ. Assume that $5 \notin \Gamma$ and $7 \in \Gamma$. If $p \neq 13$ then $(x(t), y(t), z(t))$ is equivalent to one of the following parametrizations:*

1. $(t^4\ , t^7, 0)$
2. $(t^4, t^7 + t^9, 0)$
3. $(t^4, t^7 + t^{13}, 0)$
4. (t^4, t^7, t^9)
5. $(t^4, t^7, t^9 + t^{10})$
6. (t^4, t^7, t^{10})
7. $(t^4, t^7 + t^9, t^{10})$
8. $(t^4\ , t^7, t^{13})$
9. $(t^4, t^7 + t^9, t^{13})$
10. (t^4, t^7, t^{17})
11. $(t^4, t^7 + t^9, t^{17})$
12. $(t^4, t^7 + t^{13}, t^{17})$

If $p = 13$, then $(x(t), y(t), z(t))$ is equivalent to one of the following parametrizations:

13. (t^4, t^7, t^9)
14. $(t^4, t^7, t^9 + t^{10})$

Proof. We first consider the case $p \neq 13$. We may assume that the parametrization is in weak normal form, i.e. $x(t) = t^4$, $y(t) = t^7 + \sum_{i>7, i\notin\Gamma} a_i t^i$ and $z(t) = 0$ or $z(t) = t^r + \sum_{i>r, i\notin\Gamma} b_i t^i$, $r \notin \langle 4, 7\rangle$, i.e. $r \in \{9, 10, 13, 17\}$. We apply the plane curve classification (cf.[17]) to $(x(t), y(t))$ and obtain that $(x(t), y(t))$ is equivalent to one of the following parametrization:

(i) (t^4, t^7)

(ii) $(t^4, t^7 + t^9)$

(iii) $(t^4, t^7 + t^{13})$.

If $z(t) = 0$, we obtain (1),(2) or (3).
If $z(t) \neq 0$ then the conductor of the semigroup $\langle 4, 7, r\rangle$ is smaller or equal to 14. This implies that $(x(t), y(t), z(t))$ is equivalent to (8)-(12) if $r \geq 13$.
If $r = 10$ we obtain that $(x(t), y(t), z(t))$ is equivalent to $(t^4, t^7, t^{10} + b_{13}t^{13})$ or $(t^4, t^7 + t^9, t^{10} + b_{13}t^{13})$.
We prove the second case, the first case is similar. If $b_{13} \neq 0$, we use the automorphism defined by $t \to t - \frac{1}{10}b_{13}t^4$ to obtain

$$(t^4 + \sum\nolimits_{i\geq 7, i\neq 9}\alpha_i t^i, t^7 + t^9 + \sum\nolimits_{i\geq 10}\beta_i t^i, t^{10} + \sum\nolimits_{i\geq 16}\gamma_i t^i).$$

For suitable α_i, β_i, γ_i. This is equivalent to $(t^4 + \bar{\alpha}_{13}t^{13}, t^7 + t^9 + \bar{\beta}_{13}t^{13}, t^{10})$. For suitable $\bar{\alpha}_{13}$ and $\bar{\beta}_{13}$.
Using the transformation $t \to t - \frac{1}{10}\bar{\beta}_{13}t^7$, we obtain similarly an equivalence to $(t^4 + \bar{\bar{\alpha}}_{13}t^{13}, t^7 + t^9, t^{10})$. Using the transformation $t \to t - \frac{1}{4}\bar{\bar{\alpha}}_{13}t^{10}$, we obtain the equivalence to $(t^4, t^7 + t^9, t^{10})$.
If $r = 9$ then $(x(t), y(t), z(t))$ is equivalent to $(t^4, t^7, t^9 + \alpha_{10}t^{10})$. and we obtain the cases (4) and (5).
Now it remains to prove that the parametrizations (1)-(12) are simple.
Obviously a deformation with multiplicity 4 leads either to the cases (1)-(12) or to a case with the semigroup, containing 5 or 6. If 5 is in the semigroup , we have finitely many equivalence classes in the deformation. If we obtain a semigroup $\langle 4, 6, k, s\rangle$ then $s = \infty$ and $k = 7$ or $s \leq 7$. We have finitely many equivalence classes.
Now assume that we have a deformation with multiplicity 3. If 4 is in the semigroup we obtain a semigroup $\Gamma \leq \langle 3, 4\rangle$ with obviously finitely many equivalence classes. We may assume that we have a deformation $X(u, t) = \alpha t^3 + t^4 + ...,\ Y(u, t),\ Z(u, t)$.The corresponding semigroup $\Gamma \leq \langle 3, 7\rangle$. This implies that for fixed $u \neq 0$, the parametrization $(X(u, t), Y(u, t), Z(u, t))$ is simple since $p > 3$. If $(X(u, t), Y(u, t), Z(u, t))$ is a deformation with multiplicity 2 then the semigroup is $\Gamma \leq \langle 2, 7\rangle$. This implies that there are again only finitely many different equivalence classes. All together we obtain that the parametrizations in the proposition are simple.
If $p = 13$ then Corollary 9 implies that (t^4, t^7, t^{10}) is not simple. We obtain as the only possible candidates for simple parametrizations the cases (13) and (14). Arguments as above show that they are simple. □

Proposition 11. *Let $(x(t), y(t), z(t))$ be a parametrization of a space curve singularity of multiplicity 4 with semigroup Γ. Assume that $5 \notin \Gamma$ and $7 \in \Gamma$.*

1. *If the characteristic $p = 3$ or $p = 7$ then the parametrization is not simple.*

2. *If the characteristic $p = 5$ and the parametrization is simple then it is equivalent to (t^4, t^7, t^9).*

Proof. The proposition is a consequence of Lemma 7 (if $p = 5$), Corollary 2 (if $p = 3$), and Lemma 8 (if $p = 7$). □

7 Space curves in characteristic 2

Let $\mathbb{K}$ be an algebraically closed field of characteristic 2. In [17] the simple plane curve singularities of multiplicity ≥ 2 are classified by the following proposition.

Proposition 12. *The simple plane curve singularities of multiplicity ≥ 2 are given by the following parametrizations.*

1. (t^2, t^k) , $k \geq 3$ *odd.*
2. $(t^2 + t^m, t^k)$, $0 < m < k$, k, m *odd.*
3. (t^3, t^4)
4. $(t^3, t^4 + t^5)$
5. (t^3, t^5)
6. (t^3, t^7)
7. $(t^3, t^7 + t^8)$.

Proposition 13. *The simple space curve singularities of multiplicity ≥ 2 are given by (1)-(7) of Proposition 12 with third component* 0 *and additionally*

1. (t^3, t^4, t^5)
2. (t^3, t^5, t^7)
3. (t^3, t^7, t^8)
4. (t^3, t^7, t^{11})
5. $(t^3, t^7 + t^8, t^{11})$
6. (t^3, t^8, t^{10})
7. (t^3, t^8, t^{13})
8. $(t^3, t^8 + t^{10}, t^{13})$
9. (t^3, t^{10}, t^{11})
10. (t^3, t^{10}, t^{14})
11. $(t^3, t^{10} + t^{11}, t^{14})$.

Proof. We apply Proposition 2 and obtain that the semigroup of the parametrization must be smaller than $\langle 4,5,6\rangle$ and $\langle 3,10,17\rangle$ and $\langle 3,8\rangle$ to be a candidate for a simple parametrization. We obtain as possible cases (1) to (11). We will prove here only one case. The other cases can be proved similarly. Consider a parametrization with semigroup $\Gamma = \langle 3,8,13\rangle = \{0,3,6,8,9,11,...\}$ and assume that we already know that the parametrizations (1) to (6) are simple. We may assume that the parametrization is given as

$$(t^3, t^8 + \sum_{i>8, i\notin\Gamma} a_i t^i, t^{13} + \sum_{i>13, i\notin\Gamma} b_i t^i) = (t^3, t^8 + a_{10}t^{10}, t^{13}).$$

If $a_{10} \neq 0$, we obtain using the $\mathbb{K}^*$- action $(t^3, t^8 + t^{10}, t^{13})$. If $a_{10} = 0$, we obtain (t^3, t^8, t^{13}). A parametrization in a deformation has a semi-group smaller or equal to $\langle 3,8,13\rangle$. The possibilities are $\langle 2,k\rangle$, k odd, $\langle 3,4\rangle$, $\langle 3,4,5\rangle$, $\langle 3,5\rangle$,$\langle 3,5,7\rangle$, $\langle 3,7\rangle$, $\langle 3,7,8\rangle$, $\langle 3,7,11\rangle$, $\langle 3,8,10\rangle$, $\langle 3,8,13\rangle$. We know by Proposition 12 and our assumption that the parametrizations with the first nine semigroups are simple. We obtain that $(t^3, t^8 + a_{10}t^{10}, t^{13})$ is simple since it is equivalent to (7) or (8). $\square$

8 Proof of the main theorem

Proof. The aim of this section is to give a proof of Theorem 3
We first assume that the characteristic p is different from 2, 3, 5, 7 and 13.
Lemma 5 gives two semigroups, $\langle 5,6,7\rangle$ and $\langle 4,9,10\rangle$, such that the corresponding parametrization is not simple. This implies that parametrizations with the semigroup $\Gamma \geq \langle 5,6,7\rangle$ or $\Gamma \geq \langle 4,9,10\rangle$ are not simple. Proposition 3 (for multiplicity 2), Proposition 5 (for multiplicity 3), Proposition 6 (for multiplicity 4 and semigroup $\langle 4,6,...\rangle$), Proposition 9 (for multiplicity 4 and semigroup $\langle 4,5,...\rangle$) and Proposition 10 (for multiplicity 4 and semigroup $\langle 4,7,...\rangle$) give all the simple parametrizations with semigroup $\Gamma < \langle 4,9,10\rangle$ resp. $\Gamma < \langle 5,6,7\rangle$.
Now assume that $p = 13$. Lemma 9 and Corollary 3 imply that parametrizations with semigroup $\langle 4,7,10\rangle$ and $\langle 4,6,11,13\rangle$ are not simple. From Lemma 5 we know that parametrizations with semigroup $\langle 5,6,7\rangle$ are not simple. This implies that simple parametrizations must have a semigroup Γ with $\Gamma < \langle 5,6,7\rangle$ or $\Gamma < \langle 4,7,10\rangle$ or $\Gamma < \langle 4,6,11,13\rangle$. We obtain the classification similarly as above using Proposition 3 (for multiplicity 2), Proposition 5 (for multiplicity 3), Proposition 8 (for multiplicity 4 and semigroup $\langle 4,6,...\rangle$), Proposition 9 (for multiplicity 4 and semigroup $\langle 4,5,...\rangle$) and Proposition 10 (for multiplicity 4 and semigroup $\langle 4,7,...\rangle$).
Now assume that $p = 7$. Similarly to characteristic 13 we obtain additionally to $\langle 5,6,7\rangle$ and $\langle 4,9,10\rangle$ a third semigroup $\langle 4,7,13\rangle$ such that the corresponding parametrization is not simple (Lemma 8). The simple parametrizations of multiplicity 2, 3 and 4 (with semigroup $\langle 4,5,...\rangle$, $\langle 4,6,...\rangle$) are classified as above. There are no simple parametrizations of multiplicity 4 with semigroup $\langle 4,7,...\rangle$ (Proposition 11).
Now assume that $p = 5$. Similarly to characteristic 13 we obtain additionally to $\langle 5,6,7\rangle$ and $\langle 4,9,10\rangle$ a third semigroup $\langle 4,7,10\rangle$ such that the corresponding parametrization is not simple (Lemma 7).
The simple parametrizations of multiplicity 2, 3 and 4 (with semigroup $\langle 4,5,...\rangle$, $\langle 4,6,...\rangle$)

are classified as a above. The simple parametrizations of multiplicity 4 with semigroup $\langle 4,7,...\rangle$ are classified using Proposition 11.
Now assume that $p=3$. In this case we obtain additionally to $\langle 5,6,7\rangle$, $\langle 4,9,10\rangle$ the semigroups $\langle 3,7,11\rangle$, $\langle 4,7,9\rangle$ and $\langle 4,6,k,s\rangle$, $k\geq 13$, $s\geq 9$ ($k=\infty$ included) such that the corresponding parametrizations are not simple (Lemma 6 and Corollary 2). The simple parametrizations of multiplicity 2, 3 and 4 (with semigroup $\langle 4,5,...\rangle$, $\langle 4,6,...\rangle$) are classified as above. There are no simple parametrizations of multiplicity 4 with semigroup $\langle 4,7,...\rangle$ (Proposition 11).
Now assume that $p=2$. Proposition 12 implies that parametrizations with semigroup $\langle 4,5,6\rangle$ or $\langle 3,8\rangle$ or $\langle 3,10,17\rangle$ are not simple. This implies that simple parametrizations have multiplicity ≤ 3 and semigroup $\Gamma\leq\langle 3,10,14\rangle$. The simple parametrizations are classified using Proposition 13. □

9 Classifier

In this section we want to give an example for a classifier of space curve singularities which we implemented in the computer algebra system SINGILAR. The classifier is based on the results of this paper. It computes first of all the semigroup and the weak normal form of the given curve using the sagbi basis algorithm of SINGILAR. On this basis it is decided whether the curve is simple and if this is the case the normal form is computed using the classification of this paper. Let us consider an example.

```
LIB"classify_aeq.lib";
ring R=31,t,ds;
ideal I=t10-11t11-6t12-t13+12t14+4t15-14t16+15t17-12t18+5t19+t20,
   t3+6t4+13t5-13t6+10t7+2t8-6t9-10t10-15t11-6t12+8t13-2t14+t15+8t16,
   t7+15t8+7t9-11t10-15t11-6t12+8t13-2t14+t15+8t16;
ideal J=classSpaceCurve(I);
J;
J[1]=t3
J[2]=t7+t8
J[3]=t10
```

The procedure classSpaceCurve decides if the input is a simple curve and computes the normal form in this case. In the example we consider the curve given by the parametrization $(x(t),y(t),z(t))$ over the algebraic closure of $\mathbb{Z}/31$ with

$$x(t)=t^{10}-11t^{11}-6t^{12}-t^{13}+12t^{14}+4t^{15}-14t^{16}+15t^{17}-12t^{18}+5t^{19}+t^{20}.$$
$$y(t)=t^3+6t^4+13t^5-13t^6+10t^7+2t^8-6t^9-10t^{10}-15t^{11}-6t^{12}+8t^{13}-2t^{14}+t^{15}+8t^{16}$$
$$z(t)=t^7+15t^8+7t^9-11t^{10}-15t^{11}-6t^{12}+8t^{13}-2t^{14}+t^{15}+8t^{16}$$

The normal form of this curve is (t^3,t^7+t^8,t^{10}).

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[(1)]Abdus Salam School of Mathematical Sciences, GC University,
Lahore, 68-B, New Muslim Town, Lahore 54600, Pakistan
E-mail: `nimrajaved16@gmail.com`

[(2)]Department of Mathematics, University of Kaiserslautern,
Erwin-Schrödinger-Str., 67663 Kaiserslautern, Germany
E-mail: `pfister@mathematik.uni-kl.de`