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Statistical Lie algebras of constant curvature and locally conformally Kähler Lie algebras

by
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Abstract

We show that a statistical manifold of constant non-zero curvature can be realised as a level set of Hessian potential on a Hessian cone. We construct a Sasakian structure on $TM \times \mathbb{R}$ by a statistical manifold of constant non-zero curvature on M. By a statistical Lie algebra of constant non-zero Lie algebra we construct a l.c.K. Lie algebra.

Key Words: Statistical manifolds, l.c.K. manifolds, Hessian manifolds, geometric structures on Lie groups and algebras.
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1 Introduction

A **flat affine manifold** is a differentiable manifold equipped with a flat, torsion-free connection. Equivalently, it is a manifold equipped with an atlas such that all transition maps between charts are affine transformations (see [18] or [28]). A **Hessian manifold** is a flat affine manifold with a Riemannian metric which is locally equivalent to the Hessian of a function. Equivalently, a Hessian manifold is a flat affine manifold (M, ∇) endowed with a Riemannian metric g such that the tensor ∇g is totally symmetric. Any Kähler metric can be defined as the complex Hessian of a plurisubharmonic function. Thus, the Hessian geometry is a real analogue of the Kähler one.

A Kähler structure (I, g^{r}) on TM can be constructed out of a Hessian structure (∇, g) on M (see [28]). The correspondence

$$\mathrm{r} : \{\text{Hessian manifolds}\} \to \{\text{Kähler manifolds}\}$$

$$(M, \nabla, g) \quad \to \quad (TM, I, g^{\mathrm{r}})$$

is called the **r-map**. In particular, this map associates special Kähler manifolds to special real manifolds (see [1]). In this case, the r-map describes a correspondence between the special geometries for supersymmetric theories in dimension 5 and 4. See [15] for details on the r-map and supersymmetry.

Hessian manifolds have many different application: in supersymmetry ([15], [16], [1]), in convex programming ([25], [26]), in the Monge-Ampère Equation ([17], [21]), in the WDVV equations ([29]).

A **Riemannian cone** is a Riemannian manifold $(M \times \mathbb{R}^{>0}, s^2 g_M + ds^2)$, where t is the coordinate on $\mathbb{R}^{>0}$ and g_M is a Riemannian metric on M. Riemannian cones have important applications in supegravity ([2], [4], [12], [13]). The geometry and holonomy of pseudo-Riemannian cones are studied in [3] and [6].

A Riemannian manifold (M, g) is called **Sasakian** if there exists a complex structure I on the cone $\left(M \times \mathbb{R}^{>0}, s^2 g_M + ds^2\right)$ such that $\left(M \times \mathbb{R}^{>0}, s^2 g_M + ds^2, I\right)$ is a Kähler manifold. Note that our definitions of Sasakian manifolds are not standard but equivalent. See [7] or [9] for standard definitions and [5] or [27] for equivalence of them.

A **radiant manifold** (C, ∇, ρ) is a flat affine manifold (C, ∇) endowed with a vector field ρ satisfying $\nabla \rho = \mathrm{Id}$. Equivalently, it is a manifold equipped with an atlas such that all transition maps between charts are linear transformations (see e.g. [20]). A **Hessian cone** is a Hessian manifold $(M \times \mathbb{R}^{>0}, \nabla, g = s^2 g_M + ds^2)$ such that there exists a constant $\lambda \neq 0, \frac{1}{2}$ such that $(M \times \mathbb{R}^{>0}, \nabla, \lambda s \frac{\partial}{\partial s})$ is a radiant manifold. In the case $\lambda = \frac{1}{2}$, the metric g satisfies $\iota_\xi g = 0$, i.e. can not be positive definite ([19]).

A **statistical manifold** (M, D, g) is a manifold M endowed with a torsion-free connection D and a Riemannian metric g such that the tensor Dg is totally symmetric. A statistical manifold (M, D, g) is said to be **of constant curvature** c if the curvature tensor Θ_D satisfies

$$\Theta_D(X, Y)Z = c\,(g(Y, Z)X - g(X, Z)Y)\,,$$

for any $X, Y, Z \in TM$ (see [28] or [22]). Note that the set of statistical manifolds of constant curvature zero is exactly the set of Hessian manifolds.

We show that a statistical manifold of constant curvature can be realised as a level set of a Hessian potential on a Hessian cone. A Sasakian manifold is a level set of the Kähler potential of a Kähler cone. In this sense, statistical manifolds of constant curvature are real analogue of Sasakian manifolds.

Theorem 1.1. Let (M, g, ∇) be a statistical manifold of constant curvature. Then $TM \times \mathbb{R}$ admits a structure of a Sasakian manifold.

This theorem is closely related to the r-map. Namely, we have a diagram

$$\begin{array}{ccc} M \times \mathbb{R}^{>0} & \xrightarrow{\ \mathrm{r}\ } & T(M \times \mathbb{R}^{>0}) \\ \uparrow{\scriptstyle con} & & \uparrow{\scriptstyle con} \\ M & \longrightarrow & TM \times \mathbb{R} \end{array},$$

where vertical arrows associate Riemannian cones to the corresponding Riemannian manifolds. The theorem implies that the Riemannian manifold $T(M \times \mathbb{R}^{>0})$ with the metric constructed by r-map is actually a cone over $TM \times \mathbb{R}$.

Then we work with Lie algebras and groups equipped with invariant structures on them. There are different descriptions of an **affine structure** on a Lie algebra $\mathfrak{g}$: a torsion-free flat connection on $\mathfrak{g}$, an **étale affine representation** $\mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n)$, where n is the dimension of $\mathfrak{g}$, or a structure of left symmetric algebra on $\mathfrak{g}$, that is, a multiplication on $\mathfrak{g}$ satisfying

$$XY - YX = [X, Y] \quad \text{and} \quad X(YZ) - (XY)Z = Z(XY) - (ZX)Y$$

for any $X, Y, Z \in \mathfrak{g}$ (see [10] or [11]).

An almost complex structure I on a Lie algebra $\mathfrak{g}$ is called **integrable** if the Nijenhuis tensor of I equals to zero i.e. for any $X, Y \in \mathfrak{g}$

$$N_I(X, Y) = [X, Y] + I([IX, Y] + [X, IY] - [IX, IY] = 0.$$

An almost complex structure on the Lie algebra $\mathfrak{g}$ of left invariant fields of a Lie group G sets a left invariant almost complex structure on G. It follows from Newlander–Nirenberg theorem that the almost complex structure on $\mathfrak{g}$ is integrable if and only if the left invariant almost complex structure on G is integrable.

Let $(\mathfrak{g}, \nabla)$ be a Lie algebra with a flat torsion free connection and $\mathfrak{g}_a$ the abelian Lie algebra which coincides with $\mathfrak{g}$ as a vector space. Consider the Lie algebra $\mathfrak{g} \ltimes_\nabla \mathfrak{g}_a$ that is the vector space $\mathfrak{g} \oplus \mathfrak{g}$ with the commutator

$$[X_1 \oplus Y_1, X_2 \oplus Y_2] = [X_1, X_2] \oplus (\nabla_{X_1} Y_2 - \nabla_{X_2} Y_1)\,.$$

Then the almost complex structure I on $\mathfrak{g} \ltimes_\nabla \mathfrak{g}_a$ defined by the rule

$$I\,(X_1 \oplus X_2) = -X_2 \oplus X_1.$$

is integrable (see [14] or [8]). If θ is an étale affine representation of G then the Lie algebra of left invariant fields on $G \ltimes_\theta \mathbb{R}^n$ equals $\mathfrak{g} \ltimes_\nabla \mathfrak{g}_a$.

A **Hessian Lie algebra** $(\mathfrak{g}, \nabla, g)$ is a Lie algebra $\mathfrak{g}$ endowed with a flat torsion free connection ∇ and symmetric bilinear form g such that ∇g is totally linear i.e. for any $X, Y, Z \in \mathfrak{g}$ we have

$$g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = g(\nabla_Y X, Z) + g(X, \nabla_Y Z).$$

A **Hessian Lie group** (G, ∇, g) is a Lie group G endowed with a left invariant affine structure ∇ and a left invariant Hessian metric g. A Lie group admits a Hessian structure if and only if the corresponding Lie algebra admits a Hessian structure.

We adapt the r-map to the case of Lie algebras and groups. A **Kähler Lie algebra** is a Lie algebra endowed with an integrable almost complex structure I and a closed 2-form ω such that the bilinear form $\omega(\cdot, I\cdot)$ is positive definite.

Theorem 1.2. Let $(\mathfrak{g}, \nabla, g)$ be an Hessian Lie algebra and $\pi : \mathfrak{g} \ltimes_\nabla \mathfrak{g}_a \to \mathfrak{g}$ the projection. Then $(\mathfrak{g} \ltimes_\nabla \mathfrak{g}_a, I, \omega)$ is a Kähler Lie algebra, where

$$I(X \oplus Y) = -Y \oplus X \quad \text{and} \quad \omega(X, Y) = \pi^* g(IX, Y) - \pi^* g(X, IY).$$

Corollary 1.3. Let G be an n-dimensional simply connected Lie group equipped with a left invariant affine structure ∇ and θ the linear part of the corresponding affine action of G. Then there exists a left invariant Kähler metric on $G \ltimes_\theta \mathbb{R}^n$.

Remark 1.1. Note that a Kähler structure on a the group $G \ltimes_{\theta^*} (\mathbb{R}^n)^*$ is constructed by an invariant Hessian structure on G in [23]. The corresponding complex structure on $G \ltimes_{\theta^*} (\mathbb{R}^n)^*$ depends on the Hessian metric on G. In our case, the complex structure on $G \ltimes_\theta \mathbb{R}^n$ depends only on the affine structure on G.

A **locally conformally Kähler (l.c.K.) manifold / Lie algebra** is a manifold / Lie algebra endowed with a complex structure I, closed 1-form θ, and 2-form ω such that $d\omega = \theta \wedge \omega$ and $\omega(\cdot, I\cdot)$ is positive definite. The closed form θ on a l.c.K. manifold is locally exact i.e. equals to the differential of a locally defined function f. Then the locally defined form $e^{-f}\omega$ is Kähler.

A **statistical Lie algebra** $(\mathfrak{g}, g, D)$ is a Lie algebra endowed with a bilinear symmetric positive-definite form g and torsion-free connection and D such that Dg is a totally symmetric tensor. A statistical Lie algebra $(\mathfrak{g}, g, D)$ is said to be **of constant curvature** c if the curvature tensor equals

$$\Theta_D(X,Y)Z = c\,(g(Y,Z)X - g(X,Z)Y)\,,$$

for any $X, Y, Z \in \mathfrak{g}$.

A **statistical Lie group (of constant curvature** c**)** (G, g, D) is a Lie group endowed with a left invariant statistical structure (of constant curvature c).

Obviously, there exists a one-to-one correspondence between simply connected statistical Lie groups (of constant curvature c) and statistical Lie algebras (of constant curvature c).

Theorem 1.4. Let $(\mathfrak{g}, g_{\mathfrak{g}}, D)$ be a statistical Lie algebra of constant non-zero curvature c, ρ a generator of the subalgebra $\{0\} \times \mathbb{R} \subset \mathfrak{g} \times \mathbb{R}$, ∇ a connection on the Lie algebra defined by

$$\nabla_X Y = D_X Y - cg(X,Y)\rho, \quad \nabla_X \rho = \nabla_\rho X = X, \quad \nabla_\rho \rho = \rho$$

for any $X, Y \in \mathfrak{g} \times \{0\} \subset \mathfrak{g} \times \mathbb{R}$, $\pi : (\mathfrak{g} \times \mathbb{R}) \ltimes_\nabla (\mathfrak{g} \times \mathbb{R})_a \to \mathfrak{g}$ the projection on the first factor. Denote $\rho_1 = \rho \oplus 0,\ \rho_2 = 0 \oplus \rho \in (\mathfrak{g} \times \mathbb{R}) \ltimes_\nabla (\mathfrak{g} \times \mathbb{R})_a$. Consider an almost complex structure I and a 2-form ω on $(\mathfrak{g} \times \mathbb{R}) \ltimes_\nabla (\mathfrak{g} \times \mathbb{R})_a$ defined by

$$I(X \oplus Y) = -Y \oplus X, \quad \omega(X,Y) = \pi^* g(IX,Y) - \pi^* g(X,IY),$$

for any $X, Y \in \mathfrak{g} \times \mathbb{R}$. Then for any $t \in \mathbb{R}^{>0}$

$$\left((\mathfrak{g} \times \mathbb{R}) \ltimes_\nabla (\mathfrak{g} \times \mathbb{R})_a, I, \omega_t = \omega + t\rho^1 \wedge \rho^2, -(1+ct)\rho_1^*\right)$$

is a l.c.K. Lie algebra. Moreover, if $1 + ct = 0$ then this algebra is Kähler.

If $1 + ct = 0$ then $\left(\mathfrak{g}, \nabla, g_t = g + t\,(\rho^*)^2\right)$ is a Hessian Lie algebra (see [24] or [28]). The construction of a Kähler Lie algebra from Theorem 1.4 arises from applying Theorem 1.2 to the Hessian Lie algebra $\left(\mathfrak{g}, \nabla, g_t = g + t\,(\rho^*)^2\right)$.

Corollary 1.5. Let G be an n-dimensional simply connected statistical Lie group of constant curvature c and θ the linear part of the corresponding affine representation of $G \times \mathbb{R}^{>0}$. Then there exists an étale affine representation $\mathfrak{g} \to \mathfrak{aff}\left(\mathbb{R}^{n+1}\right)$ and a left invariant l.c.K. structure on $\left(G \times \mathbb{R}^{>0}\right) \ltimes_\theta \mathbb{R}^{n+1}$. Moreover, if $c < 0$ then there exists a Kähler structure on $\left(G \times \mathbb{R}^{>0}\right) \ltimes_\theta \mathbb{R}^{n+1}$.

Further, we provide examples of statistical Lie algebras of constant curvature and apply the construction of l.c.K. Lie algebras to them. First, we consider examples of statistical Lie algebras of constant curvature called clans. Clans are a certain class of Lie algebras that can be constructed by homogeneous cones without full straight lines (see [30]). Second, we consider the Lie algebras $\mathfrak{so}(2)$ and $\mathfrak{su}(2)$.

2 Geometric structures on manifolds

2.1 Hessian and Kähler structures

Definition 2.1. A **flat affine manifold** is a differentiable manifold equipped with a flat, torsion-free connection. Equivalently, it is a manifold equipped with an atlas such that all transition maps between charts are affine transformations (see [18]).

Definition 2.2. A Riemannian metric g on a flat affine manifold (M, ∇) is called a **Hessian metric** if g is locally expressed by the Hessian of a function

$$g = \text{Hess } \varphi = \frac{\partial^2 \varphi}{\partial x^i \partial x^j} dx^i dx^j,$$

where $x^1, \ldots, x^n$ are flat local coordinates. Equivalently, g is Hessian if and only if the 3-tensor ∇g is totally symmetric. A **Hessian manifold** $(M, \nabla, g))$ is a flat affine manifold (M, ∇) endowed with a Hessian metric g. (see [28]).

Let U be an open chart on a flat affine manifold M, functions $x^1, \ldots, x^n$ be affine coordinates on U, and $x^1, \ldots, x^n, y^1, \ldots, y^n$ be the corresponding coordinates on TU. Define the complex structure I by $I(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$. The corresponding complex coordinates are given by $z^i = x^i + \sqrt{-1} y^i$. The complex structure I does not depend on the choice of flat coordinates on U. Thus, in this way, we get a complex structure on the TM.

Let $\pi : TM \to M$ be the natural projection. Consider a Riemannian metric g on M given locally by

$$g_{ij} dx^i dx^j.$$

Define a bilinear form g^{r} on TM by

$$g^{\mathrm{r}} = \pi^* g_{ij} \left(dx^i dx^j + dy^i dy^j \right)$$

or, equivalently,

$$g^{\mathrm{r}}(X, Y) = (\pi^* g)(X, Y) + (\pi^* g)(IX, IY), \tag{1}$$

for any $X, Y \in T(TM)$.

Proposition 2.3 ([28], [1])**.** Let M be a flat affine manifold, g and g^{r} as above. Then the following conditions are equivalent:

(i) g is a Hessian metric.

(ii) g^{r} is a Kähler metric.

Moreover, if $g = \text{Hess}\varphi$ locally then g^{r} is equal to a complex Hessian

$$g^{\mathrm{r}} = \text{Hess}_{\mathbb{C}}(4\pi^* \varphi) = \partial \bar{\partial}(4\pi^* \varphi).$$

Definition 2.4. The metric g^{r} is called the **Kähler metric associated to** g. The correspondence which associates the Kähler manifold (TM, g^{r}) to a Hessian manifold (M, g) is called the **(affine) r-map** (see [1]).

2.2 Hessian cones and statistical manifolds of constant non-zero curvature

Definition 2.5. A **radiant manifold** (C, ∇, ρ) is a flat affine manifold (C, ∇) endowed with a vector field ρ satisfying

$$\nabla \rho = \mathrm{Id}. \tag{2}$$

Equivalently, it is a manifold equipped with an atlas such that all transition maps between charts are linear transformations i.e. can be represented be elements of $\mathrm{GL}(\mathbb{R}^n)$ (see e.g. [20]).

Proposition 2.6 ([20])**.** Let s be the coordinate on $\mathbb{R}^{>0}$ and $(M \times \mathbb{R}^{>0}, \nabla, s\frac{\partial}{\partial s})$ a radiant manifold. Consider a natural action of $\mathbb{R}^{>0}$ on $M \times \mathbb{R}^{>0}$. Then the connection ∇ is $\mathbb{R}^{>0}$-invariant.

Definition 2.7. A **Hessian cone** is a Hessian manifold $(M \times \mathbb{R}^{>0}, \nabla, g = s^2 g_M + ds^2)$ such that there exists a constant $\lambda \neq 0, \frac{1}{2}$ such that $(M \times \mathbb{R}^{>0}, \nabla, \lambda s\frac{\partial}{\partial s})$ is a radiant manifold.

Proposition 2.8 ([19])**.** Let (C, ∇, ρ) be a radiant manifold and g a Hessian metric on M with respect to ∇. Then

$$\mathcal{L}_\rho g = g + \nabla(\iota_\rho g).$$

Proposition 2.9. Let $(M \times \mathbb{R}^{>0}, \nabla, g = s^2 g_M + ds^2)$ be a Hessian cone. Then we have

$$g = \mathrm{Hess}\left(\frac{\lambda s^2}{4\lambda - 2}\right). \tag{3}$$

Proof. Let $\rho = \lambda s\frac{\partial}{\partial s}$ be the radiant vector field. Then

$$\iota_\rho g = \iota_{\lambda s\frac{\partial}{\partial s}}\left(s^2 g_M + ds^2\right) = \lambda s ds = d\left(\frac{\lambda s^2}{2}\right).$$

Hence,

$$\mathrm{Hess}\left(\frac{\lambda s^2}{2}\right) = \nabla d\left(\frac{\lambda s^2}{2}\right) = \nabla \iota_\rho g.$$

Combining this with Proposition 2.8, we get

$$\mathcal{L}_\rho g - g = \mathrm{Hess}\left(\frac{\lambda s^2}{2}\right).$$

We have $\mathcal{L}_\rho g = \mathcal{L}_{\lambda r\frac{\partial}{\partial r}}\left(s^2 g_M + ds^2\right) = 2\lambda g$. Thus,

$$g = \mathrm{Hess}\left(\frac{\lambda s^2}{4\lambda - 2}\right).$$

∎

Definition 2.10. A **statistical manifold** (M, D, g) is a manifold M endowed with a torsion-free connection D and a Riemannian metric g such that the tensor Dg is totally symmetric. A statistical manifold (M, D, g) is said to be **of constant curvature** c if the curvature tensor Θ_D satisfies

$$\Theta_D(X, Y)Z = c\,(g(Y, Z)X - g(X, Z)Y)\,,$$

for any $X, Y, Z \in TM$.

Lemma 2.11. Let $(M \times \mathbb{R}^{>0}, \nabla, g = s^2 g_M + ds^2)$ be a Hessian cone and $\rho = \lambda s \frac{\partial}{\partial s}$ the radiant vector field. Then the following condition are satisfied:

(i) $(\nabla_X g)(Y, \rho) = (\nabla_\rho g)(X, Y) = (2\lambda - 2)\, g(X, Y)$.

(ii) $g(\nabla_X Y, \rho) = (1 - 2\lambda)\, g(X, Y)$

Proof. (i) Since g is a Hessian metric, the tensor ∇g is totally symmetric. Therefore,

$$(\nabla_X g)(Y, \rho) = (\nabla_\rho g)(X, Y) = \mathcal{L}_\rho\,(g(X, Y)) - g(\nabla_\rho X, Y) - g(X, \nabla_\rho Y). \tag{4}$$

Since $X, Y \in TM$, the value $g_M(X, Y)$ is constant along ρ. Hence,

$$\mathcal{L}_\rho\,(g(X, Y)) = \mathcal{L}_\rho\left(s^2 g_M(X, Y)\right) = 2\lambda\left(s^2 g_M(X, Y)\right) = 2\lambda\,(g(X, Y))\,.$$

Moreover, ρ commutes with X and Y. Hence, $\nabla_\rho X = \nabla_X \rho = X$ and $\nabla_\rho Y = \nabla_Y \rho = Y$. Combining this with (4), we get $(\nabla_X g)(Y, \rho) = (\nabla_\rho g)(X, Y) = (2\lambda - 2)\, g(X, Y)$.

(ii) We have

$$g(\nabla_X Y, \rho) = -(\nabla_X g)(Y, \rho) + \mathcal{L}_X\,(g(Y, \rho)) - g(Y, \nabla_X \rho) = -(\nabla_X g)(Y, \rho) - g(Y, X).$$

Combining this with the item (i), we obtain $g(\nabla_X Y, \rho) = (1 - 2\lambda)\, g(X, Y)$. ∎

Theorem 2.12. Let $(M \times \mathbb{R}^{>0}, \nabla, g = s^2 g_M + ds^2)$ be a Hessian cone, $\rho = \lambda s \frac{\partial}{\partial s}$ the radiant vector field, and $c = \frac{2\lambda - 1}{\lambda^2}$. Let us identify M with the submanifold $M \times 1 \subset M \times \mathbb{R}^{>0}$. Then for any $X, Y \in TM$, we have

$$\nabla_X Y = D_X Y - c g_M(X, Y)\rho, \tag{5}$$

where D is a torsion-free connection on M. Moreover, (M, g_M, D) is a statistical manifold of curvature c.

Conversely, if (M, g_M, D) is a statistical manifold of non-zero constant curvature $c \leq 1$, λ a solution of the equation $\frac{2\lambda - 1}{\lambda^2} = c$, and $\rho = \lambda s \frac{\partial}{\partial s}$ a field of on $M \times \mathbb{R}^{>0}$ then $(M \times \mathbb{R}^{>0}, \nabla, g = s^2 g_M + ds^2)$ is a Hessian cone, where the connection ∇ is defined by equation (5) and

$$\nabla_X \rho = \nabla_\rho X = X, \qquad \nabla_\rho \rho = \rho, \tag{6}$$

for any $X \in TM$.

Proof. Since ρ is orthogonal to M we have

$$\nabla_X Y = D_X Y + \frac{g(\nabla_X Y, \rho)}{g(\rho,\rho)}\rho = D_X Y + \frac{g(\nabla_X Y,\rho)}{\lambda^2 s^2}\rho.$$

where $D_X Y \in TM$. Combining this with item (ii) of Lemma 2.11, we get that

$$\nabla_X Y = D_X Y + \frac{g(\nabla_X Y,\rho)}{\lambda^2 s^2}\rho = D_X Y + \frac{1-2\lambda}{\lambda^2 s^2} g(X,Y)\rho.$$

Combining this with $g(X,Y) = s^2 g_M(X,Y)$ we get (5). The term $g(*,*)\rho$ is 2-1 tensor and ∇ is a connection. Therefore, D is a connection.

The connection ∇ is flat hence for any $X,Y,Z \in TM$ we have

$$\Theta_\nabla(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = 0.$$

Combining this with (5) and $\nabla\rho = \mathrm{Id}$ we get

$$\Theta_\nabla(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = \Theta_D(X,Y)Z - c\left(g_M(Y,Z)X - g_M(X,Z)Y\right) + \\ - c\left(\mathcal{L}_Y\left(g_M(X,Z)\right) - \mathcal{L}_X\left(g_M(Y,Z)\right) + g_M(Y,D_X Z) - g_M(X,D_Y Z) + g_M([X,Y],Z)\right)\rho.$$

Since $X,Y,Z \in TM$ and $\Theta_\nabla = 0$, we get that

$$\Theta_D(X,Y)Z = c\left(g_M(Y,Z)X - g_M(X,Z)Y\right)$$

and

$$\mathcal{L}_X\left(g_M(Y,Z)\right) - \mathcal{L}_Y\left(g_M(X,Z)\right) + g_M(X,D_Y Z) - g_M(Y,D_X Z) - g_M([X,Y],Z) = 0.$$

Combining the last equation with the identity $[X,Y] = D_X Y - D_Y X$ and the formula of covariant derivative of a metric we get that

$$\left(D_X g\right)(Y,Z) - \left(D_Y g\right)(X,Z) = 0.$$

Thus, the tensor Dg is totally symmetric. We proved the first part of the theorem.

Now, let (M, g_M, D) be a statistical manifold of curvature c. Then ∇ is flat by the same calculation as above (in the opposite direction). Thus, it is enough to check that the metric g is Hessian.

For any $X,Y,Z \in TM$ we have

$$\left(\nabla_X g\right)(Y,Z) = \mathcal{L}_X\left(g(Y,Z)\right) - g\left(\nabla_X Y, Z\right) - g\left(Y, \nabla_X Z\right) = \\ = \mathcal{L}_X\left(g(Y,Z)\right) - g\left(D_X Y, Z\right) - g\left(Y, D_X Z\right) = \left(D_X g\right)(Y,Z)$$

Combining this with $\left(D_X g\right)(Y,Z) - \left(D_Y g\right)(X,Z) = 0$, we get

$$\left(\nabla_X g\right)(Y,Z) = \left(\nabla_Y g\right)(X,Z).$$

We have,

$$\left(\nabla_\rho g\right)(X,Y) = \mathcal{L}_\rho\left(g(X,Y)\right) - g\left(\nabla_\rho X, Y\right) - g\left(X, \nabla_\rho Y\right) = (2a-2)g(X,Y)$$

and

$$(\nabla_X g)(\rho, Y) = \mathcal{L}_X(g(\rho, Y)) - g(\nabla_X \rho, Y) - g(\rho, \nabla_X Y) = \\ = g(X,Y) - \frac{1-2a}{a^2} g(\rho,\rho) g_M(X,Y) = (2a-2) g(X,Y).$$

Thus,

$$(\nabla_\rho g)(X, Y) = (\nabla_X g)(\rho, Y).$$

Finally,

$$(\nabla_\rho g)(\rho, X) = \mathcal{L}_\rho(g(\rho, X)) - g(\nabla_\rho \rho, X) - g(\rho, \nabla_\rho X) = 0$$

and

$$(\nabla_X g)(\rho, \rho) = \mathcal{L}_X(g(\rho, \rho)) - g(\nabla_X \rho, \rho) - g(\rho, \nabla_X \rho) = 0$$

We checked that the tensor ∇g is totally symmetric. Therefore, the metric g is Hessian. □

Corollary 2.13. Any statistical manifold of a non-zero constant curvature can be realised as a level set of a Hessian potential on a Hessian cone.

Proof. According to Theorem 2.12, a statistical manifold of a non-zero constant curvature can be realised as a level of the function s on a Hessian cone $(M \times \mathbb{R}^{>0}, \nabla, g = s^2 g_M + ds^2)$. It follows from Proposition 2.9, that the level sets of s coincide with the level sets of a Hessian potential. □

2.3 Statistical manifolds of a non-zero constant curvature and Sasakian manifolds

Definition 2.14. A **Sasakian manifold** is a Riemannian manifold (M, g_M) such that the cone metric $s^2 g_M + ds^2$ on $M \times \mathbb{R}^{>0}$ is Kähler with respect to a dilation invariant complex structure.

Proposition 2.15 ([27])**.** Let $(M \times \mathbb{R}^{>0}, g, I)$ be a Kähler manifold. For any $q \in \mathbb{R}^{>0}$ consider the map $\mu_q : M \times \mathbb{R}^{>0} \to M \times \mathbb{R}^{>0}$ defined by $\mu_q(m, s) = (m, qs)$. If $\mu_q^* g = q^2 g$ then $g = s^2 g_M + ds^2$ and (M, g_M) is a Sasakian manifold.

There exists a decomposition

$$T(M \times \mathbb{R}^{>0}) = TM \times T\mathbb{R}^{>0} = TM \times \mathbb{R} \times \mathbb{R}^{>0}.$$

If $M \times \mathbb{R}^{>0}$ possesses a Hessian structure then, according to Proposition 2.3, $T(M \times \mathbb{R}^{>0})$ admits a Kähler structure.

Proposition 2.16. Let $(M \times \mathbb{R}^{>0}, \nabla, g)$ be a Hessian cone and g^{r} the metric constructed by the r-map on $T(M \times \mathbb{R}^{>0})$. Consider $T(M \times \mathbb{R}^{>0}) = (TM \times \mathbb{R}) \times \mathbb{R}^{>0}$ as a cone over $TM \times \mathbb{R}$. Then for any $q \in \mathbb{R}^{>0}$ we have $\eta_q^* g = q^2 g$, where the map $\eta_q : (TM \times \mathbb{R}) \times \mathbb{R}^{>0} \to (TM \times \mathbb{R}) \times \mathbb{R}^{>0}$ is defined by $\mu_q(m, s, t) = (m, s, qt)$.

Proof. We have the commutative diagram

$$\begin{array}{ccc} T(M\times\mathbb{R}^{>0}) & \xrightarrow{\eta_q} & T(M\times\mathbb{R}^{>0}) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ M\times\mathbb{R}^{>0} & \xrightarrow{\mu_q} & M\times\mathbb{R}^{>0} \end{array},$$

where η_q and μ_q are multiplications of the coordinate on $\mathbb{R}^{>0}$ by q. By the definition of g^r we have

$$g^{\mathrm{r}}(X,Y)=\pi^*g(X,Y)+\pi^*g(IX,IY). \tag{7}$$

Since the diagram is commutative, it follows that

$$\eta_q^*\pi^*=\pi^*\mu_q^*. \tag{8}$$

Moreover, g is a cone metric. Hence,

$$\mu_q^*g=q^2g. \tag{9}$$

It follows from (7), (8), and (9) that

$$\eta_q^*g^{\mathrm{r}}(X,Y)=q^2g^{\mathrm{r}}(X,Y).$$

□

Theorem 2.17. Let (M,g_M,D) be a statistical manifold of a non-zero constant curvature. Then $TM\times\mathbb{R}$ admits a structure of a Sasakian manifold.

Proof. If the curvature of (M,g_M,D) is $c>1$ then $(M,\frac{1}{c}g_M,D)$ is a statistical manifold of curvature 1. Thus, we can assume that the curvature of (M,g_M,D) does not exceed 1. By Theorem 2.12, there exists a Hessian cone $(M\times\mathbb{R}^{>0},\nabla,g=r^2g_M+dr^2)$. Then the r-map defines a Kähler structure (g^{r},I) on $T(M\times\mathbb{R}^{>0})=TM\times\mathbb{R}\times\mathbb{R}^{>0}$. By Proposition 2.6, the connection ∇ is $\mathbb{R}^{>0}$ invariant. Therefore, the complex structure I constructed by ∇ is $\mathbb{R}^{>0}$-invariant. Let us identify $TM\times\mathbb{R}$ with $TM\times\mathbb{R}\times 1\subset TM\times\mathbb{R}\times\mathbb{R}^{>0}$. Combining propositions 2.15 and 2.16, we get that $(TM\times\mathbb{R},g^{\mathrm{r}}|_{TM\times\mathbb{R}})$ is a Sasakian manifold. □

3 Geometric structures on Lie groups and algebras

3.1 Flat torsion free connections and complex structure

The group of affine transformations $\mathrm{Aff}(\mathbb{R}^n)$ is given by the matrices of the form

$$\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}\in\mathrm{GL}(\mathbb{R}^{n+1}),$$

where $A\in\mathrm{GL}(\mathbb{R}^n)$, and $a\in\mathbb{R}^n$ is a column vector. The corresponding Lie algebra $\mathfrak{aff}(\mathbb{R}^n)$ is given by matrices of the form

$$\begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix}\in\mathfrak{gl}(\mathbb{R}^{n+1}).$$

The commutator of $\mathfrak{aff}(\mathbb{R}^n)$ is equal to

$$\left[\begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & b \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} [A,B] & A(b)-B(a) \\ 0 & 0 \end{pmatrix}.$$

The algebra $\mathfrak{aff}(\mathbb{R}^n)$ is the semidirect product $\mathfrak{gl}(\mathbb{R}^n) \ltimes \mathbb{R}^n$, where the commutator is given by

$$[(A,a),(B,b)] = ([A,B], Ab - Ba).$$

The group $\mathrm{Aff}(\mathbb{R}^n)$ is the semidirect product $\mathrm{GL}(\mathbb{R}^n) \ltimes \mathbb{R}^n$, where multiplication is given by

$$(A,a)(B,b) = (AB, a + Ab).$$

Definition 3.1. An affine representation $G \to \mathrm{Aff}(\mathbb{R}^n)$ is called **étale** if there exists a point $x \in \mathbb{R}^n$ such that the orbit of x is open and the stabilizer of x is discrete. A representation of a Lie algebra $\mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n)$ is called **étale** if the corresponding representation of the simply connected Lie group is étale.

Theorem 3.2 ([10] or [11])**.** Let G be an n-dimensional Lie group and $\mathfrak{g}$ the corresponding Lie algebra. Choose an identification $i : \mathfrak{g} \to \mathbb{R}^n$. Then there exists a 1-1 correspondence between left invariant flat torsion-free connections and étale affine representations. Moreover, if $\theta : G \to \mathrm{GL}(\mathbb{R}^n)$ is the linear part of the étale affine representation corresponding to a left invariant flat torsion-free connection ∇ then the differential $\eta : \mathfrak{g} \to \mathfrak{gl}(\mathbb{R}^n)$ is defined by $X \to i \circ \nabla_X \circ i^{-1}$.

Let ∇ be a flat connection on a Lie algebra $\mathfrak{g}$. Denote by $\mathfrak{g}_a$ the vector space $\mathfrak{g}$ with the structure of an abelian Lie algebra. Look on ∇ as a Lie algebras representation $\nabla : \mathfrak{g} \to \mathfrak{gl}(\mathbb{R}^n)$, $X \to \nabla_X$. Consider the semidirect product $\mathfrak{g} \ltimes_\nabla \mathfrak{g}_a$ that is the vector space $\mathfrak{g} \oplus \mathfrak{g}$ with the Lie bracket defined by

$$[X_1 \oplus Y_1, X_2 \oplus Y_2] = [X_1, X_2] \oplus (\nabla_{X_1} Y_2 - \nabla_{X_2} Y_1). \tag{10}$$

The flatness of ∇ is equivalent to the Jacobi identity on $\mathfrak{g} \ltimes_\nabla \mathfrak{g}_a$. Define an almost complex structure on $\mathfrak{g} \ltimes_\nabla \mathfrak{g}_a$ by the rule

$$I(X_1 \oplus X_2) = -X_2 \oplus X_1.$$

Theorem 3.3 ([14] or [8])**.** Let ∇ be a flat connection on a Lie algebra $\mathfrak{g}$, and I be an almost complex structure on $\mathfrak{g} \ltimes_\nabla \mathfrak{g}_a$. Then the almost complex structure I is integrable if and only if ∇ is torsion free.

Proposition 3.4. Let ∇ be a flat torsion free connection on a Lie algebra $\mathfrak{g}$ and $\theta : G \to \mathrm{GL}(\mathbb{R}^n)$ the linear part corresponding to an étale affine representation. Then the Lie algebra of left invariant fields on $G \ltimes_\theta \mathbb{R}^n$ equals $\mathfrak{g} \ltimes_\nabla \mathfrak{g}_a$.

Proof. Choose an identification $i : \mathfrak{g} \to \mathbb{R}^n$. According to Theorem 3.2, the differential of θ is defined by $\eta(X) = i \circ \nabla_X \circ i^{-1}$. Then the corresponding Lie algebra $g \ltimes_\eta \mathbb{R}^n$ is isomorphic to $\mathfrak{g} \ltimes_\nabla \mathfrak{g}_a$. $\square$

Corollary 3.5. Let G be a simply connected Lie group equipped with a left invariant affine structure, $\mathfrak{g}$ the corresponding Lie algebra, and θ the linear part of the corresponding affine action of G. Then there exists a left invariant integrable complex structure on the group $G \ltimes_\theta \mathbb{R}^n$.

Proof. The existence of a left invariant integrable complex structure follows from Theorem 3.3 and Proposition 3.4. ∎

3.2 Hessian and Kähler structures

Definition 3.6. A Hessian Lie group (G, ∇, g) is a Lie group G endowed with a left invariant affine structure ∇ and a left invariant Hessian metric g.

Definition 3.7. A **Hessian Lie algebra** $(\mathfrak{g}, \nabla, g)$ is a Lie algebra $\mathfrak{g}$ endowed with a flat torsion free connection ∇ and symmetric bilinear form g such that ∇g is totally linear i.e. for any $X, Y, Z \in \mathfrak{g}$ we have

$$g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = g(\nabla_Y X, Z) + g(X, \nabla_Y Z).$$

A Lie l.c.K. ts a left invariant Hessian structure if and only if the corresponding Lie algebra admits a Hessian structure.

Proof of Theorem 1.2. According to Theorem 3.3 the almost complex structure I is integrable. The bilinear form

$$g^{\mathrm{r}}(X, Y) = \omega(X, IY) = \pi^* g(X, Y) + \pi^* g(IX, IY)$$

is positive definite. Hence, it is enough to check that the form ω is closed. For any $X_1, X_2 \in \mathfrak{g} \oplus 0$ we have

$$\omega(X_1 \oplus 0, X_2 \oplus 0) = \omega(0 \oplus X_1, 0 \oplus X_2) = 0, \tag{11}$$

Combining this with the formula of the exterior derivative

$$d\omega(V_1, V_2, V_3) = -\omega([V_1, V_2], V_3) + \omega([V_1, V_3], V_2) - \omega([V_2, V_3], V_1) \tag{12}$$

and the definition of the Lie bracket (10)

$$d\omega(X \oplus 0, Y \oplus 0, Z \oplus 0) = 0, \quad d\omega(X \oplus 0, 0 \oplus Y, 0 \oplus Z) = 0, \quad d\omega(0 \oplus X, 0 \oplus Y, 0 \oplus Z) = 0.$$

and

$$\begin{aligned} d\omega(X \oplus 0, Y \oplus 0, 0 \oplus Z) &= -g([X, Y], Z) - g(Y, \nabla_X Z) + g(X, \nabla_Y Z) = \\ = -g(\nabla_X Y, Z) - g(Y, \nabla_X Z) + g(\nabla_Y X, Z) + g(X, \nabla_Y Z) &= -\left(\nabla_X g\right)(Y, Z) + \left(\nabla_Y g\right)(X, Z) = 0. \end{aligned}$$

∎

Corollary 1.3 follows from Proposition 3.4 and Theorem 1.2.

3.3 Statistical structures of a non-zero constant curvature and l.c.K. structures

Definition 3.8. A **statistical Lie algebra** $(\mathfrak{g}, g, D)$ is a Lie algebra endowed with a bilinear symmetric positive-definite form g and a torsion-free connection and D such that Dg is a totally symmetric tensor. A statistical Lie algebra $(\mathfrak{g}, g, D)$ is said to be **of constant curvature** c if the curvature tensor equals

$$\Theta_D(X, Y)Z = c\,(g(Y, Z)X - g(X, Z)Y)\,,$$

for any $X, Y, Z \in \mathfrak{g}$.

Definition 3.9. A **statistical Lie group (of constant curvature** c**)** (G, g, D) is a Lie group endowed with a left invariant statistical structure (of constant curvature c).

Obviously, there exists a one-to-one correspondence between simply connected statistical Lie groups (of constant curvature c) and statistical Lie algebras (of constant curvature c).

Definition 3.10. A **locally conformally Kähler (l.c.K.) manifold (Lie algebra)** is a manifold (Lie algebra) endowed with an integrable almost complex structure I, closed 1-form θ, and 2-form ω such that

$$d\omega = \theta \wedge \omega.$$

Proof of Theorem 1.4. The bilinear form $g^{\mathrm{r}}_t = \omega_t(\cdot, I\cdot)$ equals $g^{\mathrm{r}} + t\left(\left(\rho^1\right)^2 + \left(\rho^1\right)^2\right)$ where g^{r} is defined by

$$g(X, Y) = \pi^* g(X, Y) + \pi^* g(IX, IY).$$

For $t > 0$ the form g^{r}_t is positive definite.

The same computations as in the proof of Theorem 1.2 show that for any $X, Y, Z \in \mathfrak{g}$ we have

$$\omega_t(X\oplus 0, Y\oplus 0, Z\oplus 0) = \omega_t(X\oplus 0, Y\oplus 0, 0\oplus Z) = \omega_t(X\oplus 0, 0\oplus Y, 0\oplus Z) = \omega_t(0\oplus X, 0\oplus Y, 0\oplus Z) = 0. \tag{13}$$

Combining the formula of the exterior derivative (12) with (11) and the definition of the Lie bracket (10) we get that for any $X, Y \in \mathfrak{g}$

$$d\omega_t(0 \oplus X, 0 \oplus Y, 0 \oplus \rho) = 0, \quad d\omega_t(X \oplus 0, 0 \oplus Y, 0 \oplus \rho) = 0,$$

and

$$d\omega_t(X \oplus 0, Y \oplus 0, 0 \oplus \rho) = -\omega_t([X, Y] \oplus 0, 0 \oplus \rho) + \omega_t(0 \oplus X, Y \oplus 0) - \omega_t(0 \oplus Y, X \oplus 0) = 0.$$

Since $[X, Y] \in \mathfrak{g}$, we have $\omega([X, Y] \oplus 0, 0 \oplus \rho) = 0$. It follows from the definition of ω_t that

$$\omega_t(0 \oplus X, Y \oplus 0) = g(X, Y) = \omega_t(0 \oplus Y, X \oplus 0).$$

Therefore,

$$d\omega_t(X \oplus 0, Y \oplus 0, 0 \oplus \rho) = 0.$$

Also, using (11), (12), and (10) we get that for any $X, Y \in \mathfrak{g}$

$$d\omega_t(X \oplus 0, Y \oplus 0, \rho \oplus 0) = 0, \quad d\omega_t(0 \oplus X, 0 \oplus Y, \rho \oplus 0) = 0$$

and

$$g(X \oplus 0, 0 \oplus Y, \rho \oplus 0) = -\omega_t(0 \oplus \nabla_X Y, \rho \oplus 0) - \omega_t(X \oplus 0, 0 \oplus Y).$$

We have

$$\omega_t(0 \oplus \nabla_X Y, \rho \oplus 0) = \omega_t(0 \oplus D_X Y - cg(X,Y), \rho \oplus 0) = \omega_t(0 \oplus -cg(X,Y)\rho, \rho \oplus 0).$$

Combining this with the identity $\omega_t = \omega + t\rho^1 \wedge \rho^2$ and the definition of ω_t we get

$$\omega_t(0 \oplus \nabla_X Y, \rho \oplus 0) = tcg(X,Y) = -tc\omega_t(X \oplus 0, 0 \oplus Y).$$

Thus,

$$\omega(0 \oplus \nabla_X Y, \rho \oplus 0) = -(1+tc)g(X \oplus 0, 0 \oplus Y).$$

As above, using (11), (12), and (10) we get that for any $X \in \mathfrak{g}$, we have

$$\omega_t(X \oplus 0, \rho \oplus 0, 0 \oplus \rho) = 0, \quad \omega_t(0 \oplus X, \rho \oplus 0, 0 \oplus \rho) = 0.$$

We checked that

$$d\omega_t = -(1+tc)\rho^1 \wedge \omega_t.$$

If the $1 + tc = 0$. Then the form ω_t is Kähler.

□

Corollary 1.5 follows from Proposition 3.4 and Theorem 1.4.

3.4 Examples

3.4.1 Convex regular cones and clans

Definition 3.11. A subset $V \subset \mathbb{R}^n$ is called **regular** if V does not contain any straight full line.

Let $V \subset \mathbb{R}^n$ be a convex regular domain. We denote the maximal subgroup of $\mathrm{GL}(\mathbb{R}^n)$ preserving V by $\mathrm{Aut}(V)$. Note that if V is a regular convex cone then

$$\mathrm{Aut}(V) = (\mathrm{Aut}(V) \cap \mathrm{SL}(\mathbb{R}^n)) \times \mathbb{R}^{>0}.$$

The following theorem summarized known results.

Theorem 3.12 ([30], [28])**.** Let $V \subset \mathbb{R}^n$ be a homogeneous convex cone. Then there exists a function $\varphi : V \to \mathbb{R}$, a subgroup $T \subset \mathrm{Aut}(V)$ satisfying the following conditions.

(i) T acts on V simply transitively.

(ii) The bilinear form $g_{con} = \mathrm{Hess}\ (\ln \varphi)$ is a T-invariant Hessian metric.

(iii) The group $T_{\mathrm{SL}} = T \cap \mathrm{SL}(\mathbb{R}^n)$ preserves the hypersurface $M = \{\varphi = 1\}$ and acts simply transitively on it.

(iv) Let $\pi : TV|_M \to TM$ be the projection along the radiant vector field and ∇ be the standard connection on $\mathbb{R}^n$. Then (D, g_{con}) is a statistical structure of constant curvature on M, where D is defined by $D_X Y = \pi(\nabla_X Y)$.

Definition 3.13. The function φ and the hypersurface M from Theorem 3.12 are called the **characteristic function** and the **characteristic hypersurface** of the cone.

Definition 3.14. Let $V \subset \mathbb{R}^n$ be a homogeneous convex cone and a subgroup $T_{\mathrm{SL}} \subset \mathrm{Aut}(V)$ acts on V simply transitive. The corresponding Lie algebra $\mathfrak{t}$ is called a **clan**. There exists a purely algebraic definition of clans (see [30]).

By Theorem 3.12, a characteristic hypersurface M admits a T_{SL}-invariant statistical structure of constant curvature. Thus, any clan admits a statistical structure of constant curvature.

Example 3.15. Let V be the vector space of all real symmetric matrices of rank n and Ω the set of all positive definite symmetric matrices in V. Then Ω is a regular convex cone and the group of upper triangular matrix $\mathrm{T}(\mathbb{R}^n)$ acts simply transitively on Ω by $s(x) = sxs^T$, where $x \in \Omega$ and $s \in \mathrm{T}(\mathbb{R}^n)$. The characteristic function is equal to

$$\varphi(x) = (\det x)^{-\frac{n+1}{2}} \varphi(e),$$

where e is the unit matrix (see [28]). The corresponding clan $\mathfrak{t}$ is the algebra of upper triangular traceless matrices.

Consider the case $n = 3$. Here, $\mathfrak{t}$ is generated by elements

$$u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

and the relation $[u, v] = 2v$. The statistical structure (D, g) of costant curvature $-c < 0$ is defined by

$$D_u u = D_u v = 0, \quad D_v u = -2v, \quad D_v v = u, \quad g_{\mathfrak{g}} = \frac{4\,(u^*)^2 + 2\,(v^*)^2}{c}$$

(see [28]).

Then the corresponding l.c.K. Lie algebra admits generators $u_1, v_1, \rho_1, u_2, v_2, \rho_2$ and relations

$$[u_1, u_2] = -4\rho_2, \qquad [v_1, v_2] = -2\rho, \qquad [\rho_1, \rho_2] = \rho_2,$$

$$[u_1, v_1] = 2v_1, \quad [v_1, u_2] = -2v_2, \quad [v_1, v_2] = u_2, \quad [\rho_1, u_2] = u_2, \quad [\rho_1, v_2] = v_2.$$

The complex structure is defined by

$$I(u_1) = u_2, \quad I(v_1) = v_2, \quad I(\rho_1) = \rho_2.$$

For any $c, t \in \mathbb{R}^{>0}$ we have the following l.c.K. form

$$\omega_{c,t} = \frac{4}{c} u^1 \wedge u^2 + \frac{2}{c} v^1 \wedge v^2 + t\rho^1 \wedge \rho^2.$$

In particular, the form $\omega_{1,1} = 4u^1 \wedge u^2 + 2v^1 \wedge v^2 + \rho^1 \wedge \rho^2$ is Kähler.

3.4.2 $\mathfrak{so}(2)$ and $\mathfrak{su}(2)$

Example 3.16. Consider the group $\mathbb{R}$ as the universal covering of $\mathrm{U}(1) = \mathrm{SO}(2)$. The identification $\mathrm{SO}(2) \times \mathbb{R}^{>0} \simeq \mathbb{R}^2 \backslash \{0\}$ sets a Hessian structure (∇, g) on $\mathbb{R} \times \mathbb{R}^{>0}$ such that $\lambda_q^* g = q^2 g$, where $\lambda_q : \mathbb{R} \times \mathbb{R}^{>0} \to \mathbb{R} \times \mathbb{R}^{>0}$, $\lambda_q (x \times s) = x \times qs$. By the same way as in the proof of Theorem 1.5, we can define a l.c.K. structure on the group of homothetic motions of the plane $\mathrm{H}(2) = \left(\mathbb{R} \times \mathbb{R}^{>0}\right) \ltimes \mathbb{R}^2$. The group $\left(\mathrm{SO}(2) \times \mathbb{R}^{>0}\right) \ltimes \mathbb{R}^2$ is equal to the group of homothetic motions of the plane $\mathrm{H}(\mathbb{R}^2)$. Thus, we get a l.c.K. structure on the universal covering $\widetilde{\mathrm{H}(\mathbb{R}^2)}$.

The corresponding Lie algebra defined by generators v_1, ρ_1, v_2, ρ_2 and relations

$$[v_1, v_2] = -\rho_2, \quad [\rho_1, v_2] = v_2, \quad [\rho_1, \rho_2] = \rho_2.$$

The complex structure is defined by

$$I(v_1) = v_2, \quad I(\rho_1) = \rho_2.$$

For any $c, t \in \mathbb{R}^{>0}$ we have the following l.c.K. form

$$\omega_{c,t} = \frac{1}{c} v^1 \wedge v^2 + t\rho^1 \wedge \rho^2$$

In particular the form $\omega_{1,1} = v^1 \wedge v^2 + \rho^1 \wedge \rho^2$ is Kähler.

Example 3.17. There exists an identifications $\mathrm{SU}(2) \simeq S^3$ and a homogeneous statistical structure of curvature 1 on S^3. The corresponding l.c.K. Lie group $\left(\mathrm{SU}(2) \times \mathbb{R}^{>0}\right) \ltimes \mathbb{C}^2$ is equal to the group of homothetic complex motions $\mathrm{H}\left(\mathbb{C}^2\right)$.

The algebra $\mathfrak{su}(2)$ is defined by generators u, v, w and relations

$$[u, v] = 2w, \quad [v, w] = 2u, \quad [w, u] = 2v.$$

The statistical structure (g, D) of constant curvature 1 on $\mathfrak{su}(2)$ is defined by

$$D_u u = D_v v = D_w w = 0, \quad D_u v = w, \quad D_v w = u, \quad D_w u = v$$

and

$$g = (u^*)^2 + (v^*)^2 + (w^*)^2 .$$

The corresponding l.c.K. Lie algebra is defined by generators $u_1, v_1, w_1, \rho_1, u_2, v_2, w_2, \rho$ and relations

$$[u_1, v_1] = 2w_1, \quad [v_1, w_1] = 2u_1, \quad [w_1, u_1] = 2v_1, \quad [u_1, v_2] = w_2, \quad [v_1, w_2] = u_2, \quad [w_1, u_2] = v_2,$$

$$[u_1, u_2] = [v_1, v_2] = [w_1, w_2] = -\rho_2, \quad [\rho_1, v_2] = v_2, \quad [\rho_1, u_2] = u_2, \quad [\rho_1, v_2] = v_2, \quad [\rho_1, \rho_2] = \rho_2.$$

For any $c, t \in \mathbb{R}^{>0}$ we have the following l.c.K. form

$$\omega_{c,t} = \frac{1}{c}\left(u^1 \wedge u^2 + v^1 \wedge v^2 + w^1 \wedge w^2\right) + t\rho^1 \wedge \rho^2.$$

In particular, the form $\omega_{1,1} = u^1 \wedge u^2 + v^1 \wedge v^2 + w^1 \wedge w^2 + \rho^1 \wedge \rho^2$ is Kähler.

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