Bull. Math. Soc. Sci. Math. Roumanie
Tome 65 (113), No. 2, 2022, 191–202

On simultaneous approximation in fields of power series

by

Yann Bugeaud

À la mémoire de Doru Ştefănescu

Abstract

Let $\mathbb{F}_q((T^{-1}))$ denote the field of power series over the field $\mathbb{F}_q$ of q elements, equipped with the absolute value $|\cdot|$ normalised in such a way that $|T| = q$. For a power series ξ in $\mathbb{F}_q((T^{-1}))$ and a positive integer n, we denote by $\lambda_n(\xi)$ the supremum of the real numbers λ for which

$$0 < \max\{|Q(T)\xi - P_1(T)|, \dots, |Q(T)\xi^n - P_n(T)|\} < q^{-\lambda \deg(Q)}$$

has infinitely many solutions in polynomials $Q(T), P_1(T), \dots, P_n(T)$ in $\mathbb{F}_q[T]$. We study the set of values taken by the function λ_n over the power series in $\mathbb{F}_q((T^{-1}))$ and over the algebraic power series in $\mathbb{F}_q((T^{-1}))$.

Key Words: Diophantine approximation, power series field, simultaneous approximation.

2010 Mathematics Subject Classification: Primary 11J61; Secondary 11J13.

1 Introduction

Let k be a commutative field, $k[[T^{-1}]]$ the ring of power series in one indeterminate over k, and $k((T^{-1}))$ its quotient field. If k is algebraically closed and of zero characteristic, then the algebraic closure of $k((T^{-1}))$ is the infinite union of the fields $k((T^{-1/n}))$, where n runs through the positive integers. Such a result does not hold when k has characteristic p with p positive, since the algebraic closure of $k((T^{-1}))$ must then contain the element

$$T^{-1/p} + T^{-1/p^2} + T^{-1/p^3} + \cdots , \tag{1.1}$$

which is a root of the polynomial $TX^p - TX - 1$, as noted by Abhyankar [1]. This example has motivated two of the early papers of Ştefănescu [19, 20], where he introduced the notion of restrictive power series, that is, of power series $\xi = \sum_{\alpha \in \mathbb{Q}} a_\alpha T^{-\alpha}$ such that the set $S(\xi) = \{\alpha \in \mathbb{Q} : a_\alpha \neq 0\}$ is a well ordered set and there exists a positive integer $m = m(\xi)$ such that every rational number α in $S(\xi)$ can be written $s_\alpha/(mp^{n_\alpha})$, for integers s_α, n_α. Clearly, the power series (1.1) is a restrictive power series. Ştefănescu [19, 20] proved that not all restrictive power series over k are algebraic over $k((T^{-1}))$ and established several criteria of algebraicity over $k((T^{-1}))$. Among other results, he showed that the series $\sum_{n \geq 1} a_n T^{-1/p^n}$, with a_n in $\overline{\mathbb{F}_p}$, is algebraic over $\overline{\mathbb{F}_p}((T^{-1}))$ if and only if the sequence $(a_n)_{n \geq 1}$ is eventually periodic. Subsequently, Kedlaya [8, 9] constructed an algebraic closure of $k((T^{-1}))$ for any algebraically closed field k of positive characteristic in terms of certain generalized power series.

In the present paper, we focus on the case where k is a finite field and discuss simultaneous rational approximation to elements of $k((T^{-1}))$, and in particular to algebraic power series in $k((T^{-1}))$. Our setting is the following. Let p be a prime number and q an integer power of p. Any non-zero element ξ in $\mathbb{F}_q((T^{-1}))$ can be written

$$\xi = \sum_{n=N}^{+\infty} a_n T^{-n},$$

where N is in $\mathbb{Z}$, $a_N \neq 0$, and a_n is in $\mathbb{F}_q$ for $n \geq N$. We define an absolute value $|\cdot|$ on $\mathbb{F}_q((T^{-1}))$ by setting $|\xi| := q^{-N}$ and $|0| := 0$. In particular, if $R(T)$ is a non-zero polynomial in $\mathbb{F}_q[T]$ of degree $\deg(R)$, then we have $|R| = q^{\deg(R)}$. The field $\mathbb{F}_q((T^{-1}))$ is the completion with respect to $|\cdot|$ of the quotient field $\mathbb{F}_q(T)$ of the polynomial ring $\mathbb{F}_q[T]$. The sets $\mathbb{F}_q[T]$, $\mathbb{F}_q(T)$, and $\mathbb{F}_q((T^{-1}))$ are the analogues of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively.

If ξ is not in $\mathbb{F}_q(T)$, then its irrationality exponent $\mu(\xi)$ is the supremum of the real numbers μ for which

$$\Big|\xi - \frac{P(T)}{Q(T)}\Big| < \frac{1}{|Q(T)|^{\mu}}, \quad \text{for infinitely many } \frac{P(T)}{Q(T)} \text{ in } \mathbb{F}_q(T).$$

Exactly as for real numbers, any element of $\mathbb{F}_q((T^{-1}))$ has a continued fraction expansion. Here, the partial quotients are non-constant polynomials in $\mathbb{F}_q[T]$. As in the real case, we have $\mu(\xi) \geq 2$ for every ξ in $\mathbb{F}_q((T^{-1})) \setminus \mathbb{F}_q(T)$, with equality for almost all ξ; see e.g. [10, 18].

Mahler [13] established the analogue of Liouville's theorem in $\mathbb{F}_q((T^{-1}))$, which asserts that every irrational power series ξ of degree d satisfies $\mu(\xi) \leq d$. He also showed that, unlike in the real case (where Roth's theorem asserts that the irrationality exponent of an irrational, algebraic real number is always equal to 2), this inequality is best possible. Namely, the root

$$T^{-1} + T^{-p} + T^{-p^2} + \dots$$

of the polynomial $TX^p - TX + 1$ is an algebraic element of $\mathbb{F}_q((T^{-1}))$ of degree p and irrationality exponent p. More generally, for any integer $s \geq 1$, the root

$$\xi_{M,s} := T^{-1} + T^{-p^s} + T^{-p^{2s}} + \dots$$

of the polynomial $TX^{p^s} - TX + 1$ is an algebraic element of $\mathbb{F}_q((T^{-1}))$ of degree p^s and irrationality exponent p^s. Thus, Liouville's theorem is best possible for algebraic power series in $\mathbb{F}_q((T^{-1}))$ of degree p^s. This observation motivates the following problem, already formulated by Mahler [13].

Problem 1.1. *Let $d \geq 2$ be an integer. Prove or disprove that Liouville's theorem is best possible for algebraic elements in $\mathbb{F}_q((T^{-1}))$ of degree d.*

Let s be a positive integer. Baum and Sweet [5] have noticed that the power series

$$\xi_{BS,s} = [T; T^{p^s}, T^{p^{2s}}, \dots] = T + \cfrac{1}{T^{p^s} + \frac{1}{T^{p^{2s}} + \cdots}}$$

in $\mathbb{F}_q((T^{-1}))$, which satisfies $\xi_{BS,s} = T + 1/\xi_{BS,s}^{p^s}$, is a root of the polynomial

$$X^{p^s+1} - TX^{p^s} - 1.$$

The degree of $\xi_{BS,s}$ is equal to p^s+1 and we have $\mu(\xi_{BS,s}) = p^s+1$, by [11, p. 214]. This gives a positive answer to another special case of Problem 1.1.

A third family of examples was found by Osgood [16]. Let $d \ge 2$ be an integer coprime with p. Let s be the smallest positive integer such that d divides p^s-1. Then, there exists a power series $\xi_{O,d}$ in $\mathbb{F}_q((T^{-1}))$ such that

$$\xi_{O,d}^d = \frac{T}{1+T}$$

is algebraic of degree d and satisfies $\mu(\xi_{O,d}) = d$. To see this, put

$$\zeta = 1 + \sum_{k\ge 0}\bigl(T^{-p^{sk}}(1+T^{-1})(1+T^{-p^s})\dots(1+T^{-p^{s(k-1)}})\bigr)$$

and notice that

$$(1+T^{-1})\zeta^{p^s} = \zeta,$$

and that ζ is well approximable by the rational fractions

$$\zeta_K = 1 + \sum_{k=0}^{K}\bigl(T^{-p^{sk}}(1+T^{-1})(1+T^{-p^s})\dots(1+T^{-p^{s(k-1)}})\bigr), \quad K \ge 1,$$

obtained by truncation. Namely, we have

$$|\zeta - \zeta_K| = q^{-p^{s(K+1)}}, \quad |\zeta_K| = q^{(p^{s(K+1)}-1)/(p^s-1)}, \quad K \ge 1.$$

Furthermore, we check that $\xi_{O,d} = \zeta^{(p^s-1)/d}$. The power series $\xi_{M,s}$ and $\xi_{O,d}$, which will be used in the proof of Theorem 3.2, show that Problem 1.1 is solved for every degree d coprime with p or equal to a power of p. As far as we are aware, Problem 1.1 remains open for the other values of d, including for $d=6$ when p is equal to 2 or 3.

All the power series $\xi_{M,s}, \xi_{BS,s}$, and $\xi_{O,d}$ belong to the family $H(q)$ of hyperquadratic power series.

Definition 1.2. *For an integer $s \ge 0$, let $H_s(q)$ denote the set of irrational power series α in $\mathbb{F}_q((T^{-1}))$ for which there exist polynomials A, B, C, D in $\mathbb{F}_q[T]$ such that $AD - BC \neq 0$ and*

$$\alpha = \frac{A\alpha^{p^s} + B}{C\alpha^{p^s} + D}.$$

The set

$$H(q) = \bigcup_{s\ge 0} H_s(q)$$

is the set of hyperquadratic power series in $\mathbb{F}_q((T^{-1}))$.

Quadratic power series belong to the set $H_0(q)$. Cubic power series α belong to the set $H_1(q)$ since $1, \alpha, \alpha^p$, and α^{p+1} are linearly dependent over $\mathbb{F}_q[T]$.

Thakur [21] and Schmidt [17], independently, constructed, for every rational number r greater than 2, an (hyperquadratic) algebraic power series ξ_r in $\mathbb{F}_q((T^{-1}))$ such that $\mu(\xi_r) = r$. This motivates the study of the next problem, posed by Thakur [22], and which asks for an analogue of Roth's theorem over $\mathbb{F}_q((T^{-1}))$.

Problem 1.3. *To prove that the irrationality exponent of any algebraic element in $\mathbb{F}_q((T^{-1}))$ is a rational number.*

An important special case of Problem 1.3 has been solved by de Mathan [14], who established that the irrationality exponent of any hyperquadratic irrational power series in $\mathbb{F}_q((T^{-1}))$ is a rational number. For additional results and references, the reader is directed to [12, 17, 23].

Extending the study of the rational approximation to a power series ξ, we consider the simultaneous rational approximation of $\xi, \xi^2, \dots, \xi^n$ by rational fractions with the same denominator, as well as small values of the linear form $A_0 + A_1\xi + \dots + A_n\xi^n$ with coefficients in $\mathbb{F}_q[T]$. This leads to the definition of the exponents of approximation w_n and λ_n.

The height $H(P)$ of a polynomial $P(X) = b_n(T)X^n + \dots + b_1(T)X + b_0(T)$ over $\mathbb{F}_q[T]$ is the maximum of the absolute values of its coefficients, that is, of $|b_0|, |b_1|, \dots, |b_n|$. Furthermore, the 'fractional part' $\|\cdot\|$ is defined by

$$\Big\| \sum_{n=N}^{+\infty} a_n T^{-n} \Big\| = \Big| \sum_{n=\max\{1,N\}}^{+\infty} a_n T^{-n} \Big|,$$

for every power series $\xi = \sum_{n=N}^{+\infty} a_n T^{-n}$ in $\mathbb{F}_q((T^{-1}))$.

Definition 1.4. *Let ξ be in $\mathbb{F}_q((T^{-1}))$. Let $n \geq 1$ be an integer. We denote by $w_n(\xi)$ the supremum of the real numbers w for which*

$$0 < |P(\xi)| < H(P)^{-w}$$

has infinitely many solutions in polynomials $P(X)$ over $\mathbb{F}_q[T]$ of degree at most n. We denote by $\lambda_n(\xi)$ the supremum of the real numbers λ for which

$$0 < \max\{\|Q(T)\xi\|, \dots, \|Q(T)\xi^n\|\} < q^{-\lambda \deg(Q)}$$

has infinitely many solutions in polynomials $Q(T)$ in $\mathbb{F}_q[T]$. Furthermore, for positive real numbers w, λ, set

$$B_n(\xi, w) = \liminf_{H(P)\to+\infty} H(P)^w \cdot |P(\xi)|$$

and

$$B'_n(\xi, \lambda) = \liminf_{|Q|\to+\infty} |Q|^\lambda \cdot \max\{\|Q(T)\xi\|, \dots, \|Q(T)\xi^n\|\}.$$

Since $\mathbb{F}_q$ is a finite field, requiring 'infinitely many solutions in polynomials $Q(T)$ in $\mathbb{F}_q[T]$' is equivalent to requiring 'solutions in polynomials $Q(T)$ in $\mathbb{F}_q[T]$ of arbitrarily large degree'. Observe that the exponents w_1, λ_1 and $\mu - 1$ coincide.

The quantities $B_n(\xi, w_n(\xi))$ and $B'_n(\xi, \lambda_n(\xi))$ refine the information given by the values of $w_n(\xi)$ and $\lambda_n(\xi)$. De Mathan [14] showed that, for any hyperquadratic irrational power series ξ in $\mathbb{F}_q((T^{-1}))$, the quantity $B(\xi, w_1(\xi))$ is positive and finite.

By spectrum of an exponent of approximation, we mean the set of values taken by this exponent at irrational elements. As in the real case, the continued fraction algorithm allows us easily to construct an irrational power series in $\mathbb{F}_q((T^{-1}))$ with any prescribed irrationality exponent in $[2, +\infty]$. Recalling that $\mu(\xi) \geq 2$ for every irrational power series ξ, this shows that the spectrum of μ is equal to the interval $[2, +\infty]$ and that of w_1 to the interval $[1, +\infty]$.

Problem 1.5. *Let n be a positive integer. To determine the spectra of the exponents of approximation w_n and λ_n over $\mathbb{F}_q((T^{-1}))$ and over the set of algebraic power series in $\mathbb{F}_q((T^{-1}))$. To prove or disprove that the exponents w_n and λ_n take rational values at algebraic power series in $\mathbb{F}_q((T^{-1}))$.*

There are essentially two ways to attack the first question of Problem 1.5. A first one consists in constructing explicitly power series ξ with prescribed values for $w_n(\xi)$ and/or $\lambda_n(\xi)$. A second one is to use metric Diophantine approximation to compute the Hausdorff dimension of the set of power series ξ such that $w_n(\xi)$ is equal to some given value. The first approach has been worked out by Ooto [15], see Theorems 4.1 and 4.2. The second one has been initiated by Chen [7], whose results yield upper and lower bounds for the Hausdorff dimension of the set

$$\{\xi \in \mathbb{F}_q((T^{-1})) : w_n(\xi) = w_n\},$$

for any real number $w_n \geq n$. Actually, she established that the spectrum of the exponent w_n^*, which measures the quality of approximation by algebraic power series of bounded degree (see also [6]), contains the whole interval $[n, +\infty]$.

As far as we are aware, the exponents λ_n have not yet been studied in the setting of fields of power series. The purpose of the present note is to fill this gap. We establish several transference statements between the exponents λ_k, λ_n and w_n and a contribution towards the resolution of Problem 1.5. In Section 3, we briefly discuss uniform simultaneous rational approximation.

2 Results

For an algebraic power series ξ in $\mathbb{F}_q((T^{-1}))$ of degree d, an argument 'à la Liouville' shows that $w_n(\xi) \leq d-1$ for every $n \geq 1$. Since, by Dirichlet's box principle, $w_{d-1}(\xi) \geq d-1$, we deduce that $w_n(\xi) = d-1$ for $n \geq d-1$. Furthermore, $\lambda_n(\xi) \leq w_1(\xi) \leq d-1$ for every $n \geq 1$ and $\lambda_n(\xi) = \lambda_{d-1}(\xi)$ for $n \geq d-1$.

Our first result is a power series analogue of inequalities relating the exponents λ_n established in the real and in the p-adic settings [3, 4].

Theorem 2.1. *Let ξ be a power series in $\mathbb{F}_q((T^{-1}))$. For any positive integer k, we have*

$$(k+1)\big(1 + \lambda_{k+1}(\xi)\big) \geq k\big(1 + \lambda_k(\xi)\big),$$

with equality if $\lambda_{k+1}(\xi) > 1$. Consequently, for every integer n with $n \geq k$, we have

$$\lambda_n(\xi) \geq \frac{k\lambda_k(\xi) - n + k}{n}, \tag{2.1}$$

and equality holds if $\lambda_n(\xi) > 1$.

We display an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let ξ be a power series in $\mathbb{F}_q((T^{-1}))$. Then, $\lambda_n(\xi) = +\infty$ holds for every positive n if, and only if, $\lambda_1(\xi) = +\infty$.*

In a similar way as in the real case, two relations between the exponents w_n and λ_n can be deduced from Khintchine's transference principle in fields of power series [2, Theorem 2].

Proposition 2.3. *For any positive integer n and any power series ξ in $\mathbb{F}_q((T^{-1}))$ which is not algebraic of degree at most n, we have*

$$\frac{w_n(\xi)}{(n-1)w_n(\xi)+n} \le \lambda_n(\xi) \le \frac{w_n(\xi)-n+1}{n}.$$

A more precise statement follows from [2, Theorem 2]. Namely, if we have $B'_n(\xi,\lambda)=0$ for some real number $\lambda \ge 1/n$, then $B_n(\xi, n\lambda+n-1)=0$. Likewise, if $B_n(\xi,w)$ is finite for some real number $w \ge n$, then $B'_n(\xi,(w-n+1)/n)$ is also finite. We omit the details of the proof.

It follows from Proposition 2.3 that

$$n\lambda_n(\xi) \le w_n(\xi)-n+1, \tag{2.2}$$

while Theorem 2.1 with $k=1$ asserts that

$$n\lambda_n(\xi) \ge \lambda_1(\xi)-n+1. \tag{2.3}$$

Actually, this inequality is easy to prove directly. We can assume that $|\xi|=1$. Assume that there are $\lambda>0$, $c>0$, and polynomials $Q(T)$ in $\mathbb{F}_q[T]$ of arbitrarily large degree such that $\|Q\xi\| \le c|Q|^{-\lambda}$. Let $P(T)$ be in $\mathbb{F}_q[T]$ such that $\|Q\xi\| = |Q\xi-P|$ and P and Q have the same degree. Then, for $j=1,\ldots,n$, we have

$$\begin{aligned}\|Q^n\xi^j\| \le |Q^{n-j}|\cdot\|Q^j\xi^j\| &\le |Q^{n-j}|\cdot|Q^j\xi^j-P^j| \\ &\le |Q|^{(n-j)+(j-1)}\,|Q\xi-P| \le c|Q|^{-\lambda+n-1}.\end{aligned}$$

This gives (2.3) and also shows that $B'_n(\xi,(\lambda-n+1)/n)$ is positive if $B_1(\xi,\lambda)$ is positive.

If $w_n(\xi)=\lambda_1(\xi)$, then both inequalities (2.2) and (2.3) become equalities and we get the following statement.

Corollary 2.4. *Let $n \ge 1$ be an integer. Let ξ be a power series in $\mathbb{F}_q((T^{-1}))$. If $w_n(\xi) = \lambda_1(\xi)$, then*

$$\lambda_n(\xi) = \frac{w_n(\xi)-n+1}{n}.$$

By combining Corollary 2.4 with results of Ooto [15] asserting the existence of power series ξ such that $w_n(\xi)=\lambda_1(\xi)$ and reproduced as Theorems 4.1 and 4.2 below, we obtain partial results towards the determination of the spectrum of λ_n.

Theorem 2.5. *Let n be a positive integer. The spectrum of the exponent λ_n over the power series in $\mathbb{F}_q((T^{-1}))$ includes the interval $[1,+\infty]$. The spectrum of the exponent λ_n over the algebraic power series in $\mathbb{F}_q((T^{-1}))$ includes all the rational numbers in $(1,+\infty)$.*

Since the algebraic numbers $\xi_{M,s}$ and $\xi_{O,d}$ defined in Section 1 satisfy the assumptions of Corollary 2.4, they allow us to find new elements of the spectra of the exponents λ_n.

Theorem 2.6. *Let a/b be a positive rational number. Then, for every prime number p greater than $a+b$ and for every power q of p, there exists an algebraic power series $\xi_{a/b}$ in $\mathbb{F}_q((T^{-1}))$ such that $\lambda_b(\xi_{a/b})=a/b$ and $B'_b(\xi_{a/b},a/b)$ is positive and finite.*

Theorem 2.6 is not plainly satisfactory since the prime number p cannot be arbitrarily chosen and the exponent λ_b depends on the given rational number.

We conclude with a metric result. For the definition of the Hausdorff dimension, the reader is directed e.g. to [7, 10]. Among other results, the power series analogue of the Jarník–Besicovitch theorem is established in these papers: for any real number $\lambda \geq 1$, the Haudorff dimension of the set of power series ξ in $\mathbb{F}_q((T^{-1}))$ such that $\lambda_1(\xi) = \lambda$ is equal to $2/(1+\lambda)$.

It follows from Theorem 2.1 that any power series ξ such that $\lambda_n(\xi) > 1$ satisfies $\lambda_1(\xi) = n\lambda_n(\xi)+n-1$. Combined with the Jarník–Besicovitch theorem, this gives at once the following statement.

Theorem 2.7. *Let n be a positive integer and $\lambda > 1$ a real number. Then, the Hausdorff dimension of the set*

$$\{\xi \in \mathbb{F}_q((T^{-1})) : \lambda_n(\xi) = \lambda\}$$

is equal to $2/(n(1+\lambda))$.

3 Uniform simultaneous approximation

We take the opportunity of this note to discuss briefly uniform simultaneous approximation to tuples of power series in $\mathbb{F}_q((T^{-1}))$.

Definition 3.1. *Let $n \geq 1$ be an integer. Let $\underline{\xi} = (\xi_1, \ldots, \xi_n)$ be in $\mathbb{F}_q((T^{-1}))^n$. We denote by $\lambda_n(\underline{\xi})$ the supremum of the real numbers λ for which*

$$0 < \max\{\|Q(T)\xi_1\|, \ldots, \|Q(T)\xi_n\|\} < q^{-\lambda \deg(Q)}$$

has infinitely many solutions in polynomials $Q(T)$ in $\mathbb{F}_q[T]$. We denote by $\widehat{\lambda}_n(\underline{\xi})$ the supremum of the real numbers $\widehat{\lambda}$ for which there exists an integer d_0 such that, for every $d > d_0$, there exists a polynomial $Q(T)$ in $\mathbb{F}_q[T]$ of degree at most d such that

$$0 < \max\{\|Q(T)\xi_1\|, \ldots, \|Q(T)\xi_n\|\} < q^{-\widehat{\lambda} d}.$$

It easily follows from the theory of continued fractions that $\widehat{\lambda}_1((\xi)) = 1$ for every irrational power series ξ in $\mathbb{F}_q((T^{-1}))$. Consequently, we have $\widehat{\lambda}_n(\underline{\xi}) \leq 1$ for every $\underline{\xi}$ in $\mathbb{F}_q((T^{-1}))^n$ with at least one irrational coordinate. Furthermore, we have $\lambda_n(\underline{\xi}) = 1/n$ for almost all tuples $\underline{\xi}$ in $\mathbb{F}_q((T^{-1}))^n$, with respect to the Haar measure on $\mathbb{F}_q((T^{-1}))^n$; see [10]. Consequently, the set of $\underline{\xi}$ such that $\widehat{\lambda}_n(\underline{\xi}) > 1/n$ is a set of zero Haar measure. For $n = 2$, we give explicit examples of pairs of power series in this set.

Theorem 3.2. *For any $\varepsilon > 0$, there exist algebraic power series ξ_1, ξ_2 in $\mathbb{F}_q((T^{-1}))$ such that $1, \xi_1, \xi_2$ are linearly independent over $\mathbb{F}_q[T]$ and*

$$\widehat{\lambda}_2((\xi_1, \xi_2)) > 1 - \varepsilon.$$

Proof. Let s be a positive integer. Observe that

$$\|T^{p^{ks}} \xi_{M,s}\| = q^{-p^{(k+1)s} + p^{ks}}, \quad k \geq 1,$$

and that

$$\|T^{p^{ks}}T^{(p^{ks}-1)/(p^s-1)}\xi_{O,p^s-1}\| = q^{-p^{(k+1)s}+p^{ks}+(p^{ks}-1)/(p^s-1)}, \quad k \geq 1.$$

Observe that $1, \xi_{M,s}$, and ξ_{O,p^s-1} are linearly independent over $\mathbb{F}_q[T]$, since the algebraic power series $\xi_{M,s}$ and ξ_{O,p^s-1} have different degrees.

Put $Q_k(T) = T^{p^{ks}}T^{(p^{ks}-1)/(p^s-1)}$. Then,

$$\max\{\|Q_k(T)\xi_{M,s}\|, \|Q_k(T)\xi_{O,p^s-1}\|\} = q^{-p^{(k+1)s}+p^{ks}+(p^{ks}-1)/(p^s-1)}, \quad k \geq 1,$$

and $\deg Q_k(T) = (p^{(k+1)s}-1)/(p^s-1)$. We derive that

$$\widehat{\lambda}_2(\xi_{M,s}, \xi_{O,p^s-1}) \geq \lim_{k\to+\infty} \frac{p^{(k+1)s} - p^{ks} - (p^{ks}-1)/(p^s-1)}{(p^{(k+2)s}-1)/(p^s-1)} = 1 - \frac{2}{p^s},$$

and the theorem follows. Presumably, we have $\widehat{\lambda}_2(\xi_{M,s}, \xi_{O,p^s-1}) = 1 - \frac{2}{p^s}$, but this seems to be difficult to prove. □

4 Proofs

Proof of Theorem 2.1. We adapt the proofs given in [3] for the real case and in [4] for the p-adic case. Let $n \geq 2$ be an integer and ξ a power series in $\mathbb{F}_q((T^{-1}))$ with $\lambda_n(\xi) > 1$. Without any loss of generality, we assume that $|\xi| = 1$. Let λ be a real number with $1 < \lambda < \lambda_n(\xi)$. Then, there are arbitrarily large integers d and polynomials $Q, P_1, \ldots, P_n$ in $\mathbb{F}_q[T]$ of degree at most d and with no common factor, such that

$$|Q\xi^j - P_j| < q^{-\lambda d}, \quad j = 1, \ldots, n.$$

Since $|\xi| = 1$, the polynomials $Q, P_1, \ldots, P_n$ have the same degree and

$$|P_{j+1} - P_j\xi| = |P_{j+1} - Q\xi^{j+1} - \xi(P_j - Q\xi^j)| < q^{-\lambda d}, \quad j = 1, \ldots, n-1.$$

Set $P_0 = Q$. Observe that, for $j = 1, \ldots, n-1$, we have

$$\Delta_j := P_{j-1}P_{j+1} - P_j^2 = P_{j-1}(P_{j+1} - P_j\xi) - P_j(P_j - P_{j-1}\xi),$$

thus, by the triangle inequality,

$$|\Delta_j| < q^{(1-\lambda)d}.$$

Since Δ_j is in $\mathbb{F}_q[T]$, it satisfies $|\Delta_j| \geq 1$ unless it is zero. Therefore, since $\lambda > 1$, we get that

$$\Delta_1 = \ldots = \Delta_{n-1} = 0,$$

which implies that there exist coprime non-zero polynomials A, B in $\mathbb{F}_q[T]$ such that

$$\frac{P_1}{Q} = \frac{P_2}{P_1} = \ldots = \frac{P_n}{P_{n-1}} = \frac{A}{B}.$$

We deduce at once that the point

$$\Big(\frac{P_1}{Q},\dots,\frac{P_n}{Q}\Big)=\Big(\frac{A}{B},\dots,\Big(\frac{A}{B}\Big)^n\Big)$$

lies on the Veronese curve $x\mapsto(x,x^2,\dots,x^n)$ and that Q (resp., P_n) is an integer multiple of B^n (resp., of A^n). Since $\gcd(Q,P_1,\dots,P_n)=1$, we have in fact $(Q,P_1,\dots,P_n)=\pm(B^n,B^{n-1}A,\dots,A^n)$. In particular, we get

$$\Big|\xi-\frac{P_1}{Q}\Big|=\Big|\xi-\frac{A}{B}\Big|<q^{-(\lambda+1)d}\le q^{-n(1+\lambda)\deg(B)}.$$

This proves that

$$\lambda_n(\xi)>1 \text{ implies } \lambda_1(\xi)\ge n(1+\lambda_n(\xi))-1. \tag{4.1}$$

Let k be an integer with $1\le k<n$ and write $\lambda_k=\lambda_k(\xi)$. Assume first that λ_k is finite. Let ε be a positive real number with $\varepsilon<\lambda_k$. Then, there are arbitrarily large integers d and polynomials $Q,V_1,\dots,V_k$ in $\mathbb{F}_q[T]$ of degree at most d such that

$$q^{-(\lambda_k+\varepsilon)d}\le\max\{|Q\xi-V_1|,\dots,|Q\xi^k-V_k|\}\le q^{-(\lambda_k-\varepsilon)d}. \tag{4.2}$$

Take such an integer d with $d\ge 3k$. In particular, Q is nonzero. There exist polynomials $A_0,A_1,\dots,A_k$ in $\mathbb{F}_q[T]$, not all zero, such that

$$A_0Q+A_1V_1+\ldots+A_kV_k=0$$

and

$$\max_{0\le j\le d}\deg A_j\le\frac{d}{k}.$$

Indeed, setting $\delta=\lfloor d/k\rfloor$, there are $q^{(\delta+1)(k+1)}-1$ different non-zero $(k+1)$-tuples $(B_0,B_1,\dots,B_k)$ of polynomials in $\mathbb{F}_q[T]$ of degree at most δ. For each of them, the degree of the polynomial $B_0Q+B_1V_1+\ldots+B_kV_k$ is at most equal to $\delta+d$. Since

$$q^{(\delta+1)(k+1)}-1>q^{\delta+d+1},$$

we can conclude by the box principle.

We may assume $A_k\ne0$, otherwise we replace k by the largest index j with $A_j\ne0$ in the argument below. Then, we derive from (4.2) that

$$\begin{aligned}|Q(A_k\xi^k+\ldots+A_1\xi+A_0)| &=\\ |Q(A_k\xi^k+\ldots+A_1\xi+A_0)-(A_kV_k+\ldots+A_1V_1+A_0Q)| &\le q^{d/k}q^{-(\lambda_k-\varepsilon)d}.\end{aligned} \tag{4.3}$$

Using triangle inequalities, we get from (4.2) and (4.3) that

$$\begin{aligned}&|A_kQ\xi^{k+1}+A_{k-1}V_k+A_{k-2}V_{k-1}+\ldots+A_1V_2+A_0V_1|\\ &\quad\le\max\{|Q(A_k\xi^{k+1}+\ldots+A_0\xi)|,\\ &\qquad\qquad|Q(A_{k-1}\xi^k+\ldots+A_0\xi)-A_{k-1}V_k-\ldots-A_0V_1|\}\\ &\quad\le\max\{|Q(A_k\xi^k+\ldots+A_1\xi+A_0)|,q^{-(\lambda_k-\varepsilon)d}q^{d/k}\}\\ &\quad\le q^{-(\lambda_k-\varepsilon)d}q^{d/k}.\end{aligned} \tag{4.4}$$

Since $A_k \neq 0$, it now follows from

$$\deg(A_{k-1}V_k + A_{k-2}V_{k-1} + \ldots + A_1V_2 + A_0V_1) \leq d + \frac{d}{k},$$

(4.2), and (4.4) that

$$\lambda_{k+1}(\xi) \geq \frac{\lambda_k(\xi) - 1/k - \varepsilon}{1 + 1/k}. \tag{4.5}$$

As ε can be chosen arbitrarily close to 0, we deduce that

$$(k+1)\lambda_{k+1}(\xi) \geq k\lambda_k(\xi) - 1$$

or, equivalently,

$$(k+1)\big(1 + \lambda_{k+1}(\xi)\big) \geq k\big(1 + \lambda_k(\xi)\big). \tag{4.6}$$

This concludes the proof of the first inequality of the theorem if $\lambda_k(\xi)$ is finite. In the case where $\lambda_k(\xi)$ is infinite, the quantity $\lambda_k(\xi) - \varepsilon$ in (4.5) can be replaced by any arbitrarily large real number and we conclude that $\lambda_{k+1}(\xi)$ is infinite, thus (4.6) also holds in this case.

By iterating (4.6) up to $k = n-1$, we immediately get (2.1). In particular, we obtain

$$(k+1)(\lambda_{k+1}(\xi) + 1) \geq \lambda_1(\xi) + 1 \quad \text{and} \quad k(\lambda_k(\xi) + 1) \geq \lambda_1(\xi) + 1. \tag{4.7}$$

Assume now that $\lambda_{k+1}(\xi) > 1$. Then, we also have $\lambda_k(\xi) > 1$ and we get from (4.1) that $\lambda_1(\xi) \geq (k+1)(1 + \lambda_{k+1}(\xi)) - 1$ and $\lambda_1(\xi) \geq k(1 + \lambda_k(\xi)) - 1$. Combined with (4.7), this gives at once

$$(k+1)\big(1 + \lambda_{k+1}(\xi)\big) = k\big(1 + \lambda_k(\xi)\big) = 1 + \lambda_1(\xi).$$

By iterating this equality, we obtain that (2.1) is an equality when $\lambda_n(\xi)$ exceeds 1. □

The proof of Theorem 2.5 rests on the following two results of Ooto [15].

Theorem 4.1 (Ooto). *For any positive integer n and any real number w with $w \geq 2n - 1$, there exist uncountably many power series ξ_w in $\mathbb{F}_q((T^{-1}))$ such that*

$$w_1(\xi_w) = \ldots = w_n(\xi_w) = w.$$

Theorem 4.2 (Ooto). *For any positive integer n and any rational number w with $w > 2n-1$, there exist algebraic power series ξ_w in $\mathbb{F}_q((T^{-1}))$ such that*

$$w_1(\xi_w) = \ldots = w_n(\xi_w) = w.$$

The algebraic power series ξ_w constructed in Theorem 4.2 are hyperquadratic, thus $B_1(\xi_w, w)$ is positive and finite. A quick look at the proof shows that this is also the case for $B_j(\xi_w, w)$ for $j = 2, \ldots, n$.

Proof of Theorem 2.5. Let $\lambda \geq 1$ be a real number. By Theorem 4.1, there exists a power series ξ such that $w_1(\xi) = \ldots = w_n(\xi) = n(\lambda + 1) - 1$. It then follows from Corollary 2.4 that $\lambda_n(\xi) = \lambda$. If λ is rational and greater than 1, then Theorem 4.2 asserts the existence of an algebraic power series ξ such that $w_1(\xi) = \ldots = w_n(\xi) = n(\lambda + 1) - 1$. We again conclude by applying Corollary 2.4. □

Proof of Theorem 2.6. Let a/b be a rational number, p a prime number exceeding $a+b$, and q a power of p. The power series $\xi_{O,a+b}$ is algebraic of degree $a+b$ and satisfies

$$w_1(\xi_{O,a+b}) = \ldots = w_{a+b}(\xi_{O,a+b}) = a+b-1.$$

In particular, we have

$$\lambda_b(\xi_{O,a+b}) = \frac{w_b(\xi_{O,a+b}) - b + 1}{b} = \frac{a}{b}.$$

Since $\xi_{O,a+b}$ is hyperquadratic, a result of de Mathan [14] already mentioned asserts that $B_1(\xi_{O,a+b}, a+b-1)$ is positive and finite. The same holds for $B_j(\xi_{O,a+b}, a+b-1)$, where $j = 2, \ldots, a+b$. Combined with the observation following Proposition 2.3, this completes the proof of the theorem. □

References

[1] S. Abhyankar, Two notes on formal power series, *Proc. Amer. Math. Soc.*, **7**, 903–905 (1956).

[2] S. K. Aggarwal, Transference theorems in the field of formal power series, *Monatsh. Math.*, **72**, 97–106 (1968).

[3] D. Badziahin, Y. Bugeaud, On simultaneous rational approximation to a real number and its integral powers II, *New York J. Math.*, **26**, 362–377 (2020).

[4] D. Badziahin, Y. Bugeaud, J. Schleischitz, On simultaneous rational approximation to a p-adic number and its integral powers, II, *Proc. Edinb. Math. Soc.*, **64**, 317–337 (2021).

[5] L. E. Baum, M. M. Sweet, Continued fractions of algebraic power series in characteristic 2, *Ann. of Math.*, **103**, 593–610 (1976).

[6] J. Bell, Y. Bugeaud, Mahler's and Koksma's classifications in fields of power series, *Nagoya Math. J.*, **246**, 355-371 (2022).

[7] H.-J. Chen, Baker–Schmidt theorem for Hausdorff dimensions in finite characteristic, *Finite Fields Appl.*, **52**, 336–360 (2018).

[8] K. Kedlaya, The algebraic closure of the power series field in positive characteristic, *Proc. Amer. Math. Soc.*, **129**, 3461–3470 (2001).

[9] K. Kedlaya, On the algebraicity of generalized power series, *Beitr. Algebra Geom.*, **58**, 499–527 (2017).

[10] S. Kristensen, On well-approximable matrices in a field of formal series, *Math. Proc. Cambridge Philos. Soc.*, **135**, 255–268 (2003).

[11] A. Lasjaunias, A survey of diophantine approximation in fields of power series, *Monatsh. Math.*, **130**, 211–229 (2000).

[12] A. Lasjaunias, Diophantine approximation and continued fractions in power series fields, in *Analytic Number Theory*, Cambridge Univ. Press, Cambridge, 297–305 (2009).

[13] K. Mahler, On a theorem of Liouville in fields of positive characteristic, *Canad. J. Math.*, **1**, 397–400 (1949).

[14] B. de Mathan, Approximation exponents for algebraic functions in positive characteristic, *Acta Arith.*, **60**, 359–370 (1992).

[15] T. Ooto, On Diophantine exponents for Laurent series over a finite field, *J. Number Theory*, **185**, 349–378 (2018).

[16] C. Osgood, Effective bounds on the "Diophantine approximation" of algebraic functions over fields of arbitrary characteristic and applications to differential equations, *Nederl. Akad. Wetensch. Proc. Ser. A*, **78**, 105–119 (1975).

[17] W. M. Schmidt, On continued fractions and Diophantine approximation in power series fields, *Acta Arith.*, **95**, 139–166 (2000).

[18] V. G. Sprindžuk, *Mahler's problem in metric number theory*, American Mathematical Society, Providence, R.I. (1969).

[19] D. Ştefănescu, A method to obtain algebraic elements over $K((t))$ in positive characteristic, *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, **26 (74)**, 77–91 (1982).

[20] D. Ştefănescu, On meromorphic formal power series, *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, **27 (75)**, 169–178 (1983).

[21] D. S. Thakur, Diophantine approximation exponents and continued fractions for algebraic power series, *J. Number Theory*, **79**, 284–291 (1999).

[22] D. S. Thakur, *Function Field Arithmetic*, World Scientific Publishing Co., Inc., River Edge, NJ (2004).

[23] D. S. Thakur, From rationality to transcendence in finite characteristic, in *Transcendance et irrationalité*, 21–48, SMF Journ. Annu., Soc. Math. France, Paris (2012).

Received: 27.01.2022
Accepted: 27.03.2022

Université de Strasbourg, Mathématiques,
7, rue René Descartes, 67084 Strasbourg, France
and Institut Universitaire de France
E-mail: bugeaud@math.unistra.fr