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Combinatorial proofs of two q-binomial coefficient identities

by

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Abstract

We present combinatorial proofs of two q-binomial coefficient identities, which give two new q-analogues of the binomial coefficient identity:

$$\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^k\binom{2n}{n+2k}=2^n,$$

where $\lfloor x\rfloor$ denotes the integral part of real x.

Key Words: q-binomial coefficient, q-binomial theorem, combinatorial proof.
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1 Introduction

There are many q-analogues of the following binomial coefficient identity:

$$\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^k\binom{2n}{n+2k}=2^n, \tag{1.1}$$

such as

$$\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^k q^{2k^2}\begin{bmatrix}2n\\n+2k\end{bmatrix}=(-q;q^2)_n, \tag{1.2}$$

$$\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^k q^{2k^2+k}\begin{bmatrix}2n\\n+2k\end{bmatrix}=(1+q^n)(-q^2;q^2)_{n-1}, \tag{1.3}$$

$$\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^k q^{2k^2+2k}\begin{bmatrix}2n\\n+2k\end{bmatrix}=q^{n-1}(1+q)(-q;q^2)_{n-1}, \tag{1.4}$$

where $\lfloor x\rfloor$ denotes the integral part of real x. Here and throughout this paper, the q-shifted factorials are given by $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\ge 1$ and $(a;q)_0=1$, and the q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

We refer the interested reader to [2, 4, 5] for (1.2) and (1.3). In 2014, Guo and Zhang [3] gave combinatorial proofs of (1.2)–(1.4). The purpose of this note is to establish another two q-analogues of (1.1), which appear to be new.

Theorem 1. *For any positive integer n, we have*

$$\sum_{k=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^k q^{2k^2+3k} \begin{bmatrix} 2n \\ n+2k \end{bmatrix}$$
$$= \frac{q^{n-1}+q^{n+1}+q^{2n-1}+q^{2n+1}-q^{3n}+q^{2n}+q^n-1}{q}(-q^2;q^2)_{n-2}. \tag{1.5}$$

Theorem 2. *For any positive integer n, we have*

$$\sum_{k=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^k q^{2k^2+4k} \begin{bmatrix} 2n \\ n+2k \end{bmatrix}$$
$$= \frac{q^{2n-2}+q^{2n+1}+q^{2n-1}+q^{2n+2}-q^{4n}+2q^{2n}-1}{q^2}(-q;q^2)_{n-2}. \tag{1.6}$$

It is clear that letting $q \to 1$ in (1.5) and (1.6) leads us to the binomial coefficient identity (1.1). Inspired by Guo and Zhang's method [3], we shall present combinatorial proofs of (1.5) and (1.6) in Sections 2 and 3, respectively.

2 Proof of Theorem 1

Let $S = \{a_1, \cdots, a_{2n}\}$ be a set of $2n$ elements, and let

$$\mathscr{F} = \{A \subseteq S : \#A \equiv n \pmod 2\},$$

$$\mathscr{G} = \{A \subseteq S : \#(A \cap \{a_{2i-1}, a_{2i}\}) = 1, \quad \text{for all } i = 1, \cdots, n\}.$$

For any $A \in \mathscr{F}$, we associate A with a sign $\operatorname{sgn}(A) = (-1)^{(\#A-n)/2}$ and a weight $||A|| = \sum_{a \in A} a$. By the q-binomial theorem [1, Theorem 3.3]:

$$(-qz;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} z^k q^{\binom{k+1}{2}},$$

we have

$$\sum_{\substack{A\subseteq[n]\\ \#A=k}} q^{||A||} = \begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{k+1}{2}}, \tag{2.1}$$

where $[n]=\{1,\cdots,n\}$.

Let $\{a_{2i-1},a_{2i}\}=\{-i,i\}$, for $i=1,\cdots,n-2$, $\{a_{2n-3},a_{2n-2}\}=\{0,n\}$ and $\{a_{2n-1},a_{2n}\}=\{n-1,n+1\}$. Note that S is obtained by $[2n]$ by a shift $-(n-1)$:

$$S=\{2-n,3-n,\cdots,n-2,n-1,n,n+1\}.$$

By using (2.1), we obtain

$$\begin{aligned}\sum_{A\in\mathscr{F}} \operatorname{sgn}(A)q^{||A||} &= \sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} \sum_{\substack{A\subseteq S\\ \#A=n+2k}} \operatorname{sgn}(A)q^{||A||}\\ &= \sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^k \begin{bmatrix} 2n\\ n+2k\end{bmatrix} q^{\binom{n+2k+1}{2}-(n+2k)(n-1)}\\ &= q^{\frac{n(3-n)}{2}} \sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^k q^{2k^2+3k} \begin{bmatrix} 2n\\ n+2k\end{bmatrix}. \end{aligned} \tag{2.2}$$

On the other hand,

$$\sum_{A\in\mathscr{F}} \operatorname{sgn}(A)q^{||A||} = \sum_{A\in\mathscr{F}\setminus\mathscr{G}} \operatorname{sgn}(A)q^{||A||} + \sum_{A\in\mathscr{G}} \operatorname{sgn}(A)q^{||A||}. \tag{2.3}$$

It is obvious that

$$\sum_{A\in\mathscr{G}} \operatorname{sgn}(A)q^{||A||} = \sum_{A\in\mathscr{G}} q^{||A||} = (1+q^n)(q^{n-1}+q^{n+1}) \prod_{i=1}^{n-2}(q^i+q^{-i}). \tag{2.4}$$

We define the involution f on $\mathscr{F}\setminus\mathscr{G}$ as follows:

$$f(A)=\begin{cases} A\cup\{a_{2i-1},a_{2i}\}, & \text{if } \{a_{2i-1},a_{2i}\}\cap A=\emptyset,\\ A\setminus\{a_{2i-1},a_{2i}\}, & \text{if } \{a_{2i-1},a_{2i}\}\subseteq A,\end{cases}$$

where i is the first number such that $\#(A\cap\{a_{2i-1},a_{2i}\})\neq 1$. Let

$$\mathscr{H}=\{A\in\mathscr{F}\setminus\mathscr{G}: \exists\, 1\le i\le n-2, s.t.\, \#(A\cap\{a_{2i-1},a_{2i}\})\neq 1\}.$$

The involution f is closed, weight-preserving, and sign-reversing on $\mathscr{H}$. Thus,

$$\sum_{A\in\mathscr{F}\setminus\mathscr{G}} \operatorname{sgn}(A)q^{||A||} = \sum_{A\in(\mathscr{F}\setminus\mathscr{G})\setminus\mathscr{H}} \operatorname{sgn}(A)q^{||A||}. \tag{2.5}$$

Note that $A \in (\mathscr{F} \setminus \mathscr{G}) \setminus \mathscr{H}$ if and only if A belongs to one of the following types:

$$\begin{aligned}
&\{b_1, \cdots, b_{n-2}\},\\
&\{b_1, \cdots, b_{n-2}, a_{2n-3}, a_{2n-2}\},\\
&\{b_1, \cdots, b_{n-2}, a_{2n-1}, a_{2n}\},\\
&\{b_1, \cdots, b_{n-2}, a_{2n-3}, a_{2n-2}, a_{2n-1}, a_{2n}\},
\end{aligned}$$

where $b_i \in \{a_{2i-1}, a_{2i}\}$. It follows that

$$\sum_{A \in (\mathscr{F} \setminus \mathscr{G}) \setminus \mathscr{H}} \operatorname{sgn}(A) q^{||A||} = \left(-1 + q^n + q^{2n} - q^{3n}\right) \prod_{i=1}^{n-2} (q^i + q^{-i}). \tag{2.6}$$

Combining (2.3)–(2.6) gives

$$\begin{aligned}
&\sum_{A \in \mathscr{F}} \operatorname{sgn}(A) q^{||A||}\\
&= \left(q^{n-1} + q^{n+1} + q^{2n-1} + q^{2n+1} - q^{3n} + q^{2n} + q^n - 1\right) \prod_{i=1}^{n-2} (q^i + q^{-i}).
\end{aligned} \tag{2.7}$$

It follows from (2.2) and (2.7) that

$$\begin{aligned}
&\sum_{k=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^k q^{2k^2+3k} \begin{bmatrix} 2n \\ n+2k \end{bmatrix}\\
&= \left(q^{n-1} + q^{n+1} + q^{2n-1} + q^{2n+1} - q^{3n} + q^{2n} + q^n - 1\right) q^{\frac{n(n-3)}{2}} \prod_{i=1}^{n-2} (q^i + q^{-i})\\
&= \frac{q^{n-1} + q^{n+1} + q^{2n-1} + q^{2n+1} - q^{3n} + q^{2n} + q^n - 1}{q} (-q^2; q^2)_{n-2},
\end{aligned}$$

as desired.

3 Proof of Theorem 2

Let

$$\begin{aligned}
\{a_{2i-1}, a_{2i}\} &= \left\{-i + \frac{1}{2}, i - \frac{1}{2}\right\}, \quad \text{for } i = 1, \cdots, n-2,\\
\{a_{2n-3}, a_{2n-2}\} &= \left\{n - \frac{3}{2}, n + \frac{3}{2}\right\},\\
\{a_{2n-1}, a_{2n}\} &= \left\{n - \frac{1}{2}, n + \frac{1}{2}\right\}.
\end{aligned}$$

Note that S is obtained by $[2n]$ by a shift $-(n-3/2)$:

$$S=\left\{-n+\frac{5}{2},\cdots,n-\frac{5}{2},n-\frac{3}{2},n-\frac{1}{2},n+\frac{1}{2},n+\frac{3}{2}\right\}.$$

Following the notation in the previous section and using (2.1), we have

$$\begin{aligned}\sum_{A\in\mathscr{F}}\operatorname{sgn}(A)q^{||A||}&=\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}\sum_{\substack{A\subseteq S\\ \#A=n+2k}}\operatorname{sgn}(A)q^{||A||}\\&=\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^k\begin{bmatrix}2n\\n+2k\end{bmatrix}q^{\binom{n+2k+1}{2}-\frac{(n+2k)(2n-3)}{2}}\\&=q^{\frac{n(4-n)}{2}}\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^kq^{2k^2+4k}\begin{bmatrix}2n\\n+2k\end{bmatrix}.\end{aligned}\tag{3.1}$$

By using a similar method as in the previous section, we have

$$\sum_{A\in\mathscr{G}}\operatorname{sgn}(A)q^{||A||}=(q^{n-3/2}+q^{n+3/2})(q^{n-1/2}+q^{n+1/2})\prod_{i=1}^{n-2}(q^{-i+1/2}+q^{i-1/2}),\tag{3.2}$$

and

$$\begin{aligned}\sum_{A\in\mathscr{F}\setminus\mathscr{G}}\operatorname{sgn}(A)q^{||A||}&=\sum_{A\in(\mathscr{F}\setminus\mathscr{G})\setminus\mathscr{H}}\operatorname{sgn}(A)q^{||A||}\\&=\left(-1+2q^{2n}-q^{4n}\right)\prod_{i=1}^{n-2}(q^{-i+1/2}+q^{i-1/2}).\end{aligned}\tag{3.3}$$

Finally, combining (3.1)–(3.3), we complete the proof of (1.6).

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