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A topology and the frame attached to a set of primitive submodules

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Dedicated to Professors Toma Albu and Constantin Năstăsescu on the occasion of their 80th anniversary

Abstract

For a multiplication R-module M we define the primitive topology $\mathcal{T}$ on the set $Prt\,(M)$ of primitive submodules of M. We prove that if R is a commutative ring and M is a multiplication R-module, then the complete lattice $Sprt\,(M)$ of semiprimitive submodules of M is a spatial frame. When M is projective in the category $\sigma[M]$, we obtain that the topological spaces $(Prt(M), \mathcal{T})$ and $(Prt(R), \mathcal{T})$ are homeomorphic. As an application, we prove that if M is projective in the category $\sigma[M]$, then $Prt(R)$ has classical Krull dimension if and only if $Prt(M)$ has classical Krull dimension.

Key Words: Multiplication module, primitive submodule, spatial frame, Krull dimension.
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0 Introduction

Multiplication modules were introduced by Barnard [4] and have been studied by several authors [2], [3], [10], [18] and [20]. The relationship between the algebraic properties of a ring and the topological properties of the Zariski topology defined on its prime spectrum has been studied in [11] and [12]. In this paper, we consider the concept of primitive and semiprimitive modules given in [16]. Given a multiplication module M over a commutative ring R, we consider the Primitive Topology for the poset $Prt\,(M)$ of primitive submodules of M.

In [9], [13], [14] and [15] the authors introduce a framework of lattice structure theory to analyze the submodules of a given module; in particular, interesting results are obtained by specializing to the lattice $Sub\,(M)$ of submodules of M. These authors also observe some topological aspects of certain frames constructed in those papers and whose consideration eventually leads to the construction of some spatial frames. [17] A spatial frame F is a frame which is a lattice isomorphic to the set of open subsets of topological space X. In this paper we take that point of view and we extend the results to the framework of primitive submodules of a multiplication module.

The organization of the paper is as follows:

Section 1 provides the material needed for reading the subsequent sections.

Section 2 is dedicated to primitive (semiprimitive) modules. We give the relationship between primitive (semiprimitive) submodules of a multiplication R-module M and primitive ideals of the ring R. In this section we prove there exists bijective correspondence between maximal submodules of M and maximal ideals of R.

In Section 3 we define the Primitive Topology on the set Prt(M) of the primitive subdules of the a multiplication module M and we describe a basis of open sets of this topology.

Section 4 is dedicated to the spatial frame $Sprt(M)$ of the semiprimitive submodules of M. We prove that $Sprt(M)$ is a spatial frame and we prove that the topological spaces $(Prt(M), \mathcal{T})$ and $(Prt(R), \mathcal{T})$ are homeomorphic.

In section 5 we give an application. We prove that if R is a commutative ring and M is a faithful multiplication R-module and $QM \neq M$ for all maximal ideal Q of R, then $Prt(R)$ has classical Krull dimension if and only if $Prt(M)$ has classical Krull dimension, and moreover, $cl.K\dim(Prt(R)) = cl.K\dim(Prt(M))$.

In this paper all rings are associative with an identity, except for some results where R will denote a commutative ring with unity and R-Mod will denote the category of unitary left R-modules. An R-module M is a multiplication module if for every submodule N of M, there exists an ideal I of R such that $N = IM$.

Let M and X be R-modules. Then X is said to be M-generated if there exists an R-epimorphism from a direct sum of copies of M onto X. The trace of M in X is defined as $tr^M(X) = \sum_{f \in Hom_R(M,X)} f(M)$; thus X is M-generated if and only if $tr^M(X) = X$.

Let M be a module. Any module that is isomorphic to a submodule of some homomorphic image of a direct sum of copies of M is called M-subgenerated. The full subcategory of the category of all modules whose objects are all M-subgenerated modules is denoted by $\sigma[M]$. For a ring R, $\sigma[M]$ consists of all R-modules if and only if $R \in \sigma[M]$. Let M and U be modules. M is called U-projective if for every epimorphism $g: U \longrightarrow X$ and homomorphism $f: M \longrightarrow X$, there exists a homomorphism $h: M \longrightarrow U$ such that $g \circ h = f$. A module M is called projective in $\sigma[M]$ if M is U-projective for every $U \in \sigma[M]$. If N is an R-module $ann(N) = \{r \in R \mid rN = 0\}$.

1 Preliminaries

In this section we provide the material needed for reading the following sections. We use the product of modules defined in [6] and we show that if M is a multiplication R-module (when R is a ring with commutative multiplication of ideals, in particular when R is a commutative ring), then this product of modules is commutative and associative.

Definition 1.1. *[7, Definition 1.1] Let R be a ring and $M \in R$- Mod. Let K be a submodule of M and $L \in R$-Mod. We define the product*

$$K_M L = \sum \{f(K) \mid f \in Hom(M, L)\}$$

Note that if $M = R$, then Definition 1.1, then $K_M L$ is the product of left ideals of the ring R. Also note that given a submodule N of M, there exists a submodule $\overline{N} \subset M$ such that $\overline{N}$ is the least fully invariant submodule of M which contains N. In fact, we have that $\overline{N} = \sum \{f(N) \mid f \in Hom(M, M)\}$. Therefore $\overline{N} = N_M M$. Moreover, if K and L are submodules of M, then

$$\sum \left\{ f\left(\overline{K}\right) \mid f \in Hom(M, L) \right\} = \sum \left\{ f(K) \mid f \in Hom(M, L) \right\}.$$

Therefore $\overline{K}_M L = K_M L$.

Notice that if X is an R-module, then $tr^M(X) = \sum_{f \in Hom_R(M,X)} f(M) = M_M X$. Thus X is M generated if and only if $M_M X = X$.

Definition 1.2. *[7, Definition 1.1] Let M be a nonzero module.*

i) A proper fully invariant submodule N of M is called prime in M if $K_M L \subseteq N$, then $K \subseteq N$ or $L \subseteq N$ for any fully invariant submodules K, L of M. The module M is called a prime module if 0 is a prime submodule in M. Note that if M has no nonzero proper fully invariant submodules, then M is prime.

ii) A proper fully invariant submodule N of M is called semiprime in M if $K_M K \subseteq N$, then $K \subseteq N$ for any fully invariant submodule K of M. The module M is called a semiprime module if 0 is a semiprime submodule in M.

Definition 1.3. *[5, definition 1.1]Let M and X be R-modules. The annihilator of X in M is defined as*

$$Ann_M(X) = \bigcap \{Ker(f) \mid f \in Hom_R(M, X)\}.$$

Notice that by [7, Proposition 1.9] we have that $Ann_M(X)$ is a fully invariant submodule of M and is the greatest submodule of M such that $Ann_M(X)_M X = 0$. Also, notice that $Ann_M(X)_M X = M$ if and only if $Hom_R(M, X) = 0$.

Proposition 1.4. *[7, Proposition 1.3] Let $M \in R$-Mod and K, K' be submodules of M. Then:*

1) If $K \subset K'$, then $K_M X \subset K'_M X$ for every $X \in R$-Mod.

2) If $X \in R$-Mod and $Y \subseteq X$, then $K_M Y \subseteq K_M X$.

3) $M_M X = tr^M(X)$ for every $X \in R$-Mod.

4) $0_M X = 0$ for every $X \in R$-Mod.

5) $K_M X = 0$ if and only if $f(K) = 0$ for all $f \in Hom(M, X)$.

6) If X, Y are submodules for any module $N \in R$-Mod, then $K_M X + K_M Y \subseteq K_M(X + Y)$.

7) If $\{K_i\}_{i \in I}$ is a family of submodules of M, then $\left[\sum_{i \in I} K_i\right]_M N = \sum_{i \in I} K_{i\,M} N$.

8) If $\{X_i\}_{i \in I}$ is a family of R-modules, then $K_M \left[\bigoplus_{i \in I} X_i\right] = \bigoplus_{i \in I} K_M X_i$.

Remark 1.5. *By [9, Lemma 1.3] we have that if R is a commutative ring and M is a multiplication R-module, then M generates all its submodules. Thus $M_M N = N$ for all submodule N of M. Moreover by [9, Proposition 1.4 and Corollary 1.5] we have that $N_M L = L_M N$ and $(N_M L)_M K = N_M(L_M K)$ for all N, L and K submodules of M.*

Notice that the product of submodules of M is not associative in general. We consider the example in [8, Remark 1.26]. In that example we have that K is a submodule of $M = E(S)$. Moreover $K_M K = S$ and $S_M K = 0$. Therefore $(K_M K)_M K = S_M K = 0$, but $K_M (K_M K) = K_M S = S$. Hence we have that $(K_M K)_M K \neq K_M (K_M K)$.

Lemma 1.6. *Let R be a commutative ring and let I be an ideal of R . If $M \in R$-Mod is a multiplication module and S is an R-module, then $(IM)_M S = I(M_M S)$.*

Proof. We have that

$$(IM)_M S = \sum_{f:M\longrightarrow S} f(IM) = \sum_{f:M\longrightarrow S} If(M) = I(\sum_{f:M\longrightarrow S} f(M)) = I(M_M S)$$

∎

Proposition 1.7. *Let R be a commutative ring, let M be a multiplication R-module and let I be an ideal of R. If S is an R-module generated by M such that $(IM)_M S = 0$, then $IS = 0$.*

Proof. By 1.6 we have that $(IM)_M S = I(M_M S)$. As S is generated by M, then $M_M S = S$. Thus $0 = (IM)_M S = I(M_M S) = IS$. ∎

Proposition 1.8. *Let R be a commutative ring. If M is a multiplication R-module and N is a submodule of M, then N is a fully invariant submodule of M.*

Proof. By Remark 1.5 we have that $M_M N = N$ and $M_M N = N_M M$. Hence $N_M M = N$. As

$$N_M M = \sum_{f:M\longrightarrow M} f(N), \text{ then } \sum_{f:M\longrightarrow M} f(N) = N.$$

We deduce that $f(N) \subseteq N$ for all morphism $f : M \longrightarrow M$. Thus N is a fully invariant submodule of M.

∎

Notice that if R is a ring with a commutative multiplication of ideals and M is a multiplication module, then

$$N_M \textstyle\sum_{i\in I} K_i = \left[\sum_{i\in I} K_i\right]_M N = \sum_{i\in I} (K_{iM} N) = \sum_{i\in I} (N_M K_i)$$

for every family of submodules $\{K_i\}_{i\in I}$ of M.

Proposition 1.9. *Let R be a commutative ring. If M is a multiplication R-module and N is a maximal submodule of M, then N is a prime submodule of M.*

Proof. By Proposition 1.8 we have that N is a fully invariant submodule of M. Let K and T be submodules of M such that $K_M T \subseteq N$. If $K \nsubseteq N$, then $K + N = M$. By Remark 1.5 we have that $T = M_M T$. Thus

$$T = M_M T = (K + N)_M T = K_M T + N_M T = K_M T + T_M N.$$

As $K_M T \subseteq N$ and $T_M N \subseteq N$, then $T = K_M T + T_M N \subseteq N$, which implies that N is a prime submodule of M. □

Remark 1.10. *If R is a commutative ring and S is a simple R-module, then $ann(S)$ is a maximal ideal of r, which implies that $ann(S)$ is a prime ideal of R.*

Proposition 1.11. *Let R be a commutative ring and let M be a multiplication R-module projective in the category $\sigma[M]$. If S is a simple R-module generated by M, then $Ann_M(S)$ is a prime submodule of M.*

Proof. As M generates S, then $M_M S = S$. Hence $Ann_M(S) \subsetneq M$. Let K and L be submodules of M such that $K_M L \subseteq Ann_M(S)$. Thus $(K_M L)_M S = 0$. By [5, Proposition 5.6] we have that $0 = (K_M L)_M S = K_M(L_M S)$. As S is a simple module, then $L_M S = 0$ or $L_M S = S$. If $L_M S = 0$, then $L \subseteq Ann_M(S)$. If $L_M S = S$, then $0 = K_M(L_M S) = K_M S$, which implies that $K \subseteq Ann_M(S)$. Hence $Ann_M(S)$ is a prime submodule of M. □

Remark 1.12. *By [10, Section 1] we have that $N = ann(M/N)M$ for any submodule N of a multiplication module M. By [9, Proposition 2.21] we have that if N is a prime submodule of M, then $ann(M/N)$ is a prime ideal.*

Proposition 1.13. *Let R be a commutative ring and let M be a faithful multiplication R-module such that M is a projective in the category $\sigma[M]$. If S is a simple R-module generated by M, then $ann(S) = ann(M/Ann_M(S))$.*

Proof. By Proposition we have that 1.11 $Ann_M(S)$ is a prime submodule of M. By Remark 1.12 we have that $Ann_M(S) = IM$ where $I = ann(M/Ann_M(S))$ is a prime ideal of R.

We shall prove that $ann(S) = I$. To do so, we put $J := ann(S)$. Since $Ann_M(S) = IM$, then $(IM)_M S = 0$. By Proposition 1.7 we have that $IS = 0$. Thus $I \subseteq ann(S) = J$. Now we consider the submodule JM of M. By Lemma 1.6 we have that $(JM)_M S = J(M_M S) = JS = 0$. As $Ann_M(S) = IM$, then $JM \subseteq IM$. By [9, Proposition 1.6] we have that $J \subseteq I$. Hence $J = ann(S) = I = ann(M/Ann_M(S))$. □

2 Primitive, Semiprimitive and Maximal Submodules

In this section we use the concepts of primitive and semiprimitive modules. For a commutative ring we prove that if N is a primitive submodule of a multiplication R-submodule M, then $ann(M/N)$ is a primitive ideal of R. Also, we prove that there exists a bijective correspondence between the maximal submodules of M and the maximal ideals of R.

Definition 2.1. *[16, Definition 3.2]. Let M be a module and P a proper submodule of M. The module P is called a primitive submodule of M if there exists a simple module $S \in \sigma[M]$ such that $P = Ann_M(S)$. The module M is called primitive if 0 is a primitive submodule of M.*

Note that if $M = R$ and I is an ideal of R, then I is a primitive ideal in R in the sense of Definition 2.1 if and only if I is a primitive ideal.

Remark 2.2. *If R a commutative ring and M is a nonzero multiplication R-module in [10, Theorem 2.5], the authors proved that every submodule of M is contained in a maximal submodule of M. Hence M contains maximal submodules. Thus if N is a maximal submodule of M, then $(M/N) \in \sigma[M]$ is a simple module and $P = Ann_M(M/N)$ is a primitive submodule of M.*

Notice that by Proposition 1.11 we have that if M is projective in the category $\sigma[M]$, then every primitive submodule of M is a prime submodule of M.

Proposition 2.3. *Let R be a commutative ring and let M be a faithful multiplication R-module. If S is a simple R-module generated by M such that $I = ann(S)$, then $Ann_M(S) = IM$ and therefore IM is a primitive submodule of M.*

Proof. As S is generated by M, then $S \in \sigma[M]$. We claim that $Ann_M(S) = IM$. Indeed, by Lemma 1.6 we have that

$$(IM)_M S = I(M_M S) = IS = 0.$$

Hence, $IM \subseteq Ann_M(S)$. Since M is multiplication module and $Ann_M(S)$ is a submodule of M, then there exists an ideal J of R such that $Ann_M(S) = JM$. Thus $(JM)_M S = 0$. By Proposition 1.7 we have that $J_M S = 0$, which implies that $J \subseteq ann(S) = I$. Thus $JM \subseteq IM$. But $JM = Ann_M(S)$. So $Ann_M(S) \subseteq IM$. Hence, $Ann_M(S) = IM$. Therefore IM is a primary submodule of M. □

Proposition 2.4. *Let R be a commutative ring and let M be a faithful multiplication R-module such that M is projective in the category $\sigma[M]$. If N is a primitive submodule of M and M generates all the simple R-modules, then $ann(M/N)$ is a primitive ideal of R.*

Proof. As N is a primitive submodule of M, then there exists an $S \in \sigma[M]$ simple module such that $N = Ann_M(S)$. By Proposition 1.13 we have that $ann(S) = ann(M/Ann_M(S))$. Thus $ann(S) = ann(M/N)$. Hence, $ann(M/N)$ is a primitive ideal of R. □

Notice that the condition: M generates all the simple R-modules does not imply that $\sigma[M] = R$-Mod. We can see this in the following example:

Example 2.5. *Let R be a commutative ring and $M = \bigoplus_{S \text{ is aimple}} S$. Thus $\sigma[M] = \{N \mid N \text{ is semisimple } R\text{-module}\}$. So $\sigma[M] = R$-Mod if and only if R is a semisimple ring. If $R = \mathbb{Z}$ and $M = \bigoplus_{p \text{ is prime number}} \mathbb{Z}_p$, then $\sigma[M] \neq \mathbb{Z}$-Mod.*

Proposition 2.6. *Let R be a commutative ring and let M be a faithful multiplication R-module such that M is projective in the category $\sigma[M]$ and M generates all the simple R-modules. If N is a primitive submodule of M, then there exists only one primitive ideal $I = ann(M/N)$ of R, such that $N = IM$.*

Proof. Let J be a primitive ideal of R such that $M = JM$. Thus $IM = JM$. But I and J are prime ideals, so by [9, Proposition 1.9], we have that $I = J$. □

Proposition 2.7. *Let R be a commutative ring and let M be a faithful multiplication R-module. Suppose that $QM \neq M$ for all maximal ideal Q of R. If P_α is a primitive ideal of R for every $\alpha \in \mathcal{L}$, then $(\cap_{\alpha\in\mathcal{L}} P_\alpha)M = \cap_{\alpha\in\mathcal{L}} (P_\alpha M)$.*

Proof. It is well-known that every primitive ideal of R is prime ideal. Thus P_α is a prime ideal of R for all $\alpha \in \mathcal{L}$. The result follows from [9, Proposition 2.25] □

Definition 2.8. *Let M be a module and Q a submodule of M. The module Q is called a semiprimitive submodule of M if $Q = \bigcap_{\alpha\in\mathcal{L}} N_\alpha$ such that N_α is a primitive submodule of M for all $\alpha \in \mathcal{L}$. The module M is called semiprimitive if 0 is a semiprimitive submodule of M.*

Lemma 2.9. *Let R be a ring and let M be an R-module. If $\{N_i\}_{i\in\mathcal{I}}$ is a family of R-modules, then$Ann_M(\bigoplus_{i\in\mathcal{I}} N_i) = \bigcap_{i\in\mathcal{I}} Ann_M(N_i)$.*

Proof. Since $N_i \subseteq \bigoplus_{i\in\mathcal{I}} N_i$, then

$$Ann_M(\textstyle\bigoplus_{i\in\mathcal{I}} N_i) \subseteq Ann_M(N_i) \text{ for all } i \in \mathcal{I}. \text{ Thus } K_M N_i = 0 \text{ for all } i \in \mathcal{I}.$$

Thus

$$Ann_M(\textstyle\bigoplus_{i\in\mathcal{I}} N_i) \subseteq \bigcap_{\imath\in\mathcal{I}} Ann_M(N_i).$$

Now, we put $K = \bigcap_{i\in\mathcal{I}} Ann_M(N_i)$. Thus $K_M N_i = 0$ for all $i \in \mathcal{I}$. By Proposition 1.4 (8) we have that $K_M(\bigoplus_{i\in\mathcal{I}} N_i) = 0$, which implies that $K \subseteq Ann_M(\bigoplus_{i\in\mathcal{I}} N_i)$. Thus

$$\textstyle\bigcap_{\imath\in\mathcal{I}} Ann_M(N_i) \subseteq Ann_M(\bigoplus_{i\in\mathcal{I}} N_i).$$

So $\bigcap_{\imath\in\mathcal{I}} Ann_M(N_i) = Ann_M(\bigoplus_{i\in\mathcal{I}} N_i)$. □

Proposition 2.10. *Let R be a ring and let M be an R-module. If Q is a submodule of M, then the following conditions are equivalents:*
i) Q is a semiprimitive submodule of M.
ii) $Q = Ann_M(T)$ where $T \in \sigma[M]$ is a semisimple module.

Proof. *i) ⇒ ii)* As Q is a semiprimitive module, then $Q = \bigcap_{\alpha\in\mathcal{L}} N_\alpha$ where N_α is a primitive submodule of M for all $\alpha \in \mathcal{L}$. So $Ann_M(S_\alpha) = N_\alpha$ for all $\alpha \in \mathcal{L}$, where $S_\alpha \in \sigma[M]$ is a simple R-module. By Lemma 2.9 we have that

$$Ann_M(\bigoplus_{\alpha\in\mathcal{L}} S_\alpha) = \bigcap_{\alpha\in\mathcal{L}} Ann_M(S_\alpha) = \bigcap_{\alpha\in\mathcal{L}} N_\alpha = Q.$$

Thus $T = \bigoplus_{\alpha\in\mathcal{L}} S_\alpha \in \sigma[M]$ and T is a semisimple module.

ii) $\Rightarrow$ *i)* We suppose that $Q = Ann_M(T)$ with $T \in \sigma[M]$ is a semisimple module. Thus $T = \bigoplus_{\alpha\in\mathcal{L}} S_\alpha$, with $S_\alpha \in \sigma[M]$ is a simple R-module for all $\alpha \in \mathcal{L}$. By Lemma 2.9 we have that

$$Q = Ann_M(T) = Ann_M(\bigoplus_{\alpha\in\mathcal{L}} S_\alpha) = \bigcap_{\alpha\in\mathbb{C}} Ann_M(S_\alpha).$$

But $Ann_M(S_\alpha)$ is a primitive submodule of M for all $\alpha \in \mathcal{L}$. Thus Q is a semiprimitive submodule of M. □

We know that a ring R is semiprimitive provided $\mathcal{J}(R) = 0$, where $\mathcal{J}(R)$ is the Jacobson radical of R. We give the counterpart in terms of modules in the following:

Proposition 2.11. *Let R be a ring and let M be an R-module. M is a semiprimitive module if and only if $\mathcal{J}(M) = 0$.*

Proof. $\Rightarrow$) As M is a semiprimitive module, then 0 is a semiprimitive submodule of M. Thus

$0 = \bigcap_{\alpha\in\mathcal{L}} N_\alpha$, where N_α is a primitive submodule of M for all $\alpha \in \mathcal{L}$. By [16, Proposition 3.6] we have that

$$\mathcal{J}(M) = \bigcap\{N \subseteq M \mid N \textit{ is primitive}\} \subseteq \bigcap_{\alpha\in\mathcal{L}} N_\alpha = 0.$$

So $\mathcal{J}(M) = 0$.

$\Leftarrow$) As $\mathcal{J}(M) = 0$, then

$$0 = \mathcal{J}(M) = \bigcap\{N \subseteq M \mid N \textit{ is primitive}\},$$

which implies that 0 is a semiprimitive submodule of M. Thus M is a primitive module. □

We denote

$Prt(M) = \{N \subseteq M \mid N \textit{ is semiprimitive in } M\}$

$Sprt(M) = \{N \subseteq M \mid N \textit{ is semiprimitive in } M\} \bigcup \{M\}$

$Prt(R) = \{I \subseteq R \mid I \textit{ is a primitive ideal of } R\}$

$Sprt(R) = \{J \subseteq M \mid J \textit{ is a semiprimitive ideal of } R\} \bigcup \{R\}$

Proposition 2.12. *Let R be commutative and let M be a multiplication R-module. If T is a maximal ideal of R such that $TM \subsetneq M$, then TM is maximal submodule of M.*

Proof. Suppose that TM is not a maximal submodule of M. Thus there exists L a proper submodule of M such that $TM \subsetneq L$. Hence there exists $x \in L$ and $x \notin TM$, we deduce that $TM + Rx \subseteq L \subsetneq M$. Since Rx is a submodule of M, then there exists an ideal I of the R such that $Rx = IM$. As $x \notin TM$, then $IM = Rx \nsubseteq TM$, which implies that $I \nsubseteq T$. As T is a maximal ideal, then $T + I = R$. Thus

$$M = RM = (T + I)M = TM + IM = TM + Rx \subseteq L \subsetneq M,$$

a contradiction. Thus TM is a maximal submodule of M. ∎

Remark 2.13. *In [10, Theorem 2.5] the authors show that for a maximal submodule N of a multiplication R-module there exists a maximal ideal I of R such that $N = IM$.*

Proposition 2.14. *Let R be a commutative ring and M be a faithful multiplication R-module. If N is a maximal submodule of M, them there exists a unnique maximal ideal T of R, such that $N = TM$.*

Proof. Suppose that Q is another maximal ideal such that $N = QM$. Then T and Q are prime ideals of R, as they are maximal. By [9, Proposition 1.6] we have that $T = Q$. ∎

We denote

$$Max(M) = \{N \subseteq M \mid N \text{ is maximal}\} \text{ and, } Max(R) = \{I \subseteq R \mid I \text{ is maximal}\}.$$

Proposition 2.15. *Let R be a commutative ring and let M be a multiplication R-module. Suppose that $QM \neq M$ for any maximal ideal Q of R. Then there exists a bijective correspondence between $Max(M)$ and $Max(R)$.*

Proof. By Proposition 2.14 for every N maximal submodule of M there exists only one maximal ideal T of R such that $N = TM$. So we define the mapping

$$\varphi : Max(M) \longrightarrow Max(R), \;\; \varphi(N) := T$$

Now, we shall show that φ is bijective. We consider $N = TM$ and $L = QM$ such that $\varphi(N) = \varphi(L)$. Thus $TM = QM$. By [9, Corollary 1.7] we have that $T = Q$. So φ is injective. Let T be a maximal ideal of R. By Proposition 2.12 we have that $N = TM$ is a maximal submodule of M. So $\varphi(N) = T$. Hence φ is surjective. Thus φ is bijective. ∎

3 The Primitive Topology for the Set Prt(M)

In this section we define the primitive radical of an R-module and we give some properties of this radical. Also, we define a topology for the set of the primitive submodules of a module multiplication M. We describe the open sets of this topology and we give a basis of open sets for the topology.

Definition 3.1. *Let R be a commutative ring and let M be a multiplication R-module. For N a submodule of M the radical of N in M is*

$$\sqrt{N} = \cap\{P \subseteq M \mid P \text{ is a primitive submodule of } M \text{ and } N \subseteq P\}$$

If M has no primitive submodules P such that $N \subseteq P$, then $\sqrt{N} = M$. In particular $\sqrt{M} = M$.

Notice that by Remark 2.2 we have that $Ann_M(M/P)$ is a primitive submodule of M for all P maximal submodule of M. Also, note that by [10, Proposition 2.5] we have that every proper submodule M is contained in a maximal submodule of M. Moreover if M is projective in category $\sigma[M]$ and N is a proper submodule of M, which is contained in a maximal submodule P of M, then $N \subseteq P = Ann_M(M/P)$. Thus $\sqrt{N} \subsetneq M$ for all proper submodule N of M.

Proposition 3.2. *Let R be a commutative ring, let M be a multiplication R-module and let N be a proper submodule of M. If $\sqrt{N} \neq M$, then $\sqrt{N}$ is the minimal semiprimitive submodule of M such that $N \subseteq \sqrt{N}$.*

Proof. As $\sqrt{N} \neq M$, then there exists P a primitive submodule of M such that $N \subseteq P$. So it is clear that $\sqrt{N}$ is a semiprimitive module. Now let L be a semiprimitive module in M such that $N \subseteq L$. By Definition 2.8 we have that $L = \cap_{i\in I} P_i$ where P_i is primitive submodule of M for all $i \in I$. Since $N \subseteq L$, then $N \subseteq P_i$ all $i \in I$. Thus $\sqrt{N} \subseteq L$. ∎

Proposition 3.3. *Let R be a commutative ring and let M be a multiplication R-module. Suppose that N and L are submodules of M, then the following conditions hold:*

i) If $N \subseteq L$, then $\sqrt{N} \subseteq \sqrt{L}$.

ii) $\sqrt{N} = \sqrt{\sqrt{N}}$.

iii) $\sqrt{N+L} = \sqrt{\sqrt{N}+\sqrt{L}}$.

iv) $\sqrt{N \cap L} \subseteq \sqrt{N} \cap \sqrt{L}$.

v) $\sqrt{N_M L} \subseteq \sqrt{N} \cap \sqrt{L}$.

Proof. They are straightforward. ∎

Analogously we define the radical primitive of an ideal I in R.

Definition 3.4. *Let R be a commutative ring. For I an ideal of R the primitive radical of I is*

$$\sqrt{I} = \cap\{J \subseteq R \mid J \textit{ is a primitive ideal of } R \textit{ and } I \subseteq J\}$$

If I has no primitive ideals J such that $I \subseteq J$, then $\sqrt{I} = R$. In particular $\sqrt{R} = R$.

Proposition 3.5. *Let R be a commutative ring and let M be a multiplication R-module such that $QM \neq M$ for all maximal ideal Q of R. Then $\sqrt{IM} = \sqrt{I}M$ for all proper ideal I of R. Where $\sqrt{I}$ is the primitive radical of I.*

Proof. The proof follows from [9, Theorem 2.27] and Proposition 2.7. ∎

Proposition 3.6. *Let R be a commutative ring and let M be a multiplication R-module such that M is projective in category $\sigma[M]$. Then $(Prt\,(M)\,,\,\mathcal{T})$ is a topological space, where*

$$\mathcal{T} = \{\mathcal{U}\,(N) \mid N \in \text{ is a submodule of } M\}$$

is the primitive topology and $\mathcal{U}\,(N) = \{P \in Prt\,(M) \mid N \nsubseteq P\}$ *are open sets.*

Proof. It is clear that $\mathcal{U}(M) = Prt(M)$ and $\mathcal{U}(0) = Prt(M) = \emptyset$.
Now, we consider the family $\{\mathcal{U}(N_i)\}_{i\in I}$. We claim that $\bigcup_{i\in I}\mathcal{U}(N_i) = \mathcal{U}\left(\,\sum_{i\in I} N_i\,\right)$.
Indeed, as $N_i \subseteq \sum_{i\in I} N_i$, then $\mathcal{U}(N_i) \subseteq \mathcal{U}\;(\,\sum_{i\in I} N_i\,)$. Thus

$$\textstyle\bigcup_{i\in I}\mathcal{U}(N_i) \subseteq \mathcal{U}\;(\,\sum_{i\in I} N_i\,).$$

If $P \in \mathcal{U}\;(\,\sum_{i\in I} N_i\,)$, then $\sum_{i\in I} N_i \nsubseteq P$. Hence there exists $j \in I$ such that $N_j \nsubseteq P$.
Thus $P \in \mathcal{U}(N_j) \subseteq \bigcup_{i\in I}\mathcal{U}(N_i)$. So

$$\textstyle\mathcal{U}\;(\,\sum_{i\in I} N_i\,) \subseteq \bigcup_{i\in I}\mathcal{U}(N_i).$$

This proves our claim. Therefore $\bigcup_{i\in I}\mathcal{U}(N_i) \in \mathcal{T}$.
Let $\{\mathcal{U}(N_i)\}_{i\in I}$ be a finite family. We shall prove that $\bigcap_{i\in I}\mathcal{U}(N_i) \in \mathcal{T}$. To do so, it is sufficient to prove it for two elements. Let N and L be two submodules of M. We claim that $\mathcal{U}(N)\bigcap\mathcal{U}(L) = \mathcal{U}(N_M L)$. Indeed, as N and L are fully invariant submodules of M, then $N_M L \subseteq N$ and $N_M L \subseteq L$, which implies that $\mathcal{U}(N_M L) \subseteq \mathcal{U}(N)$ and $\mathcal{U}(N_M L) \subseteq \mathcal{U}(L)$. Thus $\mathcal{U}(N_M L) \subseteq \mathcal{U}(N)\bigcap\mathcal{U}(L)$. Now, if $P \in \mathcal{U}(N)\bigcap\mathcal{U}(L)$, then $N \nsubseteq P$ and $L \nsubseteq P$. By [16, Proposition 3.4] we have that P is a prime submodule of M. Thus $N_M L \nsubseteq P$, which implies that $P \in \mathcal{U}(N_M L)$. Therefore $\mathcal{U}(N)\bigcap\mathcal{U}(L) \subseteq \mathcal{U}(N_M L)$. This proves our claim. So $\mathcal{U}(N)\bigcap\mathcal{U}(L) \in \mathcal{T}$. Thus $\mathcal{T}$ is a topology. □

Corollary 3.7. *Let R be a commutative ring. Then $(Prt(R),\,\mathcal{T})$ is a topological space, where*

$$\mathcal{T} = \{\mathcal{U}\,(I) \mid I \in \text{ is an ideal of } M\}$$

is the primitive topology and $\mathcal{U}\,(I) = \{J \in Prt\,(R) \mid J \nsubseteq I\}$ *are open sets.*

Proof. It is clear. □

Corollary 3.8. *Let R be a commutative ring and let M be a multiplication R-module such that M is projective in category $\sigma[M]$. Then $\mathcal{B} = \{\mathcal{U}\,(Rm) \mid m \in M\}$ is a basis of open sets for the primitive topology of $Prt(M)$*

Proof. We know that the open sets of the primitive topology are $\mathcal{U}(N)$ where N is a submodule of M. As $N = \sum_{m\in N} Rm$, then

$$\textstyle\mathcal{U}(N) = \mathcal{U}(\,\sum_{m\in N} Rm\,) = \bigcup_{m\in N}\mathcal{U}(Rm),$$

which proves that $\mathcal{B}$ is a basis. ∎

Remark 3.9. *As $\mathcal{U}(N) = \mathcal{U}\left(\sqrt{N}\right)$ for all N sumodule of M, then we can consider the open sets of the primitive topology as $\mathcal{U}(N)$ with N a semiprimitive submodule of M or $N = M$.*

Lemma 3.10. *Let R be a commutative ring and let M be a multiplication R-module. If N and L are submodules of M, then the following conditions hold:*
i) $\mathcal{U}(L) = \varnothing$ if and only if $L \subseteq \sqrt{0}$.
ii) $\mathcal{U}(L) = \mathcal{U}(N)$ if and only if $\sqrt{L} = \sqrt{N}$.

Proof. They are straightforward. ∎

4 The Spatial Frame $Sprt(M)$

In this section we prove that the frames $Sprt(M)$ and $\Omega(Prt(M))$ are isomorphic. Hence we have that $Sprt(M)$ is a spatial frame. Also, we prove the topological spaces $(Prt(R), \mathcal{T})$ and $(Prt(M), \mathcal{T})$ are homeomorphic.

Definition 4.1. *A frame is a complete lattice L satisfying the distributivity law*

$$(\vee A) \wedge b = \vee \{a \wedge b \mid a \in A\}$$

for all subset $A \subseteq L$ and any $b \in L$.

If $(\mathbf{X}, \tau)$ is topological space, we will denote (complete) lattice of open sets of a space $\mathbf{X}$ as $\Omega(\mathbf{X})$.

Definition 4.2. *A frame L is said to be spatial if it is isomorphic to an $\Omega(\mathbf{X})$ the fame of open sets of some space topological X.*

For details about concepts and terminology concerning frames and spatial frames see [17].

We remember that for a commutative ring R and a multiplication R-module M we have defined

$$Sprt(M) = \{N \subseteq M \mid N \text{ is semiprimitive in } M\} \bigcup \{M\}$$

For N and N' in $Sprt(M)$, we have that $N \wedge N' = N \cap N'$ and $N \vee N' = \sqrt{N + N'}$ are the meet and join (respectively) of the partially ordered set $Sprt(M)$, where the order $N \leq N'$ is $N \subseteq N'$.

Notice that $\{Sprt(M), \leq\}$ is a partially ordered set. Also, note that every subset X of $Sprt(M)$ has a least upper bound, written $\bigvee_{N \in X} N$, and greatest lower bound, written $\bigwedge_{N \in X} N$. Thus $\{Sprt(M), \leq\}$ is a complete lattice.

Proposition 4.3. *Let R be a commutative ring and let M be a multiplication R-module. If M is projective in the category $\sigma[M]$, then $\{Sprt(M), \leq, \wedge, \vee\}$ is a frame.*

Proof. We have that $\{Sprt(M), \leq, \wedge, \vee\}$ is a complete lattice. Now, let $N \in Sprt(M)$ and let $\{N_i\}_{i\in I}$ be a family of submodules in $Sprt(M)$. We shall prove that $N \wedge (\vee_{i\in I} N_i) = \vee_{i\in I}(N \wedge N_i)$. To do so we have that

$$N \wedge (\vee_{i\in I} N_i) = N \cap \left(\sqrt{\textstyle\sum_{i\in I} N_i}\right) \text{ and } \vee_{i\in I}(N \wedge N_i) = \sqrt{\textstyle\sum_{i\in I}(N \bigcap N_i)}.$$

If $N = M$, then we have the result. Suppose that $N \subsetneq M$. It is clear that $N \cap N_j \subseteq N \cap \left(\sqrt{\sum_{i\in I} N_i}\right)$ for all $j \in I$. Thus

$$\textstyle\sum_{i\in I} (N \cap N_i) \subseteq N \cap \left(\sqrt{\sum_{i\in I} N_i}\right).$$

As N is a semiprimitive submodule of M, then $N \cap \left(\sqrt{\sum_{i\in \mathcal{I}} N_i}\right)$ is an intersection of primitive submodules of M. So

$$\sqrt{\textstyle\sum_{i\in I} (N \cap N_i)} \subseteq N \cap \left(\sqrt{\textstyle\sum_{i\in I} N_i}\right).$$

Now, let P be a primitive submodule of M such that $\sum_{i\in I} (N \cap N_i) \subseteq P$. Thus $N \cap N_i \subseteq P$ for all $i \in I$. Since N is a fully invariant submodule of M, we have that $N_M N_i \subseteq N \cap N_i$. So $N_M N_i \subseteq P$ for all $i \in I$. By [16, Proposition 1.3] P is prime in M, then $N \subseteq P$ or $N_i \subseteq P$. If $N \subseteq P$, then $N \cap \left(\sqrt{\sum_{i\in I} N_i}\right) \subseteq P$. Hence

$$N \cap \left(\sqrt{\textstyle\sum_{i\in I} N_i}\right) \subseteq \sqrt{\textstyle\sum_{i\in I} (N \cap N_i)}.$$

If $N \nsubseteq P$, then $N_i \subseteq P$ for all $i \in I$. Thus $\sum_{i\in I} N_i \subseteq P$. So $\sqrt{\sum_{i\in I} N_i} \subseteq P$. Therefore

$$N \cap \left(\sqrt{\textstyle\sum_{i\in I} N_i}\right) \subseteq \sqrt{\textstyle\sum_{i\in I} (N \cap N_i)},$$

which implies that

$$N \cap \left(\sqrt{\textstyle\sum_{i\in I} N_i}\right) = \sqrt{\textstyle\sum_{i\in I} (N \cap N_i)}.$$

So $N \wedge (\vee_{i\in I} N_i) = \vee_{i\in I} (N \wedge N_i)$.

□

Corollary 4.4. *If R is a commutative ring, then $\{Sprt(R), \leq, \wedge, \vee\}$ is a frame.*

Proof. It follows from Proposition 4.3. □

Let $\mathbf{X} = Prt(M)$. By Definition 4.2 we have that

$$\Omega(Prt(M)) = \{\mathcal{U}(N) \mid N \text{ is a submodule of } M\}$$

is the frame of open sets $Prt(M)$. Thus we can put the frame $\Omega(Prt(M)) = \{\mathcal{T}, \subseteq, \bigcap, \bigcup\}$, where $\mathcal{T}$ is the primitive topology.

Proposition 4.5. *Let R be a commutative ring and let M be a multiplication R-module. If M is projective in the category $\sigma[M]$, then*

$$Sprt(M) \cong \Omega(Prt(M))$$

as frames.

Proof. We define the mapping

$$\mathcal{H} : Sprt(M) \longrightarrow \Omega(Prt(M)),\ \ \mathcal{H}(N) := \mathcal{U}(N).$$

We claim that $\mathcal{H}$ is order isomorphism. Indeed, let N_1 and N_2 in $Sprt(M)$ such that $N_1 \subseteq N_2$. If $P \in \mathcal{U}(N_1)$, then $N_1 \nsubseteq P$. Thus $N_2 \nsubseteq P$, which implies that $P \in \mathcal{U}(N_2)$. Hence $\mathcal{U}(N_1) \subseteq \mathcal{U}(N_2)$. So $\mathcal{H}$ is order morphism.
We are shall prove that $\mathcal{H}$ is injective. To do so, let $\mathcal{H}(N_1) = \mathcal{H}(N_2)$. Thus $\mathcal{U}(N_1) = \mathcal{U}(N_2)$. As N_2 is a semiprimitive submodule of M, then $N_2 = \bigcap_{i\in I} P_i$, where every P_i is a primitive submodule of M. We claim that $N_1 \subseteq P_i$ for all $i \in I$. Indeed, we suppose that there exists $i \in I$ such that $N_1 \nsubseteq P_i$. Thus $P_i \in \mathcal{U}(N_1) = \mathcal{U}(N_2)$, which implies that $N_2 \nsubseteq P_i$ a contradiction. Thus $N_1 \subseteq P_i$ for all $i \in I$. Hence $N_1 \subseteq \bigcap_{i\in I} P_i = N_2$. Analogously it is proved that $N_2 \subseteq N_1$. So $N_1 = N_2$. Therefore $\mathcal{H}$ is injective.

We are going to prove that $\mathcal{H}$ is surjective. By Remark 3.9 we can consider the open sets of the primitive topology, as $\mathcal{U}(N)$ with N a semiprimitive submodule of M or $N = M$. If $\mathcal{U}(N) \in \Omega(Prt(M))$, then N is a primitive submodule of M. Thus $\mathcal{H}(N) = \mathcal{U}(N)$, which proves that $\mathcal{H}$ is surjective.

Now, we consider the mapping

$$\mathcal{H}^{-1} : \Omega(Prt(M)) \longrightarrow Sprt(M),\ \ \mathcal{H}^{-1}(\mathcal{U}(N)) := N.$$

Clearly, $\mathcal{H}^{-1}$ is the inverse mapping of $\mathcal{H}$. We shall prove that $\mathcal{H}^{-1}$ is order morphism. To do so, let $\mathcal{U}(N_1) \subseteq \mathcal{U}(N_2)$. As N_2 is a semiprimitive submodule of M, then $N_2 = \bigcap_{i\in I} P_i$, where every P_i is a primitive submodule of M. We claim that $N_1 \subseteq P_i$ for all $i \in I$. Indeed, we suppose that there exists $i \in I$ such that $N_1 \nsubseteq P_i$. Thus $P_i \in \mathcal{U}(N_1) \subseteq \mathcal{U}(N_2)$, which implies that $N_2 \nsubseteq P_i$ is a contradiction. Thus $N_1 \subseteq P_i$ for all $i \in I$. Hence $N_1 \subseteq \bigcap_{i\in I} P_i = N_2$. Thus $\mathcal{H}^{-1}$ is order morphism. By [19, Chapter III Proposition 1.1] we have that $\mathcal{H}$ is lattice isomorphism. □

Corollary 4.6. *Let R be a commutative ring and let M be a multiplication R-module. If M is projective in the category $\sigma[M]$, then $\{Sprt(M), \leq, \wedge, \vee\}$ is a spatial frame.*

Proof. By Proposition 4.3 and Proposition 4.5 we have that $\{Sprt(M), \leq, \wedge, \vee\}$ is a spatial frame. □

Corollary 4.7. *Let R be a commutative ring, then $\{Sprt(M), \leq, \wedge, \vee\}$ is a spatial frame.*

Proof. It is clear from Corollary 4.6. □

Proposition 4.8. *Let R be a commutative ring and let M be a multiplication R-module such that $QM \neq M$ for all maximal ideal Q of R. If M generates all the simple R-modules and M is projective in the category $\sigma[M]$, then the topological spaces $(Prt(R), \mathcal{T})$ and $(Prt(M), \mathcal{T})$ are homeomorphic.*

Proof. We consider the mapping

$$\psi : Prt(R) \longrightarrow Prt(M); \quad \psi(I) := IM.$$

As I is a primitive ideal of R, then $I = ann(S)$ for some simple R-module S. By Proposition 2.3 we have that IM is a submodule primitive of M. We suppose that $\psi(I) = \psi(I')$. Thus $MI = MI'$. Since I and I' are prime ideals, then by [9, Corollary 1.9] we have that $I = I'$. Hence ψ is injective. Now, let $N \in Prt(M)$. As N is a primitive module and M is a multiplication module, then $N = ann(M/N)M$. By Proposition 2.4 $ann(M/N)$ is a primitive ideal of R. Hence $\psi(ann(M/N)) = ann(M/N)M = N$. Thus ψ is surjective. By Proposition 2.6 for every $N \in Prt(M)$ there exists only one primitive ideal I of R such that $N = IM$. Thus we define the inverse mapping of ψ as:

$$\psi^{-1} : Prt(M) \longrightarrow Prt(R); \quad \psi^{-1}(IM) := I.$$

Now, we shall prove that ψ is a continuous mapping. To do so, let $\mathcal{U}(N)$ be an open set of the primitive topology of $Prt(M)$. As M is a multiplication module, then $N = IM$ with I an ideal of R. We have that

$$\psi^{-1}(\mathcal{U}(N)) = \psi^{-1}(\mathcal{U}(IM)) = \{J \in Prt(R) \mid \psi(J) \in \mathcal{U}(IM))\}$$

$$= \{J \in Prt(R) \mid JM \in \mathcal{U}(IM))\} = \{J \in Prt(R) \mid IM \nsubseteq JM)\}.$$

We claim that

$$\{J \in Prt(R) \mid IM \nsubseteq JM)\} = \{J \in Prt(R) \mid I \nsubseteq J)\}.$$

Indeed, let $J \in Prt(R)$ such that $IM \nsubseteq JM$. Thus $I \nsubseteq J$. So $\{J \in Prt(R) \mid IM \nsubseteq JM)\} \subseteq \{J \in Prt(R) \mid I \nsubseteq J)\}$. Now, let $J \in Prt(R)$ such that $I \nsubseteq J$. We suppose that $IM \subseteq JM$. By [9, Proposition 1.6] we have that $I \subseteq J$ is a contradiction. Therefore $IM \nsubseteq JM$. So $\{J \in Prt(R) \mid I \nsubseteq J)\} \subseteq \{J \in Prt(R) \mid IM \nsubseteq JM)\}$, which proves our claim. Since $\{J \in Prt(R) \mid I \nsubseteq J)\} = \mathcal{U}(I)$, then $\psi^{-1}(\mathcal{U}(N)) = \mathcal{U}(I)$, which is an open set of the primitive topology of $Prt(R)$. Similarly we show that if $\mathcal{U}(I)$ is an open set of primitive topology of the $Prt(R)$, then $\psi(\mathcal{U}(I)) = \mathcal{U}(IM)$ is an open set of the primitive topology of $Prt(M)$. Therefore ψ is a continuous mapping. So the topological spaces $(Prt(R), \mathcal{T})$ and $(Prt(M), \mathcal{T})$ are homeomorphic. □

Corollary 4.9. *Let R be a commutative ring and let M be a multiplication R-module such that $QM \neq M$ for all maximal ideal Q of R. If M generates all the simple R-modules and M is projective in the category $\sigma[M]$, then there exists a bijective correspondence (of order) between $Prt(R)$ and $Prt(M)$.*

Proof. The mapping $\psi(I) := IM$ defined in 4.8 is bijective. Moreover, if I and J are ideals of R such that $I \subseteq J$, then $IM = \psi(I) \subseteq \psi(J) = JM$. So ψ is the order. □

5 The classical Krull dimension of the set Prt(M)

In this section we give the classical Krull dimension and we prove that $cl.K\dim(Prt(M)) = cl.K\dim(Prt(R))$. Also, we show that if the topological space $(Prt(M), \mathcal{T})$ is notherian, then the poset $Prt(M)$ has classical Krull dimension.

The classical Krull dimension of a poset $(X, \leq)$ was defined in [1]. For R a commutative ring and M a multiplication R-module we use the poset $(Prt(M), \subseteq)$ and we give the classical Krull dimension of $Prt(M)$.

Set $Prt^{-1}(M) = \emptyset$, and for an ordinal $\alpha > -1$ define

$$Prt^{\alpha}(M) = \left\{ N \in Prt(M) \mid N \subsetneq Q \in Prt(M) \Rightarrow Q \in \bigcup_{\beta<\alpha} Prt^{\beta}(M) \right\}$$

If an ordinal α with $Prt^{\alpha}(M) = Prt(M)$ exists, then the smallest of such ordinals is called the classical Krull dimension of $Prt(M)$; it is denoted by $cl.K\dim(Prt(M))$.

Notice that if R is a commutative ring and M is a multiplication module projective in category $\sigma[M]$, we have that M has maximal submodules, which are primitive submodules of M. Moreover, every proper submodule of M is contained in a maximal submodule of M. Thus $Prt^0(M) = \{P \in Prt(M) \mid P \text{ is a maximal submodule of } M\}$. Also, note that $Prt^0(R) = \{I \in Prt(R) \mid P \text{ is a maximal ideal of } R\}$.

Remark 5.1. *By [1, Proposition 1.4] we have that a set X has classical Krull dimension if and only if the poset X is noetherian.*

Notice that if M is a noetherian R-module, then the poset $(Prt(M), \subseteq)$ is noetherian. Thus $Prt(M)$ has classical Krull dimension.

Proposition 5.2. *Let R be a commutative ring and M a faithful multiplication R-module such that $QM \neq M$ for all maximal ideal Q of R. If M is projective in the category $\sigma[M]$, then $Prt(R)$ has classical Krull dimension if and only if $Prt(M)$ has classical Krull dimension. Moreover. $cl.K\dim(Prt(M)) = cl.K\dim(Prt(R))$.*

Proof. By Corollary 4.9 we have that the poset $Prt(M)$ is notherian if and only if the poset $Prt(R)$ is noetherian. Thus the proof follows from Remark 5.1. □

Definition 5.3. *A topological space $(\mathbf{X}, \mathcal{T})$ is said to be noetherian if and only if every ascending (descending) chain of open (closed) subsets is stationary, equivalently if and only if every open subset is compact.*

Proposition 5.4. *Let R be a commutative ring and M a faithful multiplication R-module such that $QM \neq M$ for all maximal ideal Q of R. If M is projective in the category $\sigma[M]$ and the topological space $(Prt(M), \mathcal{T})$ is notherian, then the poset $Prt(M)$ has classical Krull dimension.*

Proof. If $P_1 \subseteq P_2 \subseteq ... \subseteq P_n.....$ is a chain in the poset $Prt(M)$, then $\mathcal{U}(P_1) \subseteq \mathcal{U}(P_2) \subseteq ... \subseteq \mathcal{U}(P_n).....$ is a chain in the primitive topology $\mathcal{T}$ of $Prt(M)$. As the primitive topology $\mathcal{T}$ is noetherian, then there exists a natural number k such that $\mathcal{U}(P_k) = \mathcal{U}(P_{k+i})$ for all natural number i. By proof of Proposition 4.5 we have that $\mathcal{U}(P_k) = \mathcal{U}(P_{k+i})$ implies that $P_k = P_{k+i}$ for all natural number i. Thus the poset $Prt(M)$ has classical Krull dimension. □

Corollary 5.5. *Let R be a commutative ring. If the topological space $(Prt(R), \mathcal{T})$ is noetherian, then the poset $Prt(R)$ has classical Krull dimension.*

Proof. It is clear. □

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