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The Osofsky-Smith Theorem in rings, modules, categories, and lattices
by

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Dedicated to Professor Ioan Tomescu on his 80th birthday

Abstract

In this survey paper we present various extensions of the renown Osofsky-Smith Theorem from Ring and Module Theory to modular lattices, Grothendieck categories, and module categories equipped with a hereditary torsion theory.

Key Words: Modular lattice, upper continuous lattice, complement, pseudo-complement, closed element, pseudo-complemented lattice, CS module, CC lattice, the Osofsky-Smith Theorem, the Latticial Osofsky-Smith Theorem, the Categorical Osofsky-Smith Theorem, the Relative Osofsky-Smith Theorem, hereditary torsion theory, Grothendieck category.
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Introduction

The renown *Osofsky-Smith Theorem* (O-ST) [26], invented in 1991, says that a cyclic (finitely generated) right R-module such that all of its cyclic (finitely generated) subfactors are CS modules is a finite direct sum of uniform submodules.

In this survey paper we present various extensions of the O-ST to Grothendieck categories (the *Categorical O-ST*), module categories equipped with a hereditary torsion theory (the *Relative O-ST*), and modular lattices (the *Latticial O-ST*); they also illustrate a general strategy which consists on putting a module-theoretical concept/result into a latticial frame (we call it *latticization*) in order to translate that concept/result to Grothendieck categories (we call it *absolutization*) and to module categories equipped with a hereditary torsion theory (we call it *relativization*).

In Section 1 we list some definitions and results about lattices from [5] and [28], that will be used throughout this paper. Section 2 presents the Latticial O-ST. Sections 3 and 4 are devoted to apply the Latticial O-ST to Grothendieck categories and module categories equipped with a hereditary torsion theory, respectively, in order to deduce at once the Categorical O-ST and the Relative O-ST, respectively.

1 Lattice background

In this section we present some known definitions and results on lattices that are needed to make the paper as self-contained as possible. For all other undefined notation and terminology on lattices, the reader is referred to [5], [12], [14], [21], and/or [28].

Basic concepts

All partially ordered sets (more briefly, posets), and in particular, all lattices considered in this paper are assumed to be *bounded,* i.e., to have a least element denoted by 0 and a greatest element denoted by 1, and L will always denote such a lattice. If the lattices L and L' are isomorphic, we denote this by $L \simeq L'$. We shall use $\mathbb{N}$ to denote the set $\{1, 2, \ldots\}$ of all positive integers.

For a lattice L and elements $a \leqslant b$ in L we write

$$b/a := [a, b] = \{\ x \in L \mid a \leqslant x \leqslant b\ \}.$$

A *subfactor* of L is any interval b/a of L with $a \leqslant b$.

We shall denote by $\mathcal{L}$ the class of all (bounded) lattices and by $\mathcal{M}$ the class of all (bounded) modular lattices.

An element $c \in L$ is a *complement in* L if there exists an element $a \in L$ such that $a \wedge c = 0$ and $a \vee c = 1$, and in this case we say that c is a *complement* of a *in* L. By $D(L)$ we denote the set of all complements of L. A *complement interval* of L is any interval $d/0$ of L with $d \in D(L)$. The lattice L is called *indecomposable* (respectively, *complemented*) if $L \neq \{0\}$ and $D(L) = \{0, 1\}$ (respectively, if every element of L has a complement in L). An element $a \in L$ is said to be *indecomposable* if‘ $a/0$ is an indecomposable lattice.

For a lattice L and $a, b, c \in L$, the notation $a = b \overset{\cdot}{\vee} c$ means that $a = b \vee c$ and $b \wedge c = 0$, and we say then that a is a *direct join* of b and c. More generally, for a non-empty subset S of L, we use the *direct join* notation $a = \overset{\cdot}{\bigvee}_{b \in S} b$ if S is an independent subset of L and $a = \bigvee_{b \in S} b$. Recall that a non-empty subset S of L is called *independent* if $0 \notin S$, and for every $x \in S$, $n \in \mathbb{N}$, and subset $T = \{t_1, \ldots, t_n\}$ of S with $x \notin T$, one has $x \wedge (t_1 \vee \cdots \vee t_n) = 0$. Alternatively, a family $(x_i)_{i \in I}$ of elements of a complete lattice L is *independent* if $x_i \neq 0$ and $x_i \wedge (\bigvee_{j \in I \setminus \{i\}} x_j) = 0, \forall\, i \in I$, and then, necessarily $x_p \neq x_q$ for each $p \neq q$ in I.

An element $b \in L$ is a *pseudo-complement in* L if there exists an element $a \in L$ such that $a \wedge b = 0$ and b is maximal with this property; in this case we say that b is a *pseudo-complement* of a. By $P(L)$ we denote the set of all pseudo-complement elements of L.

As in [5], L is called *pseudo-complemented* if every element of L has a pseudo-complement, and *strongly pseudo-complemented* if for all $a, b \in L$ with $a \wedge b = 0$, there exists a pseudo-complement p of a in L such that $b \leqslant p$. Any upper continuous modular lattice L is strongly pseudo-complemented.

An element $e \in L$ is *essential in* L if $e \wedge x \neq 0$ for every $x \neq 0$ in L. One denotes by $E(L)$ the set of all essential elements of L. The lattice L is called *uniform* if $L \neq \{0\}$ and $x \wedge y \neq 0$ for every non-zero elements $x, y \in L$. An element u of L is called *uniform* if the interval $u/0$ of L is a uniform lattice. As in [5], L is called *E-complemented* (“E” for essential) if for each $a \in L$ there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b \in E(L)$.

An element $c \in L$ is said to be *closed* if $c \notin E(a/0)$ for all $a \in L$ with $c < a$. One denotes by $C(L)$ the set of all closed elements of L. For an element $a \in L$, we say that $c \in L$ is a *closure* of a in L if $a \in E(c/0)$ and $c \in C(L)$. A lattice L is said to be *essentially closed* if for each element $a \in L$ there exists a closure of a in L. Every strongly pseudo-complemented lattice (hence every upper continuous) modular lattice is essentially closed by Theorem [5, Theorem 1.2.24].

An element $c \in L$ is called *compact in L* if whenever $c \leqslant \bigvee_{x \in A} x$ for a subset A of L, there is a finite subset F of A such that $c \leqslant \bigvee_{x \in F} x$, and $K(L)$ will denote the set of all compact elements of L. The lattice L is said to be *compact* if 1 is a compact element in L, and *compactly generated* if it is complete and every element of L is a join of compact elements.

A lattice L is called *simple* if it has exactly two elements, so, L is simple if $L = \{0, 1\}$ and $0 \neq 1$. An element $a \in L$ is said to be an *atom* if $a \neq 0$ and $a/0 = \{0, a\}$, i.e., $a/0$ is a simple lattice. We denote by $A(L)$ the set, possibly empty, of all atoms of L. The *socle* $\mathrm{Soc}(L)$ of a complete lattice L is the join of all atoms of L, i.e., $\mathrm{Soc}\,(L) := \bigvee A(L)$; if L has no atoms, then $\mathrm{Soc}\,(L) = 0$. As in [24], a lattice L is called *semi-atomic* if 1 is a join of atoms of L.

2 CC lattices

The aim of this section is to discuss CC lattices and CEK lattices that are needed in the next section. The CC lattices have been introduced in [9] as the latticial counterparts of CS modules; they are precisely the lattices satisfying the first condition (C_1) from the list below of five conditions (C_i), $i = 1, 2, 3, 11, 12$.

Throughout this section L will denote a (bounded) modular lattice with a least element 0 and a greatest element 1.

The conditions (C_i) for lattices

Recall that for any lattice L we introduced in Section 1 the following notation:

$P(L)$:= the set of all *pseudo-complement* elements of L (P for "*Pseudo*"),
$E(L)$:= the set of all *essential* elements of L (E for "*Essential*"),
$C(L)$:= the set of all *closed* elements of L (C for "*Closed*"),
$D(L)$:= the set of all *complement* elements of L (D for "*Direct summand*"),
$K(L)$:= the set of all *compact* elements of L (K for "*Kompakt*").

We present now five conditions (C_i), $i = 1, 2, 3, 11, 12$, introduced in [9] as the latticial counterparts of the well-known corresponding conditions in Module Theory.

Definitions 2.1. *For a lattice L one may consider the following conditions:*

(C_1) *For every $x \in L$ there exists $d \in D(L)$ such that $x \in E(d/0)$.*

(C_2) *For every $x \in L$ such that $x/0 \simeq d/0$ for some $d \in D(L)$, one has $x \in D(L)$.*

(C_3) *For every $d_1, d_2 \in D(L)$ with $d_1 \wedge d_2 = 0$, one has $d_1 \vee d_2 \in D(L)$.*

(C_{11}) *For every $x \in L$ there exists a pseudo-complement p of x with $p \in D(L)$.*

(C_{12}) *For every $x \in L$ there exist $d \in D(L)$, $e \in E(d/0)$, and a lattice isomorphism $x/0 \simeq e/0$.*

Definitions 2.2. *(*[9, Definitions 1.2]*). A lattice L is called* CC *or* extending *if it satisfies (C_1),* continuous *if it satisfies (C_1) and (C_2), and* quasi-continuous *if it satisfies (C_1) and (C_3).*

CC lattices

In this subsection we present some characterizations and properties of CC lattices.

Lemma 2.3. ([9, Proposition 1.7]). *The following assertions hold for a lattice $L \in \mathcal{M}$.*

(1) *$D(L) \subseteq P(L) \subseteq C(L)$.*

(2) *$D(L) \cap (a/0) \subseteq D(a/0)$ for every $a \in L$.*

(3) *$D(L) \cap (d/0) = D(d/0)$ for every $d \in D(L)$.* □

The next result, a part of [9, Proposition 1.10], explains the term of a CC lattice, acronym for Closed elements are Complements.

Proposition 2.4. *The following statements hold for a modular lattice L.*

(1) *L is uniform $\Longrightarrow$ L is CC, and, if additionally L is indecomposable, then the inverse implication "$\Longleftarrow$" also holds.*

(2) *If additionally L is essentially closed (in particular, if L is upper continu- ous) then*

$$L \text{ is CC} \Longleftrightarrow C(L) \subseteq D(L) \Longleftrightarrow C(L) = D(L).$$

(3) *If additionally L is strongly pseudo-complemented (in particular, if L is upper continuous) then*

$$L \text{ is CC} \Longleftrightarrow C(L) \subseteq D(L) \Longleftrightarrow C(L) = D(L) \Longleftrightarrow$$

$$\Longleftrightarrow P(L) \subseteq D(L) \Longleftrightarrow P(L) = D(L).$$

(5) *L satisfies $(C_2) \Longrightarrow L$ satisfies (C_3).*

(6) *L satisfies $(C_1) \Longrightarrow L$ satisfies (C_{11}).*

(7) *L satisfies $(C_{11}) \Longrightarrow L$ satisfies (C_{12}).* □

Proposition 2.5. ([9, Proposition 1.15]). *Let L be a modular strongly pseudo-complemented lattice (in particular an upper continuous lattice). If L is a CC lattice then so is also $d/0$ for any $d \in D(L)$, in other words, the CC condition is inherited by complement intervals.*
□

Corollary 2.6. *([3, Corollary 1.5])*. Let L be a modular essentially closed CC lattice. Then L has finite Goldie dimension if and only if 1 is a finite direct join of uniform elements of L. □

CEK lattices

We present now the concept of a *CEK* lattice, that is essential in proving a key lemma used in the proof of the main result of this paper.

Definitions 2.7. *(*[3, Definitions 1.6]*). Let L be a lattice.*

(1) *An element $a \in L$ is called* essentially compact *if $E(a/0) \cap K(L) \neq \varnothing$. We denote by $E_k(L)$ the set of all essentially compact elements of L.*

(2) *L is called* CEK *(acronym for* C*losed are* E*ssentially* C*ompact) if every closed element of L is essentially compact, i.e., $C(L) \subseteq E_k(L)$.*

The next result provides large classes of CEK lattices.

Proposition 2.8. ([3, Proposition 1.7]). *Let L be a non-zero complete modular lattice having the following property:*

(†) *For every $0 \neq x \in L$ there exists $0 \neq k \in K(L)$ with $k \leqslant x$.*

In particular, L can be any compactly generated lattice.

Then L has finite Goldie dimension if and only if each element of L is essentially compact, i.e., $L = E_k(L)$. In particular, any modular lattice with finite Goldie dimension satisfying (†) *is CEK.* □

3 The Latticial Osofsky-Smith Theorem

In this section we prove the latticial counterpart of the module-theoretical Osofsky-Smith Theorem. Our contention is that the natural setting for this renown result and its various extensions is Lattice Theory, being concerned as it is, with latticial concepts like essential, uniform, complement, pseudo-complements elements, and direct joins in certain lattices.

The Osofsky-Smith Theorem (O-ST): a brief (incomplete) chronology

1964 *Barbara L. Osofsky* [25] proves her nice theorem saying that a ring R is semisimple if and only if every cyclic right R-module is injective.

1965 *Yuzo Utumi* [27] introduces the conditions (C_i), $i = 1, 2, 3$, for rings.

1977 *A.V. Chatters & Charudatta R. Hajarnavis* [13] introduce *CS* rings.

1981 *Manabu Harada & Kiyochi Oshiro* [22] use the term of *extending* for modules satisfying the condition (C_1) (i.e., *CS* modules).

1990 *Saad H. Mohamed & Bruno J. Müller* publish their seminal monograph "*Continuous and Discrete Modules*" [23].

1991 *Barbara L. Osofsky & Patrick F. Smith* [26] prove their acclaimed result known as the O-ST.

1994 *Nguyen Viet Dung & Dinh Van Huynh, & Patrick F. Smith & Robert Wisbauer* publish their renown monograph "*Extending Modules*" [18].

2010 *Septimiu Crivei & Constantin Năstăsescu & Blass Torrecillas* [15] provide a categorical (incomplete) version of the O-ST.

2011 *Toma Albu* [3] proves the O-ST for arbitrary modular lattices, called the Latticial O-ST.

2014 *Toma Albu* [4] applies the Latticial O-ST to Grothendieck categories and module categories equipped with a hereditary torsion theory, obtaining so the Categorical O-ST and the Relative O-ST, respectively.

2016 *Toma Albu & Mihai Iosif & Adnan Tercan* [9] introduce and investigate the conditions (C_i), $i = 1, 2, 3, 11, 12,$ for lattices, with applications to Grothendieck categories and module categories equipped with a hereditary torsion theory.

Three lemmas

This subsection presents three basic facts that will be used in the next subsection to prove the main result of this paper.

Lemma 3.1. *Let L be a compact, compactly generated, modular lattice. Assume that all compact intervals b/a of L are CEK, i.e., every $c \in C(b/a)$ is an essentially compact element of b/a. Then $D(L)$ is a Noetherian poset.*

Proof. See [3, Lemma 2.1] for a very technical 6-page proof. □

The next result is the latticial counterpart of a well-known simple result saying that a non-zero module M_R satisfying ACC or DCC on direct summands is a finite direct sum of finitely many indecomposable submodules (see, e.g., [11, Proposition 10.14]).

Lemma 3.2. ([3, Lemma 3.1]). *Let L be a non-zero modular lattice such that the set $D(L)$ of complement elements of L is either Noetherian or Artinian. Then* 1 *is a direct join of finitely many indecomposable elements of L.* □

Lemma 3.3. ([3, Lemma 3.2]). *Any modular, upper continuous, compact, CC lattice is CEK.* □

Proof. We have to show that $C(L) \subseteq E_k(L)$, i.e., $D(L) \subseteq E_k(L)$ by Proposition 2.4(2). So, let $d \in D(L)$. Then $d \in E(d/0) \cap K(L)$ by [5, Proposition 2.1.18 (1)], and hence L is CEK, as desired. □

The main result

Theorem 3.1. (Latticial O-ST [3, Theorem 3.4]). *Let L be a compact, compactly generated, modular lattice. Assume that all compact subfactors of L are CC. Then* 1 *is a finite direct join of uniform elements of L.*

Proof. First, observe that the lattice L is upper continuous because it is compactly generated. By assumption, every compact subfactor of L is CC, so CEK by Lemma 3.3. Using now Lemma 3.1, we deduce that $D(L)$ is a Noetherian poset, and so, by Lemma 3.2, $1 = \dot{\bigvee}_{1 \leqslant i \leqslant n} d_i$ is a finite direct join of indecomposable elements d_i of L. Since L is CC, so is also any $d_i/0$ by Proposition 2.5. On the other hand, every d_i, $1 \leqslant i \leqslant n$, is uniform by Proposition 2.4(1). Consequently, 1 is a finite direct join of uniform elements of L, and we are done. □

Following [17], a right R-module M is said to be *CF* (acronym for *C*losed are *F*initely generated) if every closed submodule of M is finitely generated, and *completely CF* provided every quotient of M is also CF. More generally, we say that a lattice L is *CK* (acronym for *C*losed are *K*ompact) if every closed element of L is compact, i.e., $C(L) \subseteq K(L)$. Clearly, any CK lattice is also CEK, so we deduce at once from Lemmas 3.1 and 3.2 the following result.

Proposition 3.4. *Let L be a compact, compactly generated, modular lattice. Assume that all compact subfactors of L are CK. Then $D(L)$ is a Noetherian poset, in particular 1 is a finite direct join of indecomposable elements of L.* □

We extend now the Latticial O-ST to more general lattices, so that it can be also applied to cyclic modules (which have no latticial counterparts).

Denote by $\mathcal{K}$ the class of all compact lattices and by $\mathcal{U}$ the class of all upper continuous lattices, and let $\mathcal{P}$ be a non-empty subclass of $\mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the following three conditions (see [3, p. 4502]):

(P_1) If $L \in \mathcal{P}$, $L' \in \mathcal{L}$, and $L \simeq L'$ then $L' \in \mathcal{P}$.

(P_2) If $L \in \mathcal{P}$ then $1/a \in \mathcal{P}, \forall\, a \in L$.

(P_3) If $L \in \mathcal{P}$ and $b/a \in \mathcal{P}$ is a subfactor of L, then $\exists\, c \in L$ such that $c/0 \in \mathcal{P}$ and $b = a \vee c$.

Examples of classes $\mathcal{P}$ satisfying the conditions $(P_1) - (P_3)$ above are:

- any $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ such that

$$L \in \mathcal{P} \Longrightarrow (1/a \in \mathcal{P} \;\&\; a/0 \in \mathcal{P}, \forall\, a \in L);$$

- the class of all compact, compactly generated, modular lattices;
- the class of all compact, semi-atomic, upper continuous, modular lattices;
- the class of lattices isomorphic to lattices of all submodules of all cyclic right R-modules.

For any lattice L we set $\mathcal{P}(L) := \{\, c \in L \mid c/0 \in \mathcal{P} \,\}$. Note that $\varnothing \neq \mathcal{P}(L) \subseteq K(L)$ whenever $L \in \mathcal{U}$.

Theorem 3.2. (Latticial $\mathcal{P}$-O-ST [3, Theorem 3.7]). *Let $\mathcal{P}$ be a non-empty subclass of $\mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the conditions $(P_1) - (P_3)$ above, and let $L \in \mathcal{P}$. Assume that all subfactors of L in $\mathcal{P}$ are CC. Then 1 is a finite direct join of uniform elements of L.* □

Proof. The proof is similar to that of Theorem 3.1 by using a straightforward adaptation of Lemma 3.1 to the lattices $L \in \mathcal{P}$. □

The next result is a reformulation of Theorem 3.2.

Corollary 3.5. Let $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the conditions $(P_1) - (P_3)$ above, and let $L \in \mathcal{P}$. Assume that $c/0$ is a completely CC lattice for every $c \in \mathcal{P}(L)$. Then 1 is a finite direct join of uniform elements of L. □

Notice that Corollary 3.6 below is a latticial version of the following module-theoretical result:

A right R-module M is semisimple $\Longleftrightarrow$ every cyclic subfactor of M is M-injective

(see [18, Corollary 7.14]), which, in turn, is a "modularization" of the well-known *Osofsky's Theorem* [25] saying that a ring R is semisimple if and only if every cyclic right R-module is injective.

Corollary 3.6. *([3, Corollary 3.9]).* Let $\varnothing \neq \mathcal{P} \subseteq \mathcal{K} \cap \mathcal{M} \cap \mathcal{U}$ satisfying the conditions $(P_1) - (P_3)$ above. Then, the following statements are equivalent for a complete modular lattice L such that any of its elements is a join of elements of $\mathcal{P}(L)$.

(1) L is semi-atomic.

(2) F is CC and $K(F) \subseteq D(F)$ for every subfactor $F \in \mathcal{P}$ of L. □

Using the concept of a linear morphism of lattices, recently introduced in [6] and briefly discussed in the next subsection, we expect to provide a consequence, involving linear injective lattices (see [6, Definitions 3.1]), of the Latticial O-ST.

Linear lattice morphisms

The concept of a *linear morphism* of lattices we present below evokes the property of a linear mapping $\varphi : M \longrightarrow N$ between modules M_R and N_R to have a kernel $\mathrm{Ker}\,(\varphi)$ and to verify the Fundamental Theorem of Isomorphism $M/\mathrm{Ker}\,(\varphi) \simeq \mathrm{Im}\,(\varphi)$.

A mapping $f : L \longrightarrow L'$ between a lattice L with least element 0 and greatest element 1 and a lattice L' with least element $0'$ and greatest element $1'$ is called a *linear morphism* if there exist $k \in L$, called a *kernel* of f, and $a' \in L'$ such that the following two conditions are satisfied.

- $f(x) = f(x \vee k), \forall\, x \in L$.
- f induces a lattice isomorphism

$$\bar{f} : 1/k \xrightarrow{\sim} a'/0',\ \bar{f}(x) = f(x),\ \forall\, x \in 1/k.$$

If $f : L \longrightarrow L'$ is a linear morphism of lattices, then, by [6, Proposition 1.2] f is an increasing mapping, commutes with arbitrary joins (i.e., $f\left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I} f(x_i)$ for any family $(x_i)_{i\in I}$ of elements of L, provided both joins exist), preserves intervals (i.e., for any $u \leqslant v$ in L, one has $f(v/u) = f(v)/f(u)$), and its kernel k is uniquely determined.

As in [6], the class $\mathcal{M}$ of all (bounded) modular lattices becomes a category, denoted by $\mathcal{LM}$ ($\mathcal{L}$ for "*L*inear" and $\mathcal{M}$ for "*M*odular") if for any $L, L' \in \mathcal{M}$ one takes as morphisms from L to L' all the linear morphisms from L to L'. A basic property of this category says that the subobjects of any $L \in \mathcal{LM}$ can be viewed as the intervals $a/0$ of L, $a \in L$ (see [6, Proposition 2.2 (5)]).

4 The Categorical Osofsky-Smith Theorem

A latticial strategy

We first present a ***general strategy*** which consists on putting a *module-theoretical* result into a *latticial frame* (we call it *latticization*), in order to translate that result to *Grothendieck categories* (we call it *absolutization*) and *module categories* equipped with *hereditary torsion theories* (we call it *relativization*).

More precisely, if $\mathbb{P}$ is a problem, involving subobjects or submodules, to be investigated in Grothendieck categories or in module categories with respect to hereditary torsion theories, our strategy since more than 30 years consists of the following three steps:

I. *Translate/formulate,* if possible, the problem $\mathbb{P}$ into a *latticial setting.*

II. *Investigate* the obtained problem $\mathbb{P}$ in this *latticial frame.*

III. *Back* to Grothendieck categories and module categories equipped with hereditary torsion theories.

This approach is equally natural and simple, because we ignore the specific context, sometimes not so easy to deal with, of Grothendieck categories and module categories equipped with hereditary torsion theories, and focus only on those latticial properties which are relevant in our given specific categorical or relative module-theoretical problem $\mathbb{P}$. The renowned *Hopkins-Levitzki Theorem* and *Osofsky-Smith Theorem* from Ring and Module Theory are among the most relevant illustrations of the power of this strategy (see [2], [3], and [4]).

This section deals with the *absolutization* of the module-theoretical Osofsky-Smith Theorem [26, Theorem 1]. Thus, by applying the Latticial Osofsky-Smith Theorem from the previous section to the specific case of Grothendieck categories we obtain at once the *Categorical* or *Absolute Osofsky-Smith Theorem.*

Grothendieck categories

Throughout this section $\mathcal{G}$ denotes a *Grothendieck category*, i.e., an Abelian category with exact direct limits and with a generator. For any object X of $\mathcal{G}$, $\mathcal{L}(X)$ will denote the lattice of all subobjects of X.

As it is well-known, $\mathcal{L}(X)$ is an upper continuous modular lattice (see, e.g., [28, Chapter 4, Proposition 5.3, and Chapter 5, Section 1]). For any subobjects Y and Z of $X \in \mathcal{G}$ we denote by $Y \cap Z$ their meet and by $Y + Z$ their join in the lattice $\mathcal{L}(X)$. For all undefined notation and terminology on Abelian categories the reader is referred to [10] and [28].

Recall that an object X of $\mathcal{G}$ is said to be *Noetherian* (respectively, *Artinian*) if the lattice $\mathcal{L}(X)$ is Noetherian (respectively, Artinian). More generally, if $\mathbb{P}$ is any property on lattices, we say that an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Similarly, a subobject Y of an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the element Y of the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Thus, we obtain the concepts of a *uniform* object, *compact* object, (C_i), $i = 1, 2, 3, 11, 12$, condition for an object, *CC* object, *pseudo-complement* subobject of an object, *essential* subobject of an object, *closed* subobject of an object, *complement* subobject of an object, etc. For a complement (respectively, compact) subobject of an object $X \in \mathcal{G}$ we use in this section the well-established term of a *direct summand* (respectively, *finitely generated* subobject) of X, and for this reason, instead of saying that X is a CC object we shall say that X is a *CS* object (acronym for *C*losed subobjects are direct *S*ummands).

Recall that the category $\mathcal{G}$ is called *locally finitely generated* if it has a family of finitely generated generators, or equivalently if the lattices $\mathcal{L}(X)$ are compactly generated for all objects X of $\mathcal{G}$. We say that an object $X \in \mathcal{G}$ is *locally finitely generated* if the lattice $\mathcal{L}(X)$ is compactly generated.

Theorem 4.1. (Categorical O-ST [4, Theorem 4.2]). *Let $\mathcal{G}$ be a Gro- thendieck category, and let $X \in \mathcal{G}$ be a finitely generated, locally finitely generated object such that every finitely generated subfactor object of X is CS. Then X is a finite direct sum of uniform objects.*

Proof. Just apply Theorem 3.1 to the lattice $L = \mathcal{L}(X)$. □

An object X of a Grothendieck category $\mathcal{G}$ is called *CF* (acronym for *C*losed are *F*initely generated) if every closed subobject of X is finitely generated, and *completely CF* if every quotient object of X is CF.

Corollary 4.1. Let X be a finitely generated, locally finitely generated object of a Grothendieck category $\mathcal{G}$ such that every finitely generated subobject of X is completely CF. Then X is a finite direct sum of indecomposable subobjects.

Proof. Specialize Proposition 3.4 for the lattice $L = \mathcal{L}(X)$. □

Denote by $\mathcal{H}$ the class of all finitely generated objects of $\mathcal{G}$, and let $\mathcal{A}$ be a subclass of $\mathcal{H}$ satisfying the following three conditions:

(A_1) If $X \in \mathcal{A}$, $X' \in \mathcal{G}$, and $X \simeq X'$, then $X' \in \mathcal{A}$.

(A_2) If $X \in \mathcal{A}$ then $X/X' \in \mathcal{A}, \forall X' \subseteq X$.

(A_3) If $X \in \mathcal{A}$ and $Z \subseteq Y \subseteq X$ with $Y/Z \in \mathcal{A}$, then $\exists U \subseteq X$ such that $U \in \mathcal{A}$ and $Y = Z + U$.

The class $\mathcal{H}$ could be empty, and in this case everything that follows makes no sense.

Theorem 4.2. (Categorical $\mathcal{A}$-O-ST [4, Theorem 4.8]). *Let $\mathcal{A}$ be a class of finitely generated objects of a Grothendieck category $\mathcal{G}$ satisfying the conditions $(A_1)-(A_3)$ above, and let $X \in \mathcal{A}$. Assume that all subfactors of X in $\mathcal{A}$ are CS. Then X is a finite direct sum of uniform objects of $\mathcal{G}$.*

Proof. Specialize Theorem 3.2 for the lattice $L = \mathcal{L}(X)$. □

We present now a consequence, involving injective objects, of the Categorical O-ST. Recall that for any Grothendieck category one can define as in Mod-R the concepts of an *M-injective object, self-injective object, simple object,* and *semisimple object* (see, e.g., [10, p. 9]).

Lemma 4.2. ([4, Lemma 4.13]). *Any self-injective object of a Grothendieck category $\mathcal{G}$ is a CS object.* □

Proposition 4.3. ([4, Proposition 4.14]). *The following assertions are equivalent for a locally finitely generated object X of a Grothendieck category $\mathcal{G}$.*

(1) *X is semisimple.*

(2) *Every finitely generated subfactor of X is X-injective.* □

Remark 4.4. Observe that some statements/results of [26] and [15] related to the Categorical O-ST saying that "*basically the same proof for modules works in the categorical setting*" are not in order (see [4, p. 2670]). Such statements are very risky and may lead to incorrect results. One reason is that we cannot prove equality between two subobjects of an object in a category as we do for submodules by taking elements of them. Notice that the well-hidden errors in the statements/results occurring in the papers mentioned above on the Categorical O-ST could be spotted only by using our latticial approach of it. Consequently, we do not only correctly absolutize the module-theoretical O-ST but also provide a correct proof of its categorical extension by passing first through its latticial counterpart. □

5 The Relative Osofsky-Smith Theorem

In this section we present the relative version with respect to a hereditary torsion theory of the module-theoretical Osofsky-Smith Theorem [26, Theorem 1]. Its proofs is an immediate specialization of the Latticial O-ST to a suitable particular lattice.

Torsion theories

In this subsection, we present relative versions with respect to a hereditary torsion theory τ on Mod-R of some module-theoretical results related to CS modules M. Their proofs are immediate applications of the lattice-theoretical results obtained in the previous sections for the lattice $\text{Sat}_\tau(M)$ of all τ-saturated submodules of M.

The concept of a *torsion theory* for Abelian categories has been introduced by S.E. Dickson [16] in 1966. We present it below only for module categories in one of the many equivalent ways that can be done. Basic torsion-theoretic concepts and results can be found in [20] and [28].

All rings considered in this paper are associative with unit element $1 \neq 0$, and modules are unital right modules. If R is a ring, then Mod-R denotes the category of all right R-modules. We often write M_R to emphasize that M is a right R-module; $\mathcal{L}(M_R)$, or just $\mathcal{L}(M)$, stands for the lattice of all submodules of M. The notation $N \leqslant M$ means that N is a submodule of M.

A *hereditary torsion theory* on Mod-R is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of nonempty subclasses $\mathcal{T}$ and $\mathcal{F}$ of Mod-R such that $\mathcal{T}$ is a *localizing subcategory* of Mod-R in the Gabriel's sense [19] (this means that $\mathcal{T}$ is a Serre class of Mod-R which is closed under direct sums) and $\mathcal{F} = \{ F_R \,|\, \mathrm{Hom}_R(T, F) = 0, \forall\, T \in \mathcal{T} \}$. This means that any hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is uniquely determined by its first component $\mathcal{T}$.

Recall that a nonempty subclass $\mathcal{T}$ of Mod-R is a *Serre class* if for any short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

in Mod-R, one has

$$X \in \mathcal{T} \Longleftrightarrow X' \in \mathcal{T} \ \ \& \ \ X'' \in \mathcal{T}.$$

One says that $\mathcal{T}$ is *closed under direct sums* if for any family $(X_i)_{i \in I}$, I arbitrary set, with $X_i \in \mathcal{T}$, $\forall\, i \in I$, it follows that $\bigoplus_{i \in I} X_i \in \mathcal{T}$.

The prototype of a hereditary torsion theory is the pair $(\mathcal{A}, \mathcal{B})$ in Mod-$\mathbb{Z}$, where $\mathcal{A}$ is the class of all torsion Abelian groups, and $\mathcal{B}$ is the class of all torsion-free Abelian groups.

Throughout this paper $\tau = (\mathcal{T}, \mathcal{F})$ will be a fixed hereditary torsion theory on Mod-R. For any module M_R we denote

$$\tau(M) := \sum_{N \leqslant M,\, N \in \mathcal{T}} N.$$

Since $\mathcal{T}$ is a localizing subcategory of Mod-R, it follows that $\tau(M) \in \mathcal{T}$, and we call it the *τ-torsion submodule* of M. Note that, as for Abelian groups, we have

$$M \in \mathcal{T} \Longleftrightarrow \tau(M) = M \quad \text{and} \quad M \in \mathcal{F} \Longleftrightarrow \tau(M) = 0.$$

The members of $\mathcal{T}$ are called *τ-torsion modules*, while the members of $\mathcal{F}$ are called *τ-torsion-free modules*.

For any $N \leqslant M$ we denote by $\overline{N}$ the submodule of M such that $\overline{N}/N = \tau(M/N)$, called the *$\tau$-saturation* of N (in M). One says that N is *τ-saturated* if $\overline{N} = N$, or equivalently, if $M/N \in \mathcal{F}$, and the set of all τ-saturated submodules of M is denoted by $\mathrm{Sat}_\tau(M)$, i.e.,

$$\mathrm{Sat}_\tau(M) \ := \ \{ \ N \mid N \leqslant M \text{ and } M/N \in \mathcal{F} \ \},$$

which is an upper continuous modular lattice by [28, Chapter 9, Proposition 4.1].

A module M_R is said to be τ-CC if the lattice $\mathrm{Sat}_\tau(M)$ is CC. More generally if $\mathbb{P}$ is any property on lattices, we say that a module M_R is/has τ-$\mathbb{P}$ if the lattice $\mathrm{Sat}_\tau(M)$ is/has $\mathbb{P}$. Thus, we obtain the concepts of a *τ-Artinian* module, *τ-Noetherian* module, *τ-uniform* module, *τ-compact* module, *τ-compactly generated* module, condition τ-(C_i),

τ-CC module, etc. We say that a submodule N of M_R is/has τ-$\mathbb{P}$ if its τ-saturation $\overline{N}$, which is an element of $\mathrm{Sat}_\tau(M)$, is/has $\mathbb{P}$. Thus, we obtain the concepts of a τ-*pseudo-complement* submodule of a module, τ-*complement* submodule of a module, τ-*essential* submodule of a module, τ-*closed* submodule of a module, τ-*independent* set/family of submodules of a module, etc. Since $\overline{N} = \overline{\overline{N}}$, it follows that N is/has τ-$\mathbb{P}$ if and only if $\overline{N}$ is/has τ-$\mathbb{P}$. Also, because the lattices $\mathrm{Sat}_\tau(M)$ and $\mathrm{Sat}_\tau(M/\tau(M))$ are canonically isomorphic, we deduce that M_R is τ-$\mathbb{P}$ if and only if $M/\tau(M)$ is τ-$\mathbb{P}$.

In the sequel we shall use the well-established term of a τ-*direct summand* of a module instead of that of a τ-*complement* submodule of a module and of a τ-*CS* module instead of that of a τ-*CC* module.

All the notions and results from the previous sections for an arbitrary modular lattice L can now be immediately specialized for the particular case when $L = \mathrm{Sat}_\tau(M_R)$. We present below only a few results.

Consider the quotient category Mod-$R/\mathcal{T}$ (see, e.g., [1, Lesson 8] for a brief explanation of this concept) of Mod-R modulo its localizing subcategory $\mathcal{T}$, and let

$$T_\tau : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}$$

be the canonical functor. By [10, Proposition 7.10], for any module M_R, the mapping

$$\mathrm{Sat}_\tau(M) \longrightarrow \mathcal{L}(T_\tau(M)), \;\; N \mapsto T_\tau(N),$$

is an isomorphism of lattices, so, for any property $\mathbb{P}$ on lattices, the module M_R is/has τ-$\mathbb{P}$ if and only if the object $T_\tau(M)$ in the quotient Grothendieck category Mod-$R/\mathcal{T}$ is/has $\mathbb{P}$. This shows that any τ-relative property in Mod-R can be formulated as a categorical (or absolute) property in the Grothendieck category Mod-$R/\mathcal{T}$, and conversely, using the *Gabriel-Popescu Theorem* (see, e.g., [1, Lesson 8] for its brief explanation) any categorical property of subobjects of objects in an arbitrary Grothendieck category can be translated into relative property of submodules of modules with respect to a suitable hereditary theory.

We say that a finite family $(N_i)_{1\leqslant i\leqslant n}$ of submodules of a module M_R is τ-*independent* if $N_i \notin \mathcal{T}$ for all $1 \leqslant i \leqslant n$, and

$$N_{k+1} \cap \sum_{1\leqslant j\leqslant k} N_j \subseteq \tau(M), \, \forall\, k, \, 1 \leqslant k \leqslant n-1,$$

or, equivalently

$$\overline{N_{k+1}} \cap \overline{\sum_{1\leqslant j\leqslant k} N_j} = \overline{N_{k+1}} \wedge \Big(\bigvee_{1\leqslant j\leqslant k} \overline{N_j} \Big) = \tau(M),$$

in other words, the family $\big(\overline{N_i}\big)_{1\leqslant i\leqslant n}$ of elements of the lattice $\mathrm{Sat}_\tau(M)$ is independent. More generally, a family $(N_i)_{i\in I}$ of submodules of M is called τ-*independent* if the family $\big(\overline{N_i}\big)_{i\in I}$ of elements of the lattice $\mathrm{Sat}_\tau(M)$ is independent.

Theorem 5.1. (Relative O-ST [4, Theorem 5.8]). *Let M_R be a τ-compact, τ-compactly generated module. Assume that all τ-compact subfactors of M are τ-CS. Then there exists a finite τ-independent family $(U_i)_{1\leqslant i\leqslant n}$ of τ-uniform submodules of M such that $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.* □

A more simplified version of the Relative O-ST in case the given module M_R is τ-torsion-free is the following one.

Theorem 5.2. (Torsion-free Relative O-ST [4, Theorem 5.12]). *Let $M_R \in \mathcal{F}$ be a τ-compact, τ-compactly generated module. Assume that all τ-compact subfactors of M are τ-CS. Then, there exists a finite independent family $(U_i)_{1\leqslant i\leqslant n}$ of uniform submodules of M such that $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.* □

As noticed above, M is τ-$\mathbb{P}$ if and only if $M/\tau(M)$ is so. Therefore, in view of Theorem 5.2 we can clearly formulate the Relative O-ST in terms of essentiality and independence in the lattice $\mathcal{L}(M/\tau(M))$ instead of τ-essentiality and τ-independence in the lattice $\mathcal{L}(M)$, respectively:

Theorem 5.3. ([4, Theorem 5.13]). *Let M_R be a τ-compact, τ-compactly generated module. If all τ-compact subfactors of M are τ-CS, then there exists a finite family $(U_i)_{1\leqslant i\leqslant n}$ of submodules of M, all containing $\tau(M)$, such that $(U_i/\tau(M))_{1\leqslant i\leqslant n}$ is an independent family of uniform submodules of $M/\tau(M)$ and $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.* □

For a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ on Mod-R one denotes by

$$F_\tau := \{\, I \leqslant R_R \mid R/I \in \mathcal{T}\}$$

the *Gabriel filter* associated with τ. By a *basis* of the Gabriel filter F_τ we mean a subset B of F_τ such that every right ideal in F_τ contains some $J \in B$.

The next result provides a characterization of Grothendieck categories possessing a finitely generated generator in terms of Gabriel filters and quotient categories.

Proposition 5.1. ([8, Proposition 2.12]). *The following assertions are equivalent for a Grothendieck category $\mathcal{G}$.*

(1) *$\mathcal{G}$ has a finitely generated generator.*

(2) *There exists a unital ring A and a hereditary torsion theory $\chi = (\mathcal{H}, \mathcal{E})$ on* Mod-A *such that the Gabriel filter F_χ has a basis of finitely generated right ideals of A and $\mathcal{G} \simeq$ Mod-$A/\mathcal{H}$.*

(3) *There exists a unital ring A and a hereditary torsion theory $\chi = (\mathcal{H}, \mathcal{E})$ on* Mod-A *such that the lattice* $\mathrm{Sat}_\chi(A)$ *is compact and $\mathcal{G} \simeq$ Mod-$A/\mathcal{H}$.*

□

In case F_τ has a basis consisting of finitely generated right deals of R, we deduce from Proposition 5.1 that the Grothendieck category Mod-$R/\mathcal{T}$ is locally finitely generated, and so, any module M_R is τ-compactly generated. Therefore, the next result is an immediate consequence of Theorem 5.3.

Theorem 5.4. ([4, Theorem 5.14]). *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on* Mod-R *such that its Gabriel filter F_τ has a basis consisting of finitely generated right ideals of R (in particular, this holds when R is τ-Noetherian), and let M_R be a τ-compact module. If all τ-compact subfactors of M are τ-CS, then there exists a finite family $(U_i)_{1\leqslant i\leqslant n}$ of submodules of M, all containing $\tau(M)$, such that $(U_i/\tau(M))_{1\leqslant i\leqslant n}$ is an independent family of uniform submodules of $M/\tau(M)$ and $M/(\sum_{1\leqslant i\leqslant n} U_i) \in \mathcal{T}$.* □

According to our definitions of module-theoretical concepts relative to a hereditary torsion theory τ, a module U_R is said to be *τ-simple* if the lattice $\mathrm{Sat}_\tau(U)$ is simple, which means that it has exactly two elements, i.e., $U \notin \mathcal{T}$ and $\mathrm{Sat}_\tau(U) = \{\tau(U), U\}$. Recall that U_R is called *τ-cocritical* if it is τ-simple and $U \in \mathcal{F}$.

The *τ-socle* of a module M_R, denoted by $\mathrm{Soc}_\tau(M)$, is defined as the τ-saturation of the sum of all τ-simple (or τ-cocritical) submodules of M, and M is said to be *τ-semisimple* if $M = \mathrm{Soc}_\tau(M)$. By [7, Proposition 6.5(1)], $\mathrm{Soc}_\tau(M)$ is exactly the socle of the lattice $\mathrm{Sat}_\tau(M)$, and so, we have

$$M_R \text{ is a } \tau\text{-semisimple module} \iff \mathrm{Sat}_\tau(M) \text{ is a semi-atomic lattice} \iff$$
$$\iff T_\tau(M) \text{ is a semisimple object of the quotient category Mod-}R/\mathcal{T}.$$

The next result is a relative version of the well-known *Osofsky's Theorem* [25] we presented just before Corollary 3.6.

Proposition 5.2. ([4, Proposition 5.16]). *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on* Mod-*R such that its Gabriel filter F_τ has a basis consisting of finitely generated right ideals of R (in particular, this holds when R is τ-Noetherian). Assume that R/I is an injective R-module for each $I \in \mathrm{Sat}_\tau(R)$. Then, any right R-module is τ-semisimple.* □

References

[1] T. Albu, *Lessons on the Grothendieck Category $\sigma[M]$*, Editura Universităţii Bucureşti, (2004).

[2] T. Albu, A seventy year jubilee: The Hopkins-Levitzki Theorem, in *Ring and Module Theory*, Trends in Mathematics, Editors T. Albu, G. F. Birkenmeier, A. Erdoğan, A. Tercan, Birkhäuser, Basel, 1-26 (2010).

[3] T. Albu, The Osofsky-Smith Theorem for modular lattices, and applications (I), *Comm. Algebra*, **39**, 4488-4506 (2011).

[4] T. Albu, The Osofsky-Smith Theorem for modular lattices, and applications (II), *Comm. Algebra*, **42**, 2663-2683 (2014).

[5] T. Albu, *Chain Conditions in Modular Lattices with Applications to Grothendieck Categories and Torsion Theories*, Monograph Series of Parana's Mathematical Society No. 1, Sociedade Paranaense de Matemática, Maringá, Paraná, Brasil (2015).

[6] T. Albu, M. Iosif, The category of linear modular lattices, *Bull. Math. Soc. Sci. Math. Roumanie*, **56 (104)**, 33-46 (2013).

[7] T. Albu, M. Iosif, Lattice preradicals with applications to Grothendieck categories and torsion theories, *J. Algebra*, **444**, 339-366 (2015).

[8] T. Albu, M. Iosif, M. L. Teply, Dual Krull dimension and quotient finite dimensionality, *J. Algebra*, **284**, 52-79 (2005).

[9] T. Albu, M. Iosif, A. Tercan, The conditions (C_i) in modular lattices, and applications, *J. Algebra Appl.*, **15**, 1650001 (2016).

[10] T. Albu, C. Năstăsescu, *Relative Finiteness in Module Theory*, Marcel Dekker, Inc., New York and Basel (1984).

[11] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, Second Edition, Springer-Verlag, New York (1992).

[12] G. Birkhoff, *Lattice Theory*, 3rd ed., Providence, RI, American Mathematical Society (1967).

[13] A. W. Chatters, C. R. Hajarnavis, Rings in which every complement right ideal is a direct summand, *Quart. J. Math. Oxford*, **28**, 61-80 (1977).

[14] P. Crawley, R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall, Englewood Cliffs, New Jersey (1973).

[15] S. Crivei, C. Năstăsescu, B. Torrecillas, On the Osofsky-Smith Theorem, *Glasgow Math. J.*, **52A**, 61-87 (2010).

[16] S. E. Dickson, A torsion theory for Abelian categories, *Trans. Amer. Math. Soc.*, **121**, 223-235 (1966).

[17] N. V. Dung, Modules whose closed submodules are finitely generated, *Proc. Edinb. Math. Soc.*, **34**, 161-166 (1991).

[18] N. V. Dung, D. V. Huynh, P. F. Smith, R. Wisbauer, *Extending Modules*, Longman Scientific & Technical, Harlow (1994).

[19] P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. France*, **90**, 323-448 (1962).

[20] J. S. Golan, *Torsion Theories*, Pitman/Longman, New York (1986).

[21] G. Grätzer, *General Lattice Theory*, Second Edition, Birkhäuser Verlag, Basel Boston Berlin (2003).

[22] M. Harada, K. Oshiro, On extending property of direct sums of uniform modules, *Osaka J. Math.*, **18**, 767-785 (1981).

[23] S. H. Mohamed, B. J. Müller, *Continuous and Discrete Modules*, Cambridge University Press, Cambridge (1990).

[24] C. Năstăsescu, F. Van Oystaeyen, *Dimensions of Ring Theory*, D. Reidel Publishing Company, Dordrecht Boston Lancaster Tokyo (1987).

[25] B. L. Osofsky, Rings all of whose finitely generated modules are injective, *Pacific J. Math.*, **14**, 645-650 (1964).

[26] B. L. Osofsky, P. F. Smith, Cyclic modules whose quotients have all complement submodules direct summands, *J. Algebra*, **139**, 342-354 (1991).

[27] Y. Utumi, On continuous regular rings and semisimple self-injective rings, *Canad. J. Math.*, **12**, 597-605 (1960).

[28] B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin Heidelberg New York (1975).

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