Bull. Math. Soc. Sci. Math. Roumanie
Tome 66 (114), No. 1, 2023, 3–14

Degree conditions and path factors with inclusion or exclusion properties

by

Sizhong Zhou

Abstract

A spanning subgraph F of a graph G is called a path factor if every component of F is a path. For an integer $d \geq 2$, a $P_{\geq d}$-factor of a graph G is a spanning subgraph F such that every component is isomorphic to a path of k vertices for some $k \geq d$. A graph G is called a $P_{\geq d}$-factor covered graph if for any $e \in E(G)$, G has a $P_{\geq d}$-factor covering e. A graph G is called a $P_{\geq d}$-factor deleted graph if for any $e \in E(G)$, G has a $P_{\geq d}$-factor excluding e. In this article, we verify that (i) a k-connected graph G with at least n vertices admits a $P_{\geq 3}$-factor if G satisfies $\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k+1})\} \geq \frac{n}{3}$ for any independent subset $\{x_1, x_2, \cdots, x_{2k+1}\}$ of G, where $k \geq 1$ and $n \geq 4k+4$ are two integers; (ii) a k-connected graph G with at least n vertices is a $P_{\geq 3}$-factor covered graph if G satisfies $\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k-1})\} \geq \frac{n+2}{3}$ for any independent subset $\{x_1, x_2, \cdots, x_{2k-1}\}$ of G, where $k \geq 1$ and $n \geq 4k+2$ are two integers; (iii) a $(k+1)$-connected graph G with at least n vertices is a $P_{\geq 3}$-factor deleted graph if G satisfies $\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k-1})\} \geq \frac{n}{3}$ for any independent subset $\{x_1, x_2, \cdots, x_{2k-1}\}$ of G, where $k \geq 1$ and $n \geq 4k+2$ are two integers.

Key Words: Graph, degree condition, $P_{\geq 3}$-factor, $P_{\geq 3}$-factor covered graph, $P_{\geq 3}$-factor deleted graph.
2020 Mathematics Subject Classification: Primary 05C70; Secondary 05C38, 90B10.

1 Introduction

In this article we deal with finite, undirected and simple graphs. We denote by $G = (V(G), E(G))$ a graph, where $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. For $x \in V(G)$, the degree of x in G, denoted by $d_G(x)$, is the number of vertices adjacent to x in G. For any $X \subseteq V(G)$, we denote by $G - X$ the subgraph derived from G by removing vertices in X together with the edges incident to vertices in X. For $E' \subseteq E(G)$, $G - E'$ denotes the subgraph derived from G by removing E'. In particular, we write $G - x = G - \{x\}$ for any $x \in V(G)$ and $G - e = G - \{e\}$ for any $e \in E(G)$. A vertex subset X of G is called independent if no two vertices in X are adjacent to each other. Let $i(G)$ and $c(G)$ denote the number of isolated vertices and connected components of G, respectively. The isolated vertex set of G is denoted by $I(G)$, and so $i(G) = |I(G)|$. We denote by K_n the complete graph of order n, and by P_n the path of order n. For two graphs G_1 and G_2, we denote by $G_1 \vee G_2$ the join of G_1 and G_2.

A spanning subgraph F (i.e. $V(F) = V(G)$) of a graph G is called a 1-factor if $d_F(x) = 1$ holds for all $x \in V(G)$. A graph H is factor-critical if every induced subgraph of order $|V(H)| - 1$ has a 1-factor. A graph R is called a sun if $R = K_1$, $R = K_2$ or R is the corona

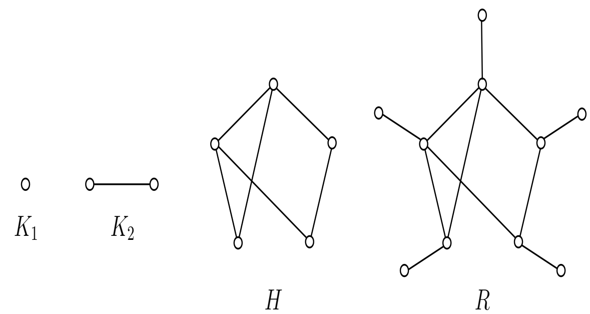

Figure 1: A factor-critical graph H and the sun R obtained from H.

of a factor-critical graph H with at least three vertices, namely, R is derived from H by adding a new vertex $u = u(v)$ together with a new edge uv for each $v \in V(H)$ to H (Figure 1, which was shown by Kano, Lu and Yu [9]). Obviously, $d_R(u) = 1$. A sun of order n with $n \geq 6$ is called a big sun. A component of a graph G is called a sun component if it is isomorphic to a sun. Let $sun(G)$ denote the number of sun components of G. In fact, $i(G) \leq sun(G) \leq c(G)$.

A spanning subgraph F of a graph G is called a path factor if every component of F is a path. For an integer $d \geq 2$, a $P_{\geq d}$-factor of a graph G is a spanning subgraph F such that every component is isomorphic to a path of k vertices for some $k \geq d$. A graph G is called a $P_{\geq d}$-factor covered graph if for any $e \in E(G)$, G has a $P_{\geq d}$-factor covering e. A graph G is called a $P_{\geq d}$-factor deleted graph if for any $e \in E(G)$, G has a $P_{\geq d}$-factor excluding e.

Wang [12] presented a criterion for a bipartite graph admitting a $P_{\geq 3}$-factor. Kaneko [6] established a criterion for a graph to have a $P_{\geq 3}$-factor. Kano, Katona and Király [7] posed a shorter proof of Kaneko's result. Ando, Egawa, Kaneko, Kawarabayshi and Matsuda [1] verified that a claw-free graph with minimum degree at least d admitted a $P_{\geq d+1}$-factor. Zhang and Zhou [16] raised a characterization for a $P_{\geq 3}$-factor covered graph. Zhou [18] derived a sufficient condition for the existence of a $P_{\geq 3}$-factor covered graph. Zhou [20, 21], Gao, Wang and Cheng [3] gave some sufficient conditions for graphs to be $P_{\geq 3}$-factor deleted graphs. Zhou, Sun and Liu [25], Hua [5], Zhou, Yang and Xu [28] got some results on the existence of $P_{\geq 3}$-factor graphs with given properties. Some other results on path factors can be referred to Kano, Lee and Suzuki [8], Kelmans [10], Egawa, Furuya and Ozeki [2], Zhou, Bian and Pan [22], Zhou, Bian and Sun [23]. Some relationships between degree conditions and graph factors were derived by Zhou, Xu and Sun [27], Zhou, Liu and Xu [24], Zhou, Zhang and Xu [29], Gao, Wang and Guirao [4], Wang and Zhang [13], Lv [11], Zhou [17], Zhou, Sun and Pan [26]. Some other results on graph factors can be found in Wang and Zhang [14], Yuan and Hao [15], Zhou [19].

The following results on path factors and path factor covered graphs are known, which play a key role in the proof of our main theorems.

Theorem 1 ([6]). *A graph G admits a $P_{\geq 3}$-factor if and only if*

$$sun(G - X) \leq 2|X|$$

for all $X \subseteq V(G)$.

Theorem 2 ([16]). *A connected graph G is a $P_{\geq 3}$-factor covered graph if and only if*

$$sun(G-X) \leq 2|X| - \varepsilon(X)$$

for all $X \subseteq V(G)$, where $\varepsilon(X)$ is defined by

$$\varepsilon(X) = \begin{cases} 2, & if\ X\ is\ not\ an\ independent\ set; \\ 1, & if\ X\ is\ a\ nonempty\ independent\ set,\ and\ G-X\ admits \\ & a\ non-sun\ component; \\ 0, & otherwise. \end{cases}$$

In this article, we study $P_{\geq 3}$-factors of graphs, $P_{\geq 3}$-factor covered graphs and $P_{\geq 3}$-factor deleted graphs. Then we establish the relationship between degree conditions and $P_{\geq 3}$-factors of graphs (or $P_{\geq 3}$-factor covered graphs, or $P_{\geq 3}$-factor deleted graphs), which are shown in Sections 2–4.

2 $P_{\geq 3}$-factors in graphs

Next, we pose the main theorem in this section.

Theorem 3. *A k-connected graph G with n vertices admits a $P_{\geq 3}$-factor if G satisfies*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k+1})\} \geq \frac{n}{3}$$

for any independent subset $\{x_1, x_2, \cdots, x_{2k+1}\}$ of G, where $k \geq 1$ and $n \geq 4k+4$ are two integers.

Proof. Suppose, to the contrary, that G has no $P_{\geq 3}$-factor. Then it follows from Theorem 1 that there exists some vertex subset X of G such that

$$sun(G-X) \geq 2|X| + 1. \tag{1}$$

Claim 1. $|X| \geq k$.
Proof. Let $|X| \leq k-1$. Then $G-X$ is connected since G is k-connected, namely, $c(G-X) = 1$. Combining this with (1), we derive

$$2|X| + 1 \leq sun(G-X) \leq c(G-X) = 1.$$

Thus, we get $|X| = 0$ and $sun(G-X) = 1$. Combining this with $n \geq 4k+4$, we see that G is a big sun. We denote by R the factor-critical graph of G with $|V(R)| = \frac{1}{2}n$. Obviously, there exists an independent set $\{x_1, x_2, \cdots, x_{2k+1}\} \subseteq V(G) \setminus V(R)$ since $n \geq 4k+4$, and so $d_G(x_i) = 1$ for $1 \leq i \leq 2k+1$. By the degree condition of Theorem 3, we admit

$$\frac{n}{3} \leq \max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k+1})\} = 1,$$

and so $n \leq 3$, which contradicts $n \geq 4k+4 \geq 8$ since $k \geq 1$. Claim 1 is verified. □
Claim 2. $i(G-X) \leq 2k$.

Proof. Assume that $i(G-X) \geq 2k+1$, which implies that there exist at least $2k+1$ isolated vertices $z_1, z_2, \cdots, z_{2k+1}$ in $G-X$. And so, $d_{G-X}(z_i) = 0$ for $1 \leq i \leq 2k+1$. Thus, we deduce

$$d_G(z_i) \leq d_{G-X}(z_i) + |X| = |X| \tag{2}$$

for $1 \leq i \leq 2k+1$.

Combining the degree condition of Theorem 3 with an independent subset $\{z_1, z_2, \cdots, z_{2k+1}\}$ of G, we admit

$$\max\{d_G(z_1), d_G(z_2), \cdots, d_G(z_{2k+1})\} \geq \frac{n}{3}. \tag{3}$$

It follows from (2) and (3) that

$$|X| \geq \max\{d_G(z_1), d_G(z_2), \cdots, d_G(z_{2k+1})\} \geq \frac{n}{3}. \tag{4}$$

In terms of (1) and (4), we get

$$n \geq |X| + sun(G-X) \geq |X| + 2|X| + 1 = 3|X| + 1 \geq 3 \cdot \frac{n}{3} + 1 = n+1,$$

which is a contradiction. This completes the proof of Claim 2. □

In view of (1) and Claim 1, we have

$$sun(G-X) \geq 2|X| + 1 \geq 2k+1. \tag{5}$$

It follows from (5) that there exist t sun components in $G-X$, denoted by $H_1, H_2, \cdots, H_t$, where $t \geq 2k+1$. Select $v_i \in V(H_i)$ with $d_{H_i}(v_i) \leq 1$, $i = 1, 2, \cdots, 2k+1$. It is obvious that $\{v_1, v_2, \cdots, v_{2k+1}\}$ is an independent set of G. Combining this with the degree condition of Theorem 3, we get

$$\max\{d_G(v_1), d_G(v_2), \cdots, d_G(v_{2k+1})\} \geq \frac{n}{3}. \tag{6}$$

Without loss of generality, assume $d_G(v_1) \geq \frac{n}{3}$ by (6). Hence, we deduce

$$d_{G[X]}(v_1) = d_G(v_1) - d_{H_1}(v_1) \geq \frac{n}{3} - 1,$$

where $G[X]$ denotes the subgraph induced by X in G, and so

$$|X| \geq d_{G[X]}(v_1) \geq \frac{n}{3} - 1. \tag{7}$$

In terms of (1), (7), Claim 2, $k \geq 1$ and $n \geq 4k+4$, we derive

$$\begin{aligned} n &\geq |X| + 2 \cdot sun(G-X) - i(G-X) \\ &\geq |X| + 2(2|X|+1) - 2k \\ &= 5|X| - 2k + 2 \\ &\geq 5\left(\frac{n}{3} - 1\right) - 2k + 2 \\ &= n + \frac{2n}{3} - 2k - 3 \\ &\geq n + \frac{2(4k+4)}{3} - 2k - 3 \\ &= n + \frac{2k}{3} - \frac{1}{3} \\ &\geq n + \frac{1}{3} \\ &> n, \end{aligned}$$

which is a contradiction. The proof of Theorem 3 is complete. □

Remark 1. Next, we claim that

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k+1})\} \geq \frac{n}{3}$$

in Theorem 3 cannot be replaced by

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k+1})\} \geq \frac{n-1}{3}.$$

Let $k \geq 1$ be an integer and r be a sufficiently large integer. Let $G = K_{kr} \vee ((2kr+1)K_1)$. Then we see that G is kr-connected, $n = 3kr+1$ and

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k+1})\} = kr = \frac{n-1}{3}$$

for any independent subset $\{x_1, x_2, \cdots, x_{2k+1}\}$ of G. Write $X = V(K_{kr})$. Thus, we get

$$sun(G-X) = 2kr+1 = 2|X|+1 > 2|X|.$$

According to Theorem 1, G has no $P_{\geq 3}$-factor.

3 $P_{\geq 3}$-factor covered graphs

Theorem 4. *A k-connected graph G with n vertices is a $P_{\geq 3}$-factor covered graph if G satisfies*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k-1})\} \geq \frac{n+2}{3}$$

for any independent subset $\{x_1, x_2, \cdots, x_{2k-1}\}$ of G, where $k \geq 1$ and $n \geq 4k+2$ are two integers.

Proof. Assume that G is not a $P_{\geq 3}$-factor covered graph. According to Theorem 2, we have

$$sun(G-X) \geq 2|X| - \varepsilon(X) + 1 \tag{1}$$

for some vertex subset X of G.

In what follows, we consider two cases by the value of $i(G-X)$.

Case 1. $i(G-X) \geq 2k-1$.

Let $\{v_1, v_2, \cdots, v_{2k-1}\} \subseteq I(G-X)$. by the degree condition of Theorem 4, we derive

$$|X| \geq \max\{d_G(v_1), d_G(v_2), \cdots, d_G(v_{2k-1})\} \geq \frac{n+2}{3}. \tag{2}$$

Using (1), (2) and $\varepsilon(X) \leq 2$, we obtain

$$\begin{aligned} n &\geq |X| + sun(G-X) \geq |X| + 2|X| - \varepsilon(X) + 1 \\ &\geq 3|X| - 1 \geq 3 \cdot \frac{n+2}{3} - 1 = n+1, \end{aligned}$$

which is a contradiction.

Case 2. $i(G-X) \leq 2k-2$.

Claim 1. $|X| \geq k$.

Proof. Assume that $|X| \leq k-1$. Then $G-X$ is connected since G is k-connected. Hence, we have $c(G-X) = 1$.

If $|X| = 0$, then $\varepsilon(X) = 0$. It follows from (1) that

$$1 = 2|X| + 1 = 2|X| - \varepsilon(X) + 1 \leq sun(G-X) \leq c(G-X) = 1.$$

Thus, we admit $sun(G) = sun(G-X) = 1$. Note that $n \geq 4k+2$. Therefore, G is a big sun of order n. Let R be the factor-critical graph of G with $|V(R)| = \frac{n}{2}$. Clearly, there exists an independent subset $\{v_1, v_2, \cdots, v_{2k-1}\} \subseteq V(G) \setminus V(R)$ of G since $n \geq 4k+2$, and so $d_G(v_i) = 1$ for $1 \leq i \leq 2k-1$. Thus, we obtain

$$1 = \max\{d_G(v_1), d_G(v_2), \cdots, d_G(v_{2k-1})\} \geq \frac{n+2}{3},$$

which contradicts $n \geq 4k+2$.

If $|X| = 1$, then $\varepsilon(X) \leq 1$. Using (1), we infer

$$2 \leq 2|X| \leq 2|X| - \varepsilon(X) + 1 \leq sun(G-X) \leq c(G-X) = 1,$$

which is a contradiction.

If $2 \leq |X| \leq k-1$, then $\varepsilon(X) \leq 2$. In terms of (1), we get

$$3 \leq 2|X| - 1 \leq 2|X| - \varepsilon(X) + 1 \leq sun(G-X) \leq c(G-X) = 1,$$

which is a contradiction. Claim 1 is proved. □

In light of (1), Claim 1 and $\varepsilon(X) \leq 2$, we deduce

$$2k-1 \leq 2|X| - 1 \leq 2|X| - \varepsilon(X) + 1 \leq sun(G-X),$$

which implies that $G-X$ admits at least $2k-1$ sun components, denoted by $H_1, H_2, \cdots, H_t$, where $t \geq 2k-1$. We choose $v_i \in V(H_i)$ with $d_{H_i} \leq 1$, $i = 1, 2, \cdots, 2k-1$. Clearly, $\{v_1, v_2, \cdots, v_{2k-1}\}$ is an independent set of G. According to the degree condition of Theorem 4, we possess

$$|X| + 1 \geq \max\{d_G(v_1), d_G(v_2), \cdots, d_G(v_{2k-1})\} \geq \frac{n+2}{3},$$

that is,

$$|X| \geq \frac{n-1}{3}. \tag{3}$$

It follows from (1), (3), $\varepsilon(X) \leq 2$ and $n \geq 4k+2$ that

$$\begin{aligned} n &\geq |X| + 2 \cdot sun(G-X) - i(G-X) \\ &\geq |X| + 2(2|X| - \varepsilon(X) + 1) - (2k-2) \\ &\geq |X| + 2(2|X| - 1) - (2k-2) \\ &= 5|X| - 2k \\ &\geq 5 \cdot \frac{n-1}{3} - 2k \\ &= n + \frac{2n}{3} - \frac{5}{3} - 2k \\ &\geq n + \frac{2(4k+2)}{3} - \frac{5}{3} - 2k \\ &= n + \frac{2k}{3} - \frac{1}{3} \\ &\geq n + \frac{2}{3} - \frac{1}{3} \\ &> n, \end{aligned}$$

which is a contradiction. Theorem 4 is verified. □

Remark 2. Next, we explain that

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k-1})\} \geq \frac{n+2}{3}$$

in Theorem 4 cannot be replaced by

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k-1})\} \geq \frac{n+1}{3}.$$

Let $k \geq 1$ be an integer and r be a sufficiently large integer. Let $G = K_{kr} \vee ((2kr-1)K_1)$. Then G is kr-connected, $n = 3kr - 1$ and

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k-1})\} = kr = \frac{n+1}{3}$$

for any independent subset $\{x_1, x_2, \cdots, x_{2k-1}\}$ of G. Set $X = V(K_{kr})$, and so $\varepsilon(X) = 2$. Thus, we derive

$$sun(G-X) = 2kr - 1 = 2|X| - \varepsilon(X) + 1 > 2|X| - \varepsilon(X).$$

In view of Theorem 2, G is not a $P_{\geq 3}$-factor covered graph.

4 $P_{\geq 3}$-factor deleted graphs

Theorem 5. *A $(k+1)$-connected graph G with n vertices is a $P_{\geq 3}$-factor deleted graph if G satisfies*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k-1})\} \geq \frac{n}{3}$$

for any independent subset $\{x_1, x_2, \cdots, x_{2k-1}\}$ of G, where $k \geq 1$ and $n \geq 4k+2$ are two integers.

Proof. For any $e \in E(G)$, let $G' = G - e$. It suffices to prove that G' has a $P_{\geq 3}$-factor. We assume that G' has no $P_{\geq 3}$-factor. Then by Theorem 1, we acquire

$$sun(G' - X) \geq 2|X| + 1 \tag{1}$$

for some $X \subseteq V(G)$.

We shall discuss the following two cases by the value of $|X|$.

Case 1. $|X| \leq k-1$.

Note that $G' = G - e$ and G is $(k+1)$-connected. Then G' is k-connected. Hence, $G' - X$ is connected, namely, $c(G' - X) = 1$.

If $1 \leq |X| \leq k-1$, then from (1), we get

$$3 \leq 2|X| + 1 \leq sun(G' - X) \leq c(G' - X) = 1,$$

which is a contradiction.

If $|X| = 0$, then by (1), we deduce

$$1 = 2|X| + 1 \leq sun(G' - X) \leq c(G' - X) = 1,$$

which implies that G' is a big sun since $n \geq 4k+2$. Let R be the factor-critical graph of G with $|V(R)| = \frac{n}{2}$. Then $d_{G'}(v_i) = 1$ for $v_i \in V(G') \setminus V(R)$. Combining this with $G' = G - e$ and $n \geq 4k+2$, there exists an independent set $\{v_1, v_2, \cdots, v_{2k-1}\} \subseteq V(G) \setminus V(R)$. Thus, we admit

$$1 \geq \max\{d_G(v_1), d_G(v_2), \cdots, d_G(v_{2k-1})\} \geq \frac{n}{3},$$

and so $n \leq 3$, which contradicts $n \geq 4k+2$.

Case 2. $|X| \geq k$.

Subcase 2.1. $i(G - X) \leq 2k-2$.

In view of (1), we have

$$sun(G' - X) \geq 2|X| + 1 \geq 2k + 1. \tag{2}$$

Note that $sun(G' - X) \leq sun(G - X) + 2$. Combining this with (2), we derive

$$sun(G - X) \geq sun(G' - X) - 2 \geq (2k+1) - 2 = 2k - 1,$$

which implies that $G - X$ possesses at least $2k-1$ sun components, denoted by $H_1, H_2, \cdots, H_t$, where $t \geq 2k-1$. Select $v_i \in V(H_i)$ with $d_{H_i}(v_i) \leq 1$ for $1 \leq i \leq 2k-1$. Obviously, $\{v_1, v_2, \cdots, v_{2k-1}\}$ is an independent set of G. Thus, we get

$$|X| + 1 \geq \max\{d_G(v_1), d_G(v_2), \cdots, d_G(v_{2k-1})\} \geq \frac{n}{3},$$

namely,

$$|X| \geq \frac{n}{3} - 1. \tag{3}$$

Note that $i(G' - X) \leq i(G - X) + 2$. Combining this with (1), (3) and $n \geq 4k + 2$, we have

$$\begin{aligned} n &\geq |X| + 2 \cdot sun(G' - X) - i(G' - X) \\ &\geq |X| + 2(2|X| + 1) - i(G - X) - 2 \\ &= 5|X| - i(G - X) \\ &\geq 5 \cdot \left(\frac{n}{3} - 1\right) - (2k - 2) \\ &= n + \frac{2n}{3} - 2k + \frac{1}{3} \\ &\geq n + \frac{2(4k + 2)}{3} - 2k + \frac{1}{3} \\ &= n + \frac{2k}{3} + \frac{5}{3} \\ &> n, \end{aligned}$$

which is a contradiction.

Subcase 2.2. $i(G - X) \geq 2k - 1$.

In this case, there exist at least $2k - 1$ isolated vertices $z_1, z_2, \cdots, z_{2k-1}$ in $G - X$, and so $d_{G-X}(z_i) = 0$ for $1 \leq i \leq 2k - 1$. Thus, we derive

$$d_G(z_i) \leq d_{G-X}(z_i) + |X| = |X|$$

for $1 \leq i \leq 2k - 1$. In light of the degree condition of Theorem 5, we get

$$\max\{d_G(z_1), d_G(z_2), \cdots, d_G(z_{2k-1})\} \geq \frac{n}{3},$$

and so

$$|X| \geq \max\{d_G(z_1), d_G(z_2), \cdots, d_G(z_{2k-1})\} \geq \frac{n}{3}. \tag{4}$$

According to (1) and (4), we infer

$$n \geq |X| + sun(G' - X) \geq |X| + 2|X| + 1 = 3|X| + 1 \geq 3 \cdot \frac{n}{3} + 1 = n + 1,$$

which is a contradiction. Theorem 5 is proved. □

Remark 3. Next, we show that

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k-1})\} \geq \frac{n}{3}$$

in Theorem 5 cannot be replaced by

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k-1})\} \geq \frac{n-1}{3}.$$

Let $k \geq 1$ be an integer and r be a sufficiently large integer. Let $G = K_{(k+1)r} \vee ((2(k+1)r+1)K_1)$. Then G is $(k+1)r$-connected, $n = 3(k+1)r+1$ and

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_{2k-1})\} = (k+1)r = \frac{n-1}{3}$$

for any independent subset $\{x_1, x_2, \cdots, x_{2k-1}\}$ of G. For any $e \in E(G)$, let $G' = G - e$. Write $X = V(K_{(k+1)r})$. Thus, we get

$$sun(G' - X) = 2(k+1)r + 1 = 2|X| + 1 > 2|X|.$$

According to Theorem 1, G has no $P_{\geq 3}$-factor, namely, G is not $P_{\geq 3}$-factor deleted.

Acknowledgement *The author would like to thank the anonymous reviewers for their careful reading and correction on this paper.*

References

[1] K. Ando, Y. Egawa, A. Kaneko, K. Kawarabayashi, H. Matsuda, Path factors in claw-free graphs, *Discrete Mathematics*, **243**, 195–200 (2002).

[2] Y. Egawa, M. Furuya, K. Ozeki, Sufficient conditions for the existence of a path-factor which are related to odd components, *Journal of Graph Theory*, **89**, 327–340 (2018).

[3] W. Gao, W. Wang, Y. Chen, Tight bounds for the existence of path factors in network vulnerability parameter settings, *International Journal of Intelligent Systems*, **36**, 1133–1158 (2021).

[4] W. Gao, W. Wang, J. Guirao, The extension degree conditions for fractional factor, *Acta Mathematica Sinica, English Series*, **36**, 305–317 (2020).

[5] H. Hua, Toughness and isolated toughness conditions for $P_{\geq 3}$-factor uniform graphs, *Journal of Applied Mathematics and Computing*, **66**, 809–821 (2021).

[6] A. Kaneko, A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two, *Journal of Combinatorial Theory, Series B*, **88**, 195–218 (2003).

[7] M. Kano, G. Y. Katona, Z. Király, Packing paths of length at least two, *Discrete Mathematics*, **283**, 129–135 (2004).

[8] M. Kano, C. Lee, K. Suzuki, Path and cycle factors of cubic bipartite graphs, *Discussiones Mathematicae Graph Theory*, **28**, 551–556 (2008).

[9] M. Kano, H. Lu, Q. Yu, Component factors with large components in graphs, *Applied Mathematics Letters*, **23**, 385–389 (2010).

[10] A. Kelmans, Packing 3-vertex paths in claw-free graphs and related topics, *Discrete Applied Mathematics*, **159**, 112–127 (2011).

[11] X. Lv, A degree condition for fractional (g, f, n)-critical covered graphs, *AIMS Mathematics*, **5**, 872–878 (2020).

[12] H. Wang, Path factors of bipartite graphs, *Journal of Graph Theory*, **18**, 161–167 (1994).

[13] S. Wang, W. Zhang, On k-orthogonal factorizations in networks, *RAIRO-Operations Research*, **55**, 969–977 (2021).

[14] S. Wang, W. Zhang, Research on fractional critical covered graphs, *Problems of Information Transmission*, **56**, 270–277 (2020).

[15] Y. Yuan, R. Hao, A neighborhood union condition for fractional ID-$[a, b]$-factor-critical graphs, *Acta Mathematicae Applicatae Sinica, English Series*, **34**, 775–781 (2018).

[16] H. Zhang, S. Zhou, Characterizations for $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs, *Discrete Mathematics*, **309**, 2067–2076 (2009).

[17] S. Zhou, A neighborhood union condition for fractional (a, b, k)-critical covered graphs, *Discrete Applied Mathematics*, **323**, 343–348 (2022).

[18] S. Zhou, Some results about component factors in graphs, *RAIRO-Operations Research*, **53**, 723–730 (2019).

[19] S. Zhou, Binding numbers and restricted fractional (g, f)-factors in graphs, *Discrete Applied Mathematics*, **305**, 350–356 (2021).

[20] S. Zhou, Remarks on path factors in graphs, *RAIRO-Operations Research*, **54**, 1827–1834 (2020).

[21] S. Zhou, Some results on path-factor critical avoidable graphs, *Discussiones Mathematicae Graph Theory*, **43**, 233–244 (2023).

[22] S. Zhou, Q. Bian, Q. Pan, Path factors in subgraphs, *Discrete Applied Mathematics*, **319**, 183–191 (2022).

[23] S. Zhou, Q. Bian, Z. Sun, Two sufficient conditions for component factors in graphs, *Discussiones Mathematicae Graph Theory*, DOI: 10.7151/dmgt.2401.

[24] S. Zhou, H. Liu, Y. Xu, A note on fractional ID-$[a, b]$-factor-critical covered graphs, *Discrete Applied Mathematics*, **319**, 511–516 (2022).

[25] S. Zhou, Z. Sun, H. Liu, Isolated toughness and path-factor uniform graphs, *RAIRO-Operations Research*, **55**, 1279–1290 (2021).

[26] S. Zhou, Z. Sun, Q. Pan, A sufficient condition for the existence of restricted fractional (g, f)-factors in graphs, *Problems of Information Transmission*, **56**, 332–344 (2020).

[27] S. Zhou, Y. Xu, Z. Sun, Degree conditions for fractional (a, b, k)-critical covered graphs, *Information Processing Letters*, **152**, 105838 (2019).

[28] S. Zhou, F. Yang, L. Xu, Two sufficient conditions for the existence of path factors in graphs, *Scientia Iranica*, **26**, 3510–3514 (2019).

[29] S. Zhou, T. Zhang, Z. Xu, Subgraphs with orthogonal factorizations in graphs, *Discrete Applied Mathematics*, **286**, 29–34 (2020).

Received: 15.05.2021
Revised: 18.06.2021
Accepted: 04.07.2021

School of Science,
Jiangsu University of Science and Technology,
Zhenjiang, Jiangsu 212100, China
E-mail: zsz_cumt@163.com