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An average equal to 1 for every non-trivial finite group

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Dedicated to Professor Ioan Tomescu on his 80th birthday

Abstract

For a non-trivial finite group G and for T^* a set of representatives for the conjugacy classes of non-trivial elements in G we define a canonical integer attached to every element in T^* and show that the average value of this integer over T^* is equal to 1.

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These authors learned a long time ago from Ioan Tomescu's book [3] that the average number of fixed points of a permutation in the symmetric group S_n is equal to 1. That elegant result is a particular case of the Cauchy-Frobenius lemma and a consequence of transitive group actions. In this short note we prove a similar result, which holds for every non-trivial finite group - the similarity refers to the fact that some average is equal to 1. Our unexplained notation is largely standard.

Let G be an arbitrary finite group with at least 2 elements and with identity 1. For $g \in G$ we write $Cl(g) = \{g^x \mid x \in G\}$, $C_G(g) = \{x \in G \mid [g,x] = 1\}$, so $|G| = |Cl(g)||C_G(g)|$. We also let $k = k(G)$ denote the number of conjugacy classes of G and recall the fact that $k = |I|$, where $I = Irr(G)$ is the set of irreducible complex characters of G.

We consider a set T of representatives for the conjugacy classes of G and write $T^* = T \setminus \{1\}$, so $|T^*| = k-1$. For every element $g \in G$, we consider the non-negative integer $m(g) = |\{(x,y) \in G \times G \mid [x,y] = g\}|$ and, finally, related to $m(g)$ we consider the number $x(g) = \dfrac{|G| - m(g)}{|C_G(g)|}$.

With these notations in mind, we prove the following

Theorem *Let G be a finite non-trivial group. Then:*
i) $m(g)$ is a multiple of $|C_G(g)|$.
ii) The average value of $x(g)$ as $g \in T^*$ is equal to 1.

Proof. We start with mentioning that because of commutator properties both $m(g)$ and $x(g)$ take constant values as g runs over a conjugacy class of G.

i) This depends, essentially, on an old and less known result of Frobenius which gives a formula for $m(g)$ in terms of the characters in I - for a simple proof, see [1]:

$$(*) \qquad m(G) = |G| \sum_{\chi \in I} \frac{\chi(g)}{\chi(1)}\,.$$

One can thus rewrite $(*)$ as $m(g) = |C_G(g)| \sum_{\chi \in I} |Cl(g)| \frac{\chi(g)}{\chi(1)}$. Since every term of the last sum is an algebraic integer by Theorem 3.7 of [2], the rational number $\frac{m(g)}{|C_G(g)|}$ is an algebraic integer and, as a consequence, it is an ordinary integer. The assertion follows.

ii) The idea is to express $|G|$ in two ways - an idea dear to Ioan Tomescu. Definitely, $|G| = \sum_{g \in T} |Cl(g)|$. Also, by the definition of $m(g)$, we obtain $|G|^2 = \sum_{g \in G} m(g)$, so that $|G| = \sum_{g \in G} \frac{m(g)}{|G|}$.

But $m(g)$ is constant on conjugacy classes, so $|G| = \sum_{g \in T} |Cl(g)| \frac{m(g)}{|G|}$. From $(*)$ we derive that $m(1) = |G|k$ and this gives

$$1 + \sum_{g \in T^*} |Cl(g)| = |G| = k + \sum_{g \in T^*} \frac{m(g)}{|C_G(g)|}\,.$$

Since $|Cl(g)| = \frac{|G|}{|C_G(g)|}$, we are now ready for the coup de grâce:

$$k - 1 = \sum_{g \in T^*} \frac{|G| - m(g)}{|C_G(g)|} = \sum_{g \in T^*} x(g).$$

This completes the proof, because $|T^*| = k - 1$. $\square$

Remark. It follows from part i) of the Theorem that if g belongs to the center of G, then $m(g)$ is a multiple of $|G|$. The converse is false and a very simple example shows this. Let $G = S_3 = \langle a, b \,|\, a^3 = b^2 = 1, a^b = a^2 \rangle$. The elements of G are $1, a, a^2, b, ab, a^2b$ and G has 3 conjugacy classes, with representatives $1, a, b$. Since b, ab, a^2b are not in the commutator subgroup $G' = \langle a \rangle$, we get $m(b) = m(ab) = m(a^2b) = 0$. And, since $m(1) = k \cdot |G| = 18$, we deduce that $m(a) = m(a^2) = 9 > 6 = |G|$.

Ore's conjecture (now, a theorem), states that if G is a finite, simple, non-abelian group, then every element in G is a commutator. Combined with the above observation, it suggests the following conjecture: if G is a finite, non-abelian group and if there exists in G a non-central element g with $m(g) > |G|$, then G is not simple.

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