Bull. Math. Soc. Sci. Math. Roumanie
Tome 67 (115), No. 3, 2024, 277–286

Some remarks on multivariate Schur-constant distributions

by

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Abstract

In this paper, we study multivariate Schur-constant distributions. We provide some interesting properties of these distributions. The Laplace transform of partial sum of Schur-constant vectors is also investigated.

Key Words: Schur-constant distribution, order statistics, survival analysis.
2020 Mathematics Subject Classification: Primary 60E05; Secondary 62E20, 60J60.

1 Introduction

An n-dimensional positive random vector $(X_1, \ldots, X_n)$ is said to have a Schur-constant distribution if its joint survival function has the particular form

$$\mathbb{P}(X_1 > x_1, \ldots, X_n > x_n) = S(x_1 + \cdots + x_n), \quad x_1, \ldots, x_n \geq 0, \tag{1}$$

for some admissible function $S : \mathbb{R}_+ \to [0, 1]$. The function S is called generator of Schur-constant distributions.

Schur-constant models are utilized to analyze random lifetimes. These models play a crucial role in Bayesian analysis of lifetimes because they possess a no-aging property, i.e., the distribution of lifetimes is exchangeable, see, e.g., Barlow and Mendel [2, 3], Caramellino and Spizzichino [4, 5], Unnikrishnan and Sankaran [16, 17]. Recently, Schur-constant models have a wide range of applications in, e.g, life sciences, actuarial sciences, finance, telecommunication and reliability, let us mention, among others, Ta et al [14, 15], Chi et al [7], Castañer et al [6], and Lefèvre and Simon [9]. Bivariate Schur-constant models, particularly those used by Kozlova and Salminen [8], Salminen and Vallois [11], and Salminen et al. [12], have been employed to investigate the starting and ending times of the busy periods in diffusion local time storage. These investigations have made significant contributions to the applications of the models in the field of telecommunications. Furthermore, Ta and Van [15] studied the Laplace transform of bivariate Schur-constant models, leading to the development of a new family of copulas. Utilizing these newly derived copulas, they investigated the dependence structure between the starting and ending times of the busy periods in diffusion local time storage.

It is noteworthy that most research in the field has focused on bivariate Schur-constant distributions. In this paper we provide some remarks and interesting properties of multivariate Schur-constant distributions. The paper is organized as follows. In the following section we introduce the concept and some important properties of order statistics, and the last section provides main results on multivariate Schur-constant models.

2 Order Statistics

Order statistics is a branch of statistical science. It has a wide range of applications in, e.g., actuarial sciences, medicine, and auction. In the next section, we will explore the connection between Schur-constant random variables and order statistics. To better understand this connection, we will review some important properties of order statistics. For more rigorous treatments and properties of order statistics, we refer to Arnold et al. [1].

Let $X_1, X_2, ..., X_n$ be independent copies of a real random variable X with distribution function F. We define $X_{(1)} := \min\{X_1, X_2, \dots, X_n\}$, $X_{(2)} := \min\{\{X_1, X_2, \dots, X_n\} \setminus \{X_{(1)}\}\}$, and $X_{(n)} := \max\{X_1, X_2, ..., X_n\}$. The vector $(X_{(1)}, X_{(2)}, \dots, X_{(n)})$ is called order statistics of X. We have

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}.$$

In actuarial sciences and extreme value theory, the distributions of extreme values $X_{(1)}$ and $X_{(n)}$ are of significant importance. It is straightforward to check the following identities

$$F_{X(n)}(x) = \mathbb{P}(X_{(n)} \leq x) = [F(x)]^n, \tag{2}$$

$$F_{X(1)}(x) = \mathbb{P}(X_{(1)} \leq x) = 1 - [1 - F(x)]^n, \quad x \in \mathbb{R}. \tag{3}$$

Furthermore, the distribution of the kth order variable $X_{(k)}$ is determined as follows.

Proposition 2.1. *For $k = 1, \dots, n$, and $x \in \mathbb{R}$, it holds*

$$F_{X(k)}(x) = \mathbb{P}(X_{(k)} \leq x) = \sum_{j=k}^{n} \binom{n}{j} F^j(x)(1 - F(x))^{n-j}. \tag{4}$$

By using the identity

$$\sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{(k-1)!(n-k)!} \int_0^p y^{j-1}(1-y)^{n-j} dy, \quad 0 < p < 1,$$

we obtain the new identity for $F_{X(k)}$

$$F_{X(k)}(x) = \frac{n!}{(k-1)!(n-k)!} \int_0^{F(x)} y^{j-1}(1-y)^{n-j} dy. \tag{5}$$

If X is a continuous random variable with density function f, then from (5) the density function of $X_{(k)}$ can be determined as follows.

Corollary 2.2. *If X is absolutely continuous, then for $k = 1, \dots, n$, the density function of $X(k)$ is*

$$f_{X(k)}(x) = \frac{n!}{(k-1)!(n-k)!} F^k(x)(1 - F(x))^{n-k} f(x). \tag{6}$$

Denote $f_{1:n}$ the joint probability density of all n order statistics. Then we have (see, e.g., [1])

$$f_{1:n}(x_1, x_2, \dots, x_n) = n! f(x_1) f(x_2) \cdots f(x_n) 1_{\{-\infty < x_1 < x_2 < \cdots < x_n < \infty\}}. \tag{7}$$

Now consider a special case when X is uniformly distributed random variable U on $(0, 1)$. Then we get the marginal probability density of $U_{(k)}$ and joint probability density of $U_{(1)}, U_{(2)}, \dots, U_{(n)}$.

Proposition 2.3. *(i) For $k = 1, \dots, n$, the marginal density of $U_{(k)}$ is*

$$f_{U(k)}(u_k) = \frac{n!}{(k-1)!(n-k)!} u_k^{k-1}(1-u_k)^{n-k}, \quad 0 < u_k < 1.$$

(ii) The joint probability density of $U_{(1)}, U_{(2)}, \dots, U_{(n)}$ is

$$f_{1:n}(u_1, u_2, ..., u_n) = n! 1_{\{0<u_1<u_2<...<u_n<1\}}.$$

3 Main results

In this section, we give a necessary and sufficient condition for n-dimensional positive random variables $(X_1, X_2, \dots, X_n)$ to be Schur-constant. More precisely, we provide a necessary and sufficient condition such that the distribution probability of a non-negative vector random variable can be represented via order statistics of the uniform distribution. Then, we derive the fundamental property of the class of Schur-constant distributions and some useful properties.

Let U be a uniformly distributed random variable on $(0, 1)$ and V an arbitrary positive random variable independent of U. Denote $U_{(1)}, U_{(2)}, \dots, U_{(n-1)}$ the order statistics of U. We have the following result.

Theorem 3.1. *Let $(X_1, X_2, ..., X_n)$ be a positive random vector, and $(U_1, U_2, \cdots, U_n)$ be independent and uniformly distributed on $(0, 1)$. Then there exists a positive random variable V independent of $(U_1, U_2, \dots, U_n)$ such that*

$$(X_1, X_2, ..., X_n) \overset{(d)}{=} (U_{(1)}V, (U_{(2)} - U_{(1)})V, \dots, (1 - U_{(n-1)})V), \tag{8}$$

if and only if the following representation holds

$$\begin{aligned} &\mathbb{P}(X_1 \in dt_1, X_2 \in dt_2, ..., X_n \in dt_n) \\ &\qquad = (n-1)! dt_1 dt_2 \cdots dt_{n-1} \nu(t_1 + \cdots + t_{n-1}, dt_n), \end{aligned} \tag{9}$$

where ν is a positive, finite measure on $\mathbb{R}_+$ satisfying

$$\nu(t, B) = \mu(t + B), \quad t + B := \{t + b : b \in B\},$$

for all Borel set B on $\mathbb{R}_+$, whilst μ is some measure on $\mathbb{R}_+$. Moreover, if (9) holds, then

$$\int_0^\infty x^{n-1} \mu(dx) = 1.$$

Proof. Assume $(X_1, X_2, ..., X_n) \overset{(d)}{=} (U_{(1)}V, (U_{(2)} - U_{(1)})V, \dots, (1 - U_{(n-1)})V)$. Let F_V denote the distribution function of V. For any bounded and positive measurable function g on $\mathbb{R}^n$, we have

$$\mathbb{E}(g((X_1, X_2, ..., X_n)) = \mathbb{E}(g(U_{(1)}V, (U_{(2)} - U_{(1)})V, ..., (1 - U_{(n-1)})V))$$

$$= \int_{\mathbb{R}_+^n} g(u_1 v, (u_2 - u_1)v, ..., (1 - u_{n-1})v)(n-1)!$$
$$1_{\{0 \le u_1 < u_2 < ... < u_{n-1} \le 1\}} du_1 du_2 ... du_{n-1} F_V(dv).$$

Change of variables $a_1 = u_1 v, a_2 = (u_2 - u_1)v, \ldots, a_{n-1} = (u_{n-1} - u_{n-2})v$ and $a_n = (1 - u_{n-1})v$ yields

$$v = \sum_{k=1}^{n} a_k,$$

and

$$u_k = \frac{\sum_{i=1}^{k} a_i}{\sum_{j=1}^{n} a_j}, \quad k = 1, \ldots, n-1.$$

We have the partial derivatives

$$\frac{\partial u_1}{\partial a_1} = \frac{\sum_{k=2}^{n} a_k}{v^2}, \quad \frac{\partial u_1}{\partial a_2} = \frac{\partial u_1}{\partial a_3} = \cdots = \frac{\partial u_1}{\partial a_{n-1}} = \frac{\partial u_1}{\partial a_n} = -\frac{a_1}{v^2},$$
$$\frac{\partial u_2}{\partial a_1} = \frac{\partial u_2}{\partial a_2} = \frac{\sum_{k=3}^{n} a_k}{v^2}, \quad \frac{\partial u_2}{\partial a_3} = \cdots = \frac{\partial u_2}{\partial a_{n-1}} = \frac{\partial u_2}{\partial b} = -\frac{(a_1 + a_2)}{v^2},$$
$$\cdots$$
$$\frac{\partial u_{n-1}}{\partial a_1} = \cdots = \frac{\partial u_{n-1}}{\partial a_{n-1}} = \frac{a_n}{v^2}, \quad \frac{\partial u_{n-1}}{\partial a_n} = -\frac{\sum_{k=1}^{n-1}}{v^2},$$
$$\frac{\partial v}{\partial a_1} = \frac{\partial v}{\partial a_2} = \cdots = \frac{\partial v}{\partial a_{n-1}} = \frac{\partial v}{\partial a_n} = 1.$$

and, hence, the Jacobian determinant

$$|J| = \frac{1}{v^{n-1}}.$$

So we obtain

$$\mathbb{E}(g(X_1, X_2, \cdots, X_n))$$
$$= \int_{\mathbb{R}_+^n} (n-1)! g(a_1, a_2, .., a_{n-1}, a_n) da_1 da_2 \cdots da_{n-1} \nu(a_1 + a_2 + \cdots + a_{n-1}, da_n),$$

where

$$\nu(a_1 + a_2 + \cdots + a_{n-1}, da_n) := \frac{1}{v^{n-1}} F_V(a_1 + a_2 + \cdots + a_{n-1} + da_n).$$

Consequently, the identity (9) holds. Moreover, putting $\mu(dv) := v^{-(n-1)} F_V(dv)$, it yields

$$\int_0^\infty x^{n-1} \mu(dx) = 1.$$

Now assume that the distribution of random vector $(X_1, X_2, \cdots, X_n)$ has the representation (9), i.e.,

$$\mathbb{P}(X_1 \in dt_1, X_2 \in dt_2, \cdots, X_n \in dt_n) = (n-1)! dt_1 dt_2 \cdots dt_{n-1} \nu(t_1 + \cdots + t_{n-1}, dt_n).$$

For every $0 < t_i \leqslant 1, \quad i = 1, \ldots, n-1, \quad t_n > 0$, we have

$$\mathbb{P}(\frac{X_1}{\sum_{k=1}^n X_k} \leqslant t_1, \frac{X_2}{\sum_{k=1}^n X_k} \leqslant t_2, \ldots, \frac{X_{n-1}}{\sum_{k=1}^n X_k} \leqslant t_{n-1}, \sum_{k=1}^n X_k \leqslant t_n)$$
$$= (n-1)! \int_D dx_1 dx_2 \cdots dx_{n-1} \nu(x_1 + x_2 + \cdots + x_{n-1}, dx_n),$$

where

$$D = \{(x_1, x_2, \ldots, x_n) : \frac{x_i}{\sum_{k=1}^n x_k} \leqslant t_i, \sum_{k=1}^n x_k \leqslant t_n, \quad t_i \in (0,1), \quad i = 1, 2, \ldots, n-1\}.$$

Putting

$$u_i = \frac{x_i}{\sum_{k=1}^n x_k}, \quad i = 1, \ldots, n-1,$$

and

$$u_n = \sum_{k=1}^n x_k,$$

we get

$$x_i = u_i u_n, \quad i = 1, \ldots, n-1,$$
$$x_n = (1 - \sum_{k=1}^{n-1} u_k) u_n,$$

and the Jacobian determinant

$$|J| = u_n^{n-1}.$$

So we have

$$\mathbb{P}(\frac{X_1}{\sum_{k=1}^n X_k} \leqslant t_1, \frac{X_2}{\sum_{k=1}^n X_k} \leqslant t_2, \cdots, \frac{X_{n-1}}{\sum_{k=1}^n X_k} \leqslant t_{n-1}, \sum_{k=1}^n X_k \leqslant t_n)$$
$$= \int_{D_1} (n-1)! 1_{\{u_1 < u_2 < \cdots < u_{n-1}\}} du_1 du_2 \cdots du_{n-1} u_n^{n-1} \mu(du_n), \tag{10}$$

where

$$D_1 = \{(u_1, u_2, \cdots, u_n) : 0 < u_1 \leqslant t_1, 0 < u_2 - u_1 \leqslant t_2, \cdots, \\ 0 < u_{n-1} - u_{n-2} \leqslant t_{n-1}, 0 < u_n \leqslant t_n\}.$$

From (10) we see that

$$(\frac{X_1}{\sum_{k=1}^n X_k}, \frac{X_2}{\sum_{k=1}^n X_k}, \ldots, \frac{X_{n-1}}{\sum_{k=1}^n X_k}) \stackrel{(d)}{=} (U_{(1)}, U_{(2)} - U_{(1)}, \ldots, U_{(n-1)} - U_{(n-2)}).$$

The random variable $V := \sum_{k=1}^n X_k$ is independent of $(\frac{X_1}{\sum_{k=1}^n X_k}, \frac{X_2}{\sum_{k=1}^n X_k}, \ldots, \frac{X_{n-1}}{\sum_{k=1}^n X_k})$, and hence, V is independent of uniformly distributed U on $(0,1)$.

$\square$

The theory of n-times monotone functions plays a crucial role in studying continuous distribution functions. A function $f(x)$ defined on $\mathbb{R}_+$ is said to be n-times monotone, where n is an integer and $n \geq 2$, if it is differentiable there up to the order $n-2$, and the derivatives

$$(-1)^k f^{(k)}(x) \geq 0, \quad k = 0, \dots, n-2,$$

and further $(-1)^{n-2} f^{(n-2)}$ is non-increasing and convex. Williamson [18] provides a characterization of an n-times monotone function as an integral transform. This transform is called the Williamson n-transform, denote by $\mathfrak{W}_n$.

Lemma 3.2. *A function f defined on $\mathbb{R}_+$ is n-times monotone, $n \geq 2$ if and only if for all $x > 0$, there exists a positive measure ν such that*

$$f(x) = \mathfrak{W}_n \nu(x) := \int_0^\infty \left[\left(t-x\right)_+\right]^{n-1} \nu(dt),$$

where $(u)_+ \equiv \max(u, 0)$.

The following proposition provides a connection between Schur-constant distribution and n-times monotone functions (see, e.g.,[9], and see also [13, Lemma 3.2]).

Proposition 3.3. *A function S on $\mathbb{R}_+$ is a survival function of a Schur-constant vector $(X_1, \dots, X_n)$ if and only if S is n-times monotone, e.g., $(-1)^k S^{(k)}(x) \geq 0, \quad k = 0, \dots, n-2,$, and $(-1)^{n-2} S^{(n-2)}$ is non-increasing and convex.*

In the following theorem we show that the class of vector random variables $\{(U_{(1)}V, (U_{(2)} - U_{(1)})V, \dots, (1 - U_{(n-1)})V)\}$ coincides with the class of Schur-constant. This result is the fundamental property of the class of Schur-constant distributions. It can be found in, e.g., [7, Theorem 2.1] and [10, Proposition 3.2, 3.3]. Section 3 in [10] also shows that the Schur-constant distributions are exactly the ℓ_1-norm symmetric distributions. However, by using Theorem 3.1 we give here a new proof for this result.

Denote $\bar{F}$ the survival function of random variables $X_1, X_2, \dots, X_n$, i.e.,

$$\bar{F}(x_1, x_2, \dots, x_n) := \mathbb{P}(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n).$$

Theorem 3.4. *The positive random vector $X = (X_1, X_2, \dots, X_n)$ is Schur-constant if and only if for every $U_1, \dots, U_n$ independent and uniformly distributed on $(0, 1)$ there exists a positive random variable V independent of $U_1, \dots, U_n$ such that*

$$(X_1, X_2, ..., X_n) \overset{(d)}{=} (U_{(1)}V, (U_{(2)} - U_{(1)})V, \dots, (1 - U_{(n-1)})V), \tag{11}$$

where $U_{(1)}, \dots, U_{(n)}$ are the order statistics associated with $U_1, \dots, U_n$.

Proof. Assume that $(X_1, X_2, \dots, X_n)$ has the representation (11). From Theorem 3.1 we have, for every $x_i > 0, i = 1, \dots, n$

$$\begin{aligned}
&\mathbb{P}(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) \\
&= (n-1)! \int_{x_1}^{\infty} du_1 \int_{x_2}^{\infty} du_2 \cdots \int_{x_{n-1}}^{\infty} du_{n-1} \int_{x_n}^{\infty} \nu(u_1 + u_2 + \cdots + u_{n-1}, du_n) \\
&= (n-1)! \int_{x_1}^{\infty} du_1 \int_{x_2}^{\infty} du_2 \cdots \int_{x_{n-1}}^{\infty} du_{n-1} \int_{u_1+u_2+\cdots+u_{n-1}+x_n}^{\infty} \mu(du_n) \\
&= (n-1)! \int_{(\sum_{k=1}^{n} x_k, \infty)} \mu(du_n) \int_C du_1 du_2 \dots du_{n-1},
\end{aligned}$$

where

$$C = \{(u_1, u_2, \dots, u_{n-1}) : u_1 + u_2 + \cdots + u_{n-1} < u_n - x_n, \quad x_i < u_i, i = 1, \dots n-1\}$$

So we get

$$\begin{aligned}
\mathbb{P}(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) = & \\
&= (n-1)! \int_{(\sum_{k=1}^{n} x_k, \infty)} \frac{(u_n - \sum_{k=1}^{n} x_k)^{n-1}}{(n-1)!} \mu(du_n) \\
&= \int_{(\sum_{k=1}^{n} x_i, \infty)} (u_n - \sum_{k=1}^{n} x_k)^{n-1} \mu(du_n). \qquad (12)
\end{aligned}$$

The identity (12) can be written as follows

$$\mathbb{P}(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) = \int_0^{\infty} \left(1 - \frac{\sum_{k=1}^{n} x_k}{v}\right)_+^{n-1} v^{n-1} \mu(dv),$$

and, hence, we see that Schur-constant vector X is the Williamson n-transform $\mathfrak{W}_n$ of the random variable V, i.e.,

$$\bar{F}(x_1, x_2, \dots, x_n) = \mathfrak{W}_n F_V(\sum_{i=1}^{n} x_i) = \int_0^{\infty} \left(1 - \frac{\sum_{k=1}^{n} x_i}{t}\right)_+^{n-1} F_V(dt). \qquad (13)$$

We refer to [10, Section 3] for more discussions regarding the connections between Schur-constant distributions and Williamson n-transforms.

Now assume that $(X_1, \dots, X_n)$ is Schur-constant with the joint distribution $\bar{F}$ and the generator S. Then S is n-times monotone. By Proposition 3.3 and Lemma 3.2 there is a measure μ such that

$$\bar{F}(x_1, x_2, \dots, x_n) = S(x_1 + x_2 + \cdots + x_n) = \int_0^{\infty} \left(1 - \frac{\sum_{k=1}^{n} x_k}{t}\right)_+^{n-1} t^{n-1} \mu(dt).$$

Consequently, $\bar{F}$ is also the joint survival distribution function of the random vector

$$(U_{(1)}V, (U_{(2)} - U_{(1)})V, \dots, (1 - U_{(n-1)})V),$$

where, V has distribution $F_V(x) = \int_x^{\infty} t^{n-1} \mu(dt)$.

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Remark 3.5. *From (13), wee see that the generator function S can be represented in terms of the distribution of the random variable V as follows*

$$S(x) = \mathbb{E}\Big[\Big(1 - \frac{x}{V}\Big)_+^{n-1}\Big], \quad x \geq 0.$$

Now consider the partial sums

$$S_k := \sum_{i=1}^{k} X_i, \qquad T_k := \sum_{i=k+1}^{n} X_i.$$

Then we obtain the Laplace transform of S_k and T_k.

Theorem 3.6. *Let $(X_1, X_2, \ldots, X_n)$ be Schur-constant vector. Then for all $\alpha \neq \beta$, it holds*

$$\mathbb{E}(e^{-\alpha S_k - \beta T_k}) = \frac{(n-1)!}{(k-1)!(n-k-1)!(\alpha-\beta)^{n-1}} \int_\beta^\alpha \mathbb{E}(e^{-tV})(\alpha - t)^{n-k-1}(t-\beta)^{k-1} dt \quad (1)$$

for all $k = 1, 2, \ldots, n-1$.

Proof. Since $(X_1, X_2, \ldots, X_n)$ is Schur-constant then

$$(X_1, X_2, \ldots, X_n) \overset{(d)}{=} (U_{(1)}V, (U_{(2)} - U_{(1)})V, \ldots, (1 - U_{(n-1)})V).$$

Hence

$$S_k = \sum_{i=1}^{k} X_i = U_{(k)}V,$$

$$T_k = \sum_{i=k+1}^{n} X_i = (1 - U_{(k)})V.$$

Density distribution of $U_{k:n-1}$ is given by

$$f_{k:n-1}(x) = \frac{(n-1)!}{(k-1)!(n-k-1)!} x^{k-1}(1-x)^{n-k-1}, \quad 0 \leqslant x \leqslant 1.$$

So we get

$$\begin{aligned}\mathbb{E}(e^{-\alpha S_k - \beta T_k}) &= \mathbb{E}\Big(e^{-V[(\alpha-\beta)U_{(k)}+\beta]}\Big) \\ &= \frac{(n-1)!}{(k-1)!(n-k-1)} \int_0^1 \mathbb{E}(e^{-V[(\alpha-\beta)x+\beta]}) x^{k-1}(1-x)^{n-k-1} dx.\end{aligned}$$

Putting

$$t = (\alpha - \beta)x + \beta,$$

we obtain

$$\mathbb{E}(e^{-\alpha S_k - \beta T_k}) = \frac{(n-1)!}{(k-1)!(n-k-1)!(\alpha-\beta)^{n-1}} \int_\beta^\alpha \mathbb{E}(e^{-tV})(\alpha - t)^{n-k-1}(t-\beta)^{k-1} dt.$$

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Acknowledgement *The author would like to thank the anonymous reviewer for a very detailed report and valuable comments which led to an improvement of the paper.*

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Received: 08.05.2023
Revised: 29.03.2024
Accepted: 08.04.2024

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