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Families of isotropic subspaces in a symplectic $\mathbf{Z}/2$-vector space

by
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Abstract

For a symplectic vector space over $\mathbf{Z}/2$ we give a non-inductive definition of a family of isotropic subspaces with remarkable properties.

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0 Introduction

0.1. Let $F = \mathbf{Z}/2$ be the field with two elements. Let $\bar{V}$ be an F-vector space of finite dimension $2n \geq 2$ endowed with a nondegenerate symplectic form $<,>$ and with a collection of vectors $\bar{e}_0, \bar{e}_1, \bar{e}_2, \dots, \bar{e}_{2n}$ such that

$< \bar{e}_0, \bar{e}_1 >=< \bar{e}_1, \bar{e}_2 >= \dots =< \bar{e}_{2n-1}, \bar{e}_{2n} >=< \bar{e}_{2n}, \bar{e}_0 >= 1,$

$< \bar{e}_1, \bar{e}_0 >=< \bar{e}_2, \bar{e}_1 >= \dots =< \bar{e}_{2n}, \bar{e}_{2n-1} >=< \bar{e}_0, \bar{e}_{2n} >= 1$

and $< \bar{e}_i, \bar{e}_j >= 0$ for all other pairs i, j. (Such a collection is called a "circular basis" in [3].)

In [3] we have introduced a family $\mathcal{F}(\bar{V})$ of isotropic subspaces of $\bar{V}$ with remarkable properties:

There is a unique bijection $\mathcal{F}(\bar{V}) \xrightarrow{\sim} \bar{V}$ such that any $x \in \bar{V}$ is contained in the corresponding subspace of $\bar{V}$. The characteristic functions of the various subspaces in $\mathcal{F}(\bar{V})$ form a new basis of the complex vector space $\bar{V}^{\mathbf{C}}$ of functions $\bar{V} \to \mathbf{C}$ which is related to the obvious basis of $\bar{V}^{\mathbf{C}}$ by an upper triangular matrix with 1 on diagonal (in some partial order $\leq$ on $\mathcal{F}(\bar{V})$).

(In fact the collection $\mathcal{F}(\bar{V})$ was already introduced in [2], but in a less symmetric form.)

A further property of $\mathcal{F}(\bar{V})$ was found in [3], namely that the matrix of the Fourier transform $\bar{V}^{\mathbf{C}} \to \bar{V}^{\mathbf{C}}$ with respect to the new basis is upper triangular with ± 1 on diagonal. The proof of this property was based on the observation that the new basis admits a dihedral symmetry which was not visible in the definition of [2].

In this paper we give a new non-inductive definition of $\mathcal{F}(\bar{V})$ which is visibly compatible with the dihedral symmetry (the definition of [2] has no such a symmetry property; the definition in [3] did have the symmetry property but was inductive). We also give a formula for the bijection $\mathcal{F}(\bar{V}) \xrightarrow{\sim} \bar{V}$ above which is clearly compatible with the dihedral symmetry. (See Theorem 1.4.)

Let V be an F-vector space with basis $e_0, e_1, \dots, e_{2n}$ such that $\bar{V}$ is the quotient of V by the line $F(e_0 + e_1 + \dots + e_{2n})$ and $\bar{e}_i$ is the image of e_i under the obvious map $V \to \bar{V}$. In Section 4 we define an analogue $\tilde{\mathcal{F}}(V)$ of $\mathcal{F}(\bar{V})$ which is a refinement of $\mathcal{F}(\bar{V})$ and has several properties of $\mathcal{F}(\bar{V})$.

In Sections 5 - 7 we study a modification of the family $\mathcal{F}(\bar{V})$ which plays the same role in the theory of unipotent representations of orthogonal groups over a finite field as that played by $\mathcal{F}(\bar{V})$ in the analogous theory for symplectic groups over a finite field.

1 Statement of the Theorem

1.1. Let V be an F-vector space endowed with a symplectic form $<,>: V \times V \to F$ and a map $e : S \to V$, $s \mapsto e_s$ where S is a finite set. Let $\mathfrak{E}$ be the set of unordered pairs $s \neq s'$ in S such that $< e_s, e_{s'} >= 1$. This is the set of edges of a graph with set of vertices S. For any $I \subset S$ we set $e_I = \sum_{s \in I} e_s \in V$ and we denote by $\underline{I}$ the full subgraph of $(S, \mathfrak{E})$ whose set of vertices is I. Let $\mathcal{I}$ be the set of all $I \subset S$ such that $\underline{I}$ is a graph of type A_m for some $m \geq 1$. We have $\mathcal{I} = \mathcal{I}^0 \sqcup \mathcal{I}^1$ where $\mathcal{I}^0 = \{I \in \mathcal{I}; |I| = 0 \mod 2\}$, $\mathcal{I}^1 = \{I \in \mathcal{I}; |I| = 1 \mod 2\}$. For I, I' in $\mathcal{I}^1$ we write $I \prec I'$ whenever $I \subsetneqq I'$ and $\underline{I' - I}$ is disconnected. For I, I' in $\mathcal{I}^1$ we write $I \spadesuit I'$ whenever $I \cap I' = \emptyset$ and $\underline{I \cup I'}$ is disconnected. For $I \in \mathcal{I}^1$ let I^{ev} be the set of all $s \in I$ such that $I - \{s\} = I' \sqcup I''$, with $I' \in \mathcal{I}^1$, $I'' \in \mathcal{I}^1$, $I' \spadesuit I''$. Let $I^{odd} = I - I^{ev}$. We have $|I^{ev}| = (|I| - 1)/2$.

1.2. Let R be the set whose elements are finite unordered sequences of objects of $\mathcal{I}^1$. For $B \in R$ let L_B be the subspace of V generated by $\{e_I; I \in B\}$; for a subspace L of V let $B_L = \{I \in \mathcal{I}^1; e_I \in L\} \subset R$. For $s \in S$, $B \in R$ we set

$$g_s(B) = |\{I \in B; s \in I\}|$$

(here $|?|$ denotes the number of elements of ?) and

$$\epsilon_s(B) = (1/2) g_s(B)(g_s(B) + 1) \in F.$$

For $B \in R$ we set
$$\epsilon(B) = \sum_{s \in S} \epsilon_s(B) e_s \in V.$$

For $B \in R$ we set $\mathrm{supp}(B) = \cup_{I \in B} I \subset S$.

Let $\phi(V)$ be the set consisting of all $B \in R$ such that $(P_0), (P_1)$ below hold.

(P_0) If $I \in B, I' \in B$, then $I = I'$, or $I \spadesuit I'$, or $I \prec I'$, or $I' \prec I$.

(P_1) Let $I \in B$. There exist $I_1, I_2, \dots, I_k$ in B such that $I^{ev} \subset I_1 \cup I_2 \cup \dots \cup I_k$ (disjoint union), $I_1 \prec I$, $I_2 \prec I, \dots, I_k \prec I$.

We say that $(V, <,>, e)$ is *perfect* if properties (i)-(iv) below hold.

(i) If $B \in \phi(V)$, then $\{e_I; I \in B\}$ is a basis of $L := L_B$; moreover, $B = B_L$.

(ii) For any $B \in \phi(V)$ we have $\epsilon(B) \in L_B$. Hence ϵ restricts to a map $\phi(V) \to V_0$ (denoted again by ϵ) where $V_0 = \cup_{B \in \phi(V)} L_B \subset V$.

(iii) The map $\epsilon : \phi(V) \to V_0$ is a bijection.

(iv) If B, B' in $\phi(V)$ are such that $\epsilon(B') \in L_B$, then $g_s(B') \leq g_s(B)$ for any $s \in S$.

For B', B in $\phi(V)$ we say that $B' \leq B$ if there exist $B_0, B_1, B_2, \dots, B_k$ in $\phi(V)$ such that $B_0 = B', B_k = B$, $\epsilon(B_0) \in L_{B_1}, \epsilon(B_1) \in L_{B_2}, \dots, \epsilon(B_{k-1}) \in L_{B_k}$.

We show:

(a) *If* $(V, <,>, e)$ *is perfect, then* $\leq$ *is a partial order on* $\phi(V)$.

Assume that we have elements $B_0, B_1, \dots, B_k, B'_0, B'_1, \dots, B'_l$ in $\phi(V)$ such that

$\epsilon(B_0) \in L_{B_1}, \epsilon(B_1) \in L_{B_2}, \dots, \epsilon(B_{k-1}) \in L_{B_k}$,
$\epsilon(B'_0) \in L_{B'_1}, \epsilon(B'_1) \in L_{B'_2}, \dots, \epsilon(B'_{l-1}) \in L_{B'_l}$,
and $B_0 = B'_l, B'_0 = B_k$. We must prove that $B_0 = B'_0$. Using (iv) and our assumptions we have for any $s \in S$:
$g_s(B_0) \le g_s(B_1) \le g_s(B_2) \le \dots \le g_s(B_k) = g_s(B'_0)$,
$g_s(B'_0) \le g_s(B'_1) \le g_s(B'_2) \le \dots \le g_s(B'_l) = g_s(B_0)$.
It follows that $g_s(B_0) \le g_s(B'_0), g_s(B'_0) \le g_s(B_0)$, so that $g_s(B_0) = g_s(B'_0)$. Since this holds for any s, we see that $\epsilon(B_0) = \epsilon(B'_0)$. Using the injectivity of ϵ (see (iii)), we deduce that $B_0 = B'_0$, as desired.

1.3. We will consider three cases:

(a) $V, <,>, e : S \to V$ are such that $\{e_s; s \in S\}$ is a basis of V and $(S, \mathfrak{E})$ is a graph of type A_{N-1}, $N \in \{3,5,7,\dots\}$;

(b) $V, <,>, e : S \to V$ are such that $\{e_s; s \in S\}$ is a basis of V and $(S, \mathfrak{E})$ is a graph of affine type A_{N-1}, $N \in \{3,5,7,\dots\}$;

(c) $V, <,>, e : S \to V$ in (b) are replaced by $\bar{V} = V/Fe_S$, by the symplectic form induced by $<,>$ (denoted again by $<,>$), and by $\pi e : S \to \bar{V}$, where $\pi : V \to \bar{V}$ is the obvious map.

In cases (b),(c) we note that the automorphism group of the graph $(S, \mathfrak{E})$ is a dihedral group Di_{2N} of order $2N$. It acts naturally on V in (b) by permutations of the basis; this induces an action of Di_{2N} on $\bar{V}$ in (c).

Let $I \subset S$; in cases (b),(c) we assume that $I \ne S$. There is a well defined subset $c(I)$ of $\mathcal{I}$ such that $I' \spadesuit I''$ for any $I' \ne I''$ in $c(I)$ and $I = \sqcup_{I' \in c(I)} I'$. Note that $\{\underline{I'}; I' \in c(I)\}$ are the connected components of the graph $\underline{I}$.

We now state the following result.

Theorem 1.4. *In each of the cases 1.3(a),(b),(c), $(V, <,>, e)$ is perfect.*

1.5. In case 1.3(a), Theorem 1.4 is contained in [1]. Let $\mathcal{F}(V)$ be the set of subspaces of V of the form L_B for some $B \in \phi(V)$. Note that $B \mapsto L_B$ is a bijection $\phi(V) \xrightarrow{\sim} \mathcal{F}(V)$.

We can write the elements of S as a sequence $s_1, s_2, \dots, s_{N-1}$ in which any two consecutive elements are joined in the graph $(S, \mathfrak{E})$. Let $I \subset S$. Let $c(I)$ be as in 1.3. Let $c(I)^{0+}$ (resp. $c(I)^{0-}$) be the set of all $I' \in c(I)$ such that $I' = \{s_k, s_{k+1}, \dots, s_l\}$ where k is even, l is odd (resp. k is odd, l is even). Let V_0 be the subset of V consisting of all e_I where $I \subset S$ satisfies $|c(I)^{0+}| = |c(I)^{0-}|$. From [1] it is known that V_0 coincides with the subset of V appearing in 1.2(ii) that is,

(a) $\cup_{L \in \mathcal{F}(V)} L = V_0$.

2 The case 1.3(c)

2.1. In this section we assume that we are in case 1.3(c). For $s \in S$ we set $\bar{e}_s = \pi(e(s))$. For $I \subset S$ we set $\bar{e}_I = \sum_{s \in I} \bar{e}_s$. Note that $\{\bar{e}_s; s \in S\}$ is a circular basis of $\bar{V}$ (in the sense of [3]) and to this we can attach a collection $\mathcal{F}(\bar{V})$ of subspaces of $\bar{V}$ as in [3]. We recall how this was done. For any $s \in S$ we set

$$\hat{s} = \{s' \in S; <\bar{e}_s, \bar{e}_{s'}> = 1\} \cup \{s\} \subset S.$$

We have $|\hat{s}|=3$. We set $\bar{e}_s^{\perp}=\{x\in\bar{V};<x,\bar{e}_s>=0\}$ and $\bar{V}_s=\bar{e}_s^{\perp}/F\bar{e}_s$. This is a symplectic F-vector space with circular basis $\{\bar{e}_{s'};s'\in S-\hat{s}\}\sqcup\{\bar{e}_{\hat{s}}\}$. Thus the analogue of S when $\bar{V}$ is replaced by $\bar{V}_s$ is $S_s=(S-\hat{s})\sqcup\{\hat{s}\}$ (a set with $|S|-2$ elements). Let $\bar{p}_s:\bar{e}_s^{\perp}\to\bar{V}_s$ be the obvious linear map. We define a collection $\mathcal{F}(\bar{V})$ of subspaces of $\bar{V}$ by induction on N. If $N=3$, $\mathcal{F}(\bar{V})$ consists of 0 and of $\bar{p}_s^{-1}(0)$ for various $s\in S$. If $N\ge 5$, $\mathcal{F}(\bar{V})$ consists of 0 and of $\bar{p}_s^{-1}(L')$ for various $s\in S$ and various $L'\in\mathcal{F}(\bar{V}_s)$ (which is defined by the induction hypothesis). In [3], $\mathcal{F}(\bar{V})$ is also identified with a collection of subspaces of $\bar{V}$ introduced in [2] in terms of a chosen element $t\in S$. From this identification we see that:

(a) *if $L\in\mathcal{F}(\bar{V})$ and $B_L^t:=\{I\in\mathcal{I};I\subset S-\{t\},\bar{e}_I\in L\}$, then $\{\bar{e}_I;I\in B_L^t\}$ is an F-basis of L, so that $L=L_{B_L^t}$.*

Now if $I\in\mathcal{I}$, then $S-I\in\mathcal{I}$ and we have $\bar{e}_I=\bar{e}_{S-I}$. Moreover, exactly one of $I,S-I$ is contained in $S-\{t\}$ and exactly one of $I,S-I$ is in $\mathcal{I}^1$. We deduce that:

(b) *If $L\in\mathcal{F}(\bar{V})$, and*

$$B_L:=\{I\in\mathcal{I}^1;\bar{e}_I\in L\}=\{I\in\mathcal{I}^1;I\in B_L^t\}\sqcup\{I\in\mathcal{I}^1;S-I\in B_L^t\}$$

then $\{\bar{e}_I;I\in B_L\}$ is an F-basis of L, so that $L=L_{B_L}$.

2.2. We show that for $B\in R$:

(a) *we have $B\in\phi(\bar{V})$ if and only if $L_B\in\mathcal{F}(\bar{V})$.*

The proof is analogous to that of the similar result in case 1.3(a) given in [1]. We argue by induction on N. If $N=3$, (a) is easily verified. In this case, B is either $\emptyset$ or it is of the form $\{s\}$ for some $s\in S$. We now assume that $N\ge 5$. For $s\in S$ we denote by $\mathcal{I}_s^1,R_s$ the analogues of $\mathcal{I}^1,R$ when S is replaced by S_s (see 2.1). For $J\in\mathcal{I}_s^1$ we write $\bar{e}_J\in\bar{V}_s$ for the analogue of $\bar{e}_I\in\bar{V},I\in\mathcal{I}^1$. We have

$$\bar{p}_s^{-1}(\bar{e}_J)=\{\bar{e}_I,\bar{e}_I+\bar{e}_s\}$$

for a well defined $I\in\mathcal{I}^1$ such that $s\notin I$; we set $I=\xi_s(J)$. There is a well defined map $\tau_s:R_s\to R$, $B'_1\mapsto B_1$ where B_1 consists of $\{s\}$ and of all $\xi_s(J)$ with $J\in B'_1$. From the definitions we see that (assuming that $B'_1\in R_s$ and $B_1=\tau_s(B'_1)$), the following holds.

(b) *B'_1 satisfies (P_0) if and only if B_1 satisfies (P_0); B'_1 satisfies (P_1) if and only if B_1 satisfies (P_1).*

Assume now that B is such that $L:=L_B\in\mathcal{F}(\bar{V})$, so that $B=B_L$. We show that B satisfies $(P_0),(P_1)$. If $B=\emptyset$, this is obvious. If $B\ne\emptyset$, we have $L=\bar{p}_s^{-1}(L')$ where $s\in S,L'\in\mathcal{F}(\bar{V}_s)$. From the definition we have $\tau_s(B_{L'})=B_L$. By the induction hypothesis, $B_{L'}$ satisfies $(P_0),(P_1)$; using (b), we see that $B=B_L$ satisfies $(P_0),(P_1)$.

Conversely, assume that B satisfies $(P_0),(P_1)$. We show that $B=B_L$ for some $L\in\mathcal{F}(\bar{V})$. If $B=\emptyset$ this is obvious. Thus we can assume that $B\ne\emptyset$. Let $I\in B$ be such that $|I|$ is minimum. If $s\in I^{ev}$ (see 1.1) then by (P_1) we can find $I'\in B$ with $s\in I',|I'|<|I|$, a contradiction. We see that $I^{ev}=\emptyset$. Thus, $I=\{s\}$ for some $s\in S$. Using (P_0) and $\{s\}\in B$, we see that for any $I'\in B-\{s\}$ we have $\{s\}\prec I'$ or I'♠$\{s\}$. It follows that $B=\tau_s(B')$ for some $B'\in R_s$. From (b) we see that B' satisfies $(P_0),(P_1)$. From the induction hypothesis we see that $B'=B_{L'}$ for some $L'\in\mathcal{F}(\bar{V}_s)$. Let $L=\bar{p}_s^{-1}(L')$. We have $L\in\mathcal{F}(\bar{V})$ and $B=B_L$. This proves (a).

We see that we have a bijection

(c) $\phi(\bar{V}) \xrightarrow{\sim} \mathcal{F}(\bar{V})$, $B \mapsto L_B$.
Using now 2.1(b) we see that 1.2(i) holds for any $B \in \phi(\bar{V})$.

2.3. We now fix $t \in S$. Let $B \in \mathcal{F}(\bar{V})$, let $L = L_B \in \mathcal{F}(\bar{V})$ and let $B^t = B^t_L$ (see 2.1). For any $s \in S - \{t\}$ we set

$$f_s(B) = |\{I \in B^t \cap \mathcal{I}^1; s \in I\}| - |\{I \in B^t \cap \mathcal{I}^0; s \in I\}| - \underline{|B^t \cap \mathcal{I}^0|}$$

where for any $m \in \mathbf{Z}$ we set $\underline{m} = 0$ if m is even, $\underline{m} = 1$ if m is odd. We also set

$$\epsilon'(B) = \sum_{s \in S - \{t\}} (1/2) f_s(B)(f_s(B) + 1)\bar{e}_s \in \bar{V}.$$

From [2],[3] we see using 2.2(c) that:
(a) *we have $\epsilon'(B) \in L_B$ for any $B \in \phi(\bar{V})$ and $B \mapsto \epsilon'(B)$ defines a bijection $\epsilon' : \phi(\bar{V}) \xrightarrow{\sim} \bar{V}$.*

2.4. We wish to rewrite the bijection $\epsilon' : \phi(\bar{V}) \xrightarrow{\sim} \bar{V}$ without reference to $t \in S$. Recall that for any $B \in \phi(\bar{V})$ and any $s \in S$ we have
(a) $g_s(B) = |\{I \in B; s \in I\}| \in \mathbf{N}$.
Setting $\beta = |B^t \cap \mathcal{I}^0|$ where $B^t = B^t_L$, $L = L_B$ (see 2.1) we have
(b) $g_t(B) = \beta$.
For $s \in S - \{t\}$ we show:
(c) $f_s(B) = g_s(B) - \beta - \underline{\beta}$
that is,

$$|\{I \in B^t \cap \mathcal{I}^1; s \in I\}| - |\{I \in B^t \cap \mathcal{I}^0; s \in I\}| = |\{I \in B; s \in I\}| - \beta.$$

To prove this, we substitute $|\{I \in B; s \in I\}|$ by

$$|\{I \in B^t \cap \mathcal{I}^1; s \in I\}| + |\{I \in B^t \cap \mathcal{I}^0; s \notin I\}|.$$

We see that desired equality becomes

$$\begin{aligned} &|\{I \in B^t \cap \mathcal{I}^1; s \in I\}| - |\{I \in B^t \cap \mathcal{I}^0; s \in I\}| \\ &= |\{I \in B^t \cap \mathcal{I}^1; s \in I\}| + |\{I \in B^t \cap \mathcal{I}^0; s \notin I\}| - \beta \end{aligned}$$

which is obvious.

We shall prove the following formula for $\epsilon'(B)$:

(d.) $$\epsilon'(B) = \textstyle\sum_{s \in S} (1/2) g_s(B)(g_s(B) + 1)\bar{e}_s$$

Using (c) we have for $s \in S - \{t\}$:

$$\begin{aligned} (1/2) f_s(B)(f_s(B) + 1) &= (1/2)(g_s(B) - \beta - \underline{\beta})(g_s(B) - \beta - \underline{\beta} + 1) \\ &= (1/2) g_s(B)(g_s(B) + 1) + H \end{aligned}$$

where

$$H = (1/2)(g_s(B)(-2\beta - 2\underline{\beta}) + (\beta + \underline{\beta})^2 - \beta - \underline{\beta}).$$

Note that

$$-2\beta - 2\underline{\beta} = 0 \mod 4, (\beta + \underline{\beta})^2 = 0 \mod 4, -\beta - \underline{\beta} = -\beta(\beta + 1) \mod 4$$

hence $H = -\beta(\beta+1) \mod 2$. Thus,

$$\begin{aligned} \epsilon'(B) &= \sum_{s \in S - \{t\}} (1/2) g_s(B)(g_s(B)+1)\bar{e}_s + \sum_{s \in S-\{t\}} (1/2) g_t(B)(g_t(B)+1)\bar{e}_s \\ &= \sum_{s \in S} (1/2) g_s(B)(g_s(B)+1)\bar{e}_s. \end{aligned}$$

We have used that $\sum_{s \in S} \bar{e}_s = 0$. This proves (d).

From (d) and 2.3(a) we see that 1.2(ii),(iii) hold in our case with $\bar{V}_0 = \bar{V}$; moreover, ϵ' in 2.3 is the same as ϵ in 1.2.

2.5. From the results in [2],[3] it is known that if B, B' in $\phi(\bar{V})$ satisfy $\epsilon'(B') \in L_B$ (that is, $\epsilon(B') \in L_B$), then $f_s(B') \le f_s(B)$ for any $s \in S - \{t\}$ and $|B^t_{L'} \cap \mathcal{I}^0| \le |B^t_L \cap \mathcal{I}^0|$. (Notation of 2.1 with $L = L_B, L' = L_{B'}$.) We show that

(a) $g_s(B') \le g_s(B)$ *for any* $s \in S$.

When $s = t$ this follows from 2.4(b). We now assume that $s \ne t$. Using 2.4(c) we have

$$g_s(B') + g_t(B') + \underline{g_t(B')} \le g_s(B) + g_t(B) + \underline{g_t(B)}$$

hence it is enough to show that

(b) $g_t(B) - g_t(B') + \underline{g_t(B)} - \underline{g_t(B')} \ge 0$.

If $g_t(B') = g_t(B)$, then (b) is obvious. Assume now that $g_t(B') \ne g_t(B)$. As we have seen above, we have $g_t(B') \le g_t(B)$ hence $g_t(B) - g_t(B') \ge 1$. We have $\underline{g_t(B)} - \underline{g_t(B')} \in \{0, 1, -1\}$, hence (b) holds. This proves (a).

We see that 1.2(iv) holds in our case. Thus Theorem 1.4 is proved in case 1.3(c).

In the remainder of this paper we write $\bar{\epsilon}$ instead of $\epsilon : \phi(\bar{V}) \to \bar{V}$ to distinguish it from ϵ in cases 1.3(a),(b).

2.6. We note:

(a) *If* $B \in \phi(\bar{V})$, *then* $\text{supp}(B) \ne S$.

This holds since B has property (P_0).

2.7. For $t \in S$ let $V(t)$ be the F-subspace of V with basis $\{e_s; s \in S - \{t\}\}$. Then $V(t)$ with this basis and the restriction of $<,>$ is as in 1.3(a). Let $R(t)$ be the analogue of R when V in 1.3(a) is replaced by $V(t)$; we have $R(t) \subset R$. Then $\phi(V(t))$ (a collection of elements of $R(t)$) is defined. From the definition we have $\phi(V(t)) \subset \phi(\bar{V})$. Now let $B \in \phi(\bar{V})$. By 2.6(a) we can find $t \in S$ such that $\text{supp}(B) \subset S - \{t\}$. Now B satisfies $(P_0), (P_1)$ relative to $V(t)$. Hence we have $B \in \phi(V(t))$. We see that

(a) $\phi(\bar{V}) = \cup_{t \in S} \phi(V(t))$.

From the definitions we see that for any $t \in S$ the following diagram is commutative:

$$\begin{array}{ccc} \phi(V(t)) & \longrightarrow & \phi(\bar{V}) \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \bar{\epsilon}} \\ V(t)_0 & \longrightarrow & \bar{V} \end{array}$$

Here the left vertical maps are as in 1.2; the horizontal maps are the obvious inclusions.

2.8. We wish to compare the approach to $\phi(\bar{V})$ given in this paper with that in [4]. Let $S' = \mathfrak{E}$. We can regard S' as a set of vertices of a graph in which $\{s_1, s_2\} \in \mathfrak{E}, \{s_3, s_4\} \in \mathfrak{E}$ are joined whenever $|\{s_1, s_2\} \cap \{s_3, s_4\}| = 1$. Thus the set $\mathfrak{E}'$ of edges of this graph is in obvious bijection with S. Note that the graph $(S', \mathfrak{E}')$ is isomorphic to $(S, \mathfrak{E})$ hence the analogues $\bar{V}', \mathcal{I}'^1, \phi(\bar{V}')$ of $\bar{V}, \mathcal{I}^1, \phi(\bar{V})$ when $(S, \mathfrak{E})$ is replaced by $(S', \mathfrak{E}')$ are defined. We can view $\bar{V}'$ as the F-vector space consisting of all subsets of S of even cardinal in which the sum of X, X' is $(X \cup X') - (X \cap X')$, which is endowed with the symplectic form $X, X' \mapsto |X \cap X'| \mod 2$ and with a circular basis consisting of all two elements subsets of S which are in $\mathfrak{E}$. This circular basis is therefore indexed by S'. Now an object of $\mathcal{I}'^1$ is a subgraph of type A_{2k+1} $(k \ge 0)$ of S', that is with vertices of the form $\{s_1, s_2\}, \{s_2, s_3\}, \dots, \{s_{2k+1}, s_{2k+2}\}$; this is the same as a graph of type A_{2k+2} of S (with vertices $s_1, s_2, \dots, s_{2k+2}$) and is completely determined by the pair of (distinct) elements s_1, s_{2k+2}. Thus $\mathcal{I}'^1$ can be identified with the set of two element subsets of S. In this way $\mathcal{I}'^1$ appears as a subset of $\bar{V}'$ and each X in $\mathcal{I}'^1$ determines a subgraph of type A_{2k+2} $(k \ge 0)$ of S; the set of vertices of this subgraph is denoted by $\underline{X}$. (We have $\underline{X} \subset \bar{V}'$ and $X \subset \underline{X}$.)

Now $\phi(\bar{V}')$ becomes the set of all unordered pairs $X_1, X_2, \dots, X_k$ of two element subsets of S such that $X_i \cap X_j = \emptyset$ for $i \ne j$ and such that for any $i \in \{1, 2, \dots, k\}$ there exists $j_1 < j_2 < \dots < j_s$ in $\{1, 2, \dots, k\}$ such that

$$\underline{X_i} - X_i = \underline{X_{j_1}} \sqcup \underline{X_{j_2}} \sqcup \dots \sqcup \underline{X_{j_s}}.$$

This approach appears in [4] (in a less symmetric and more complicated way) where S is taken to be $S_N = \{1, 2, \dots, N\}$ with $\mathfrak{E}$ consisting of $\{1, 2\}, \{2, 3\}, \dots, \{N-1, N\}, \{N, 1\}$.

The set $\mathcal{X}_{N-1}$ defined in [4, 1.3] is the same as $\phi(\bar{V}')$ although its definition is less symmetric and more complicated. Hence it is the same as $\phi(\bar{V})$ if $\bar{V}, \bar{V}'$ are identified by $\bar{e}_s \mapsto \{s, s+1\}$ if $s \in \{1, 2, \dots, N-1\}$ and $\bar{e}_N \mapsto \{N, 1\}$.

3 The case 1.3(b)

3.1. In this section we assume that we are in the setup of 1.3(b). Let V_0 be the set of all vectors of V which are of the form e_I with $I \subset S, I \ne \emptyset, I \ne S$ such that $|c(I) \cap \mathcal{I}^0|$ is even (here $c(I) \subset \mathcal{I}$ is as in 1.4); let V_1 be the set of all vectors of V which are of the form e_S or e_I with $I \subset S, I \ne \emptyset, I \ne S$ such that $|c(I) \cap \mathcal{I}^0|$ is odd. We have clearly:

(a) $V = V_0 \sqcup V_1$.

We show:

(b) *If $I \subset S, I \ne \emptyset, I \ne S$, then $e_I \in V_0$ if and only if $e_{S-I} \in V_1$. In particular, $x \mapsto x + e_S$ is a bijection $V_0 \xrightarrow{\sim} V_1$.*

We have $c(I) = \{I_1, I_3, \dots, I_{2r-1}\}$, $c(S-I) = \{I_2, I_4, \dots, I_{2r}\}$ and (if $r > 1$) we have $I_1 \cup I_2 \in \mathcal{I}$, $I_2 \cup I_3 \in \mathcal{I}$, $\dots$, $I_{2r-1} \cup I_{2r} \in \mathcal{I}$, $I_{2r} \cup I_1 \in \mathcal{I}$; in particular, we have $|c(I)| = |c(S-I)|$. (This remains true also when $r = 1$.) Hence, setting $c^0(I) = c(I) \cap \mathcal{I}^0$, $c^1(I) = c(I) \cap \mathcal{I}^1$, we have

$$|c^0(I)| - |c^0(S-I)| = -|c^1(I)| + |c^1(S-I)|.$$

Modulo 2 this equals

$$\begin{aligned}|c^1(I)|+|c^1(S-X)| &= \sum_{I'\in c^1(I)}|I'|+\sum_{I'\in c^1(S-I)}|I'| \\ &= \sum_{I'\in c^1(I)}|I'|+\sum_{I'\in c^1(S-I)}|I'|+\sum_{I'\in c^0(I)}|I'|+\sum_{I'\in c^0(S-I)}|I'| \\ &= \sum_{I'\in c(I)}|I'|+\sum_{I\in c(S-I)}|I'|=|I|+|S-I|=|S|.\end{aligned}$$

Since $|S|$ is odd, we see that

(c) $|c^0(I)|-|c^0(S-I)|=1 \mod 2$

so that (b) holds.

We show:

(d) *Let $\pi_0 : V_0 \to \bar{V}$ be the restriction of $\pi : V \to \bar{V}$. Then π_0 is a bijection.*

Assume that $v \neq v'$ in V_0 satisfy $\pi(v)=\pi(v')$. If $v=0$, then $v'\in\pi^{-1}(0)-\{0\}$ hence $v'=e_S$. But $e_S\notin V_0$, a contradiction. If $v\neq 0$, then $v=e_I, v'=e_{S-I}$ with $I\subset S, I\neq\emptyset, I\neq S$. Now $|c^0(I)|$ is even, $|c^0(S-I)|$ is even; but the sum of these numbers is odd by (c), a contradiction. We see that π_0 is injective.

From (b) we see that $|V_0|=|V_1|$ so that both of these numbers are equal to $(1/2)|V|=2^{N-1}$. We see that π_0 is an injective map between two finite sets with 2^{N-1} elements; hence it is a bijection. This proves (d).

3.2. Note that the sets $R,\mathcal{I}$ for this V and for $\bar{V}$ in 1.3(c) are the same. Hence we have $\phi(V)=\phi(\bar{V})$. For $B\in\phi(V)$ we denote by M_B (resp. L_B) the subspace of V (resp. $\bar{V}$) generated by $\{e_I; I\in B\}$ (resp. $\{\bar{e}_I; I\in B\}$). Since $\{\bar{e}_I; I\in B\}$ is a basis of L_B, we see that $\{e_I; I\in B\}$ is a basis of M_B and that π restricts to an isomorphism $M_B\xrightarrow{\sim}L_B$. If $I\in\mathcal{I}$ is such that $e_I\in M_B$, then $\bar{e}_I=\pi(e_I)\in L_B$ and by 1.2(i) for $\bar{V}$ we have $I\in B$. We see that $\phi(V)$ satisfies 1.2(i).

For $B\in\phi(V)$ we show:

(a) *We have $M_B\subset V_0$ (notation of 3.1). Moreover, $\pi^{-1}(L_B)=M_B\oplus Fe_S$.*

By 2.7(a) we can find $t\in S$ such that $B\in\phi(V(t))$. By 1.5(a) the subspace of V (or $V(t)$) spanned by $\{e_I; I\in B\}$ is contained in $V(t)_0$. Thus, $M_B\subset V(t)_0$.

Let $x\in M_B$. We have $x\in V(t)_0$; since $e_S\notin V(t)$ we have $x=e_I$ for some $I\subset S$, $I\neq S$. By the definition of $V(t)_0$ we have $|c(I)^{0+}|=|c(I)^{0-}|$ (see 1.5) so that $|c^0(I)|=|c(I)^{0+}|+|c(I)^{0-}|$ is even and $e_I\in V_0$. Thus $x\in V_0$. This proves the first assertion of (a). For the second assertion we note that M_B is a hyperplane in $\pi^{-1}(L_B)$ and that $e_S\in\pi^{-1}(L_B)$. It remains to note that $e_S\notin M_B$ (since $e_S\notin V(t)$).

3.3. Consider the map $\epsilon:\phi(V)\to V$ in 1.2(ii). For $B\in\phi(V)$ we show:

(a) *We have $\epsilon(B)\in M_B$. In particular we have $\epsilon(B)\in V_0$.*

(See 3.2(a).) As in the proof of 3.2(a) we can assume that $B\in\phi(V(t))$ where $t\in S$. Using the commutative diagram in 2.7 we are reduced to property 1.2(ii) for $V(t)$ which is already known.

We show:

(b) *The map $\epsilon:\phi(V)\to V$ restricts to a bijection $\phi(V)\xrightarrow{\sim}V_0$.*

The composition $\pi\epsilon:\phi(V)\to\bar{V}$ is the same as the map ϵ for $\bar{V}$ hence is a bijection. It follows that $\epsilon : \phi(V) \to V$ is injective and its image has exactly 2^{N-1} elements. Since this image is contained in V_0 (see (a)) and $|V_0| = 2^{N-1}$, we see that (b) holds.

We show:

(c) $V_0 = \cup_{B\in\phi(V)} M_B$

The right hand side is contained in the left hand side by 3.2(a). Now let $x \in V_0$. By [2] we have $\bar{V} = \cup_{L\in\mathcal{F}(\bar{V})} L$. Thus, we have $\pi(x) \in L_B$ for some $B \in \phi(V)$. It follows that we have $x \in \pi^{-1}(L_B) = M_B \oplus Fe_S$. It is enough to show that $x \in M_B$. If $x \notin M_B$, then $x + e_S \in M_B$ so that by (a) we have $x + e_S \in V_0$. Using 3.1(b) we then have $x \in V_1$, contradicting $x \in V_0$. This proves (c).

We see that $\phi(V)$ satisfies 1.2(ii),(iii).

Now let B, B' in $\phi(V)$ be such that $\epsilon(B') \in M_B$. Applying π we see that $\pi\epsilon(B') \in L_B$. Note that $\pi\epsilon$ is the same as ϵ relative to $\bar{V}$. Since $\phi(\bar{V})$ satisfies 1.2(iv), we see that $g_s(B') \le g_s(B)$ for any $s \in S$. (The function g_s is the same for V as for $\bar{V}$.) Thus, 1.2(iv) holds for $\phi(V)$. This completes the proof of Theorem 1.4.

3.4. Let $B \in \phi(V) = \phi(\bar{V})$ be such that $B \ne \emptyset$. Then $\mathrm{supp}(B) \ne \emptyset$ and by 2.6 we have $\mathrm{supp}(B) \ne S$ hence the subset $c(\mathrm{supp}B)$ of $\mathcal{I}$ is defined as in 1.3. As in the proof of 3.1(b) we have $c(\mathrm{supp}(B)) = \{I_1, I_3, \dots, I_{2r-1}\}$, $c(S-\mathrm{supp}(B)) = \{I_2, I_4, \dots, I_{2r}\}$ for some $r \ge 1$. Since $e_{I_1\cup I_3\cup\dots I_{2r-1}} \in V_0$, from 3.1(b) we see that $e_{I_2\cup I_4\cup\dots I_{2r}} \in V_1$, so that

(a) $|I_k|$ *is even for some* $k \in \{2, 4, \dots, 2r\}$. *In particular there exist* s, s' *in* S *such that* $\{s, s'\} \in \mathfrak{E}$ *and* $\mathrm{supp}(B) \cap \{s, s'\} = \emptyset$.

We show:

(b) $|B| \le (|S| - 1)/2$.

A proof identical to that of [2, 1.3(g)] shows:

(c) *If* $I \in B$ *then* $|\{I' \in B; I' \subset I\}| = (|I| + 1)/2$.

Using (c) we have

$$\begin{aligned} |B| &= \sum_{I\in c(\mathrm{supp}(B)} = \sum_{I\in\chi(\mathrm{supp}B} |\{I' \in B; I' \subset I\}| \\ &\le \sum_{I\in\chi(\mathrm{supp}B} (|I| + 1)/2 = (|I_1| + 1)/2 + (|I_3| + 1)/2 + \dots + (|I_{2r-1}| + 1)/2 \\ &= (|I_1| + |I_3| + \dots + |I_{2r-1}| + r)/2 = (|S| - |I_2| - |I_4| - \dots - |I_{2r}| + r)/2 \le |S|/2. \end{aligned}$$

Thus $|B| \le |S|/2$. Since $|B| \in \mathbf{N}$ and $|S|$ is odd we see that (b) holds.

We show:

(d) *We have* $|B| = (|S| - 1)/2$ *if and only if we have* $|I_k| = 1$ *for all* $k \in \{2, 4, \dots, 2r\}$ *except for a single value of* k *for which* $|I_k| = 2$.

Assume first that $|B| = (|S|-1)/2$. The proof of (c) shows that in our case $(|S|-|I_2|-|I_4|-\dots-|I_{2r}|+r)/2$ is equal to $(|S|-1)/2$ or to $|S|/2$, hence $(|I_2|-1)+(|I_4|-1)+\dots+(|I_{2r}|-1)$ is equal to 1 or 0. Thus either (d) holds or else we have $|I_k| = 1$ for all $k \in \{2, 4, \dots, 2r\}$ without exception. This last possibility is excluded by (a). This proves one implication of (d). The reverse implication follows from the proof of (c).

3.5. Let $\mathbf{e}$ be a two element subset of S such that $\mathbf{e} \in \mathfrak{E}$. Let $[\mathbf{e}] = \bar{e}_{(S-\mathbf{e})^{odd}} \in \bar{V}$. We define a linear function $z_{\mathbf{e}} : \bar{V} \to F$ by $z_{\mathbf{e}}(\bar{e}_s) = 1$ if $s \in \mathbf{e}$, $z_{\mathbf{e}}(\bar{e}_s) = 0$ if $s \in S - \mathbf{e}$. Note that the radical of $<,>|_{z_{\mathbf{e}}^{-1}(0)}$ is $F[\mathbf{e}]$.

Let $B \in \phi(\bar{V})$. The following result is used in [4, 3.5].

(a) *If* $[\mathbf{e}] \in L_B$ *then* $\mathrm{supp}(B) \cap \mathbf{e} = \emptyset$ *and* $|B| = (|S|-1)/2$.
Let $B^* \in \phi(\bar{V})$ be the subset of R consisting of the various $\{s\}$ with $s \in (S-\mathbf{e})^{odd}$. We have $[\mathbf{e}] = \epsilon(B^*)$ so that $B^* \le B$. Using 1.2(iv), we see that $g_s(B^*) \le g_s(B)$ for all $s \in S$. It follows that $g_s(B) \ge 1$ for all $s \in (S-\mathbf{e})^{odd}$. Thus $(S-\mathbf{e})^{odd} \subset \mathrm{supp}(B)$.

Let $\{I_{i_1}, I_{i_2}, \dots, I_{i_l}\}$ be the subset of $\{I_2, I_4, \dots, I_{2r}\}$ consisting of those I_k (k even) such that $|I_k| \ge 2$. This subset is nonempty by 3.4(a). Let $I \in \{I_{i_1}, I_{i_2}, \dots, I_{i_l}\}$. We have $I \cap \mathrm{supp}(B) = \emptyset$ hence $I \cap (S-\mathbf{e})^{odd} = \emptyset$. If $I \ne \mathbf{e}$ then, since $|I| \in \{2,4,6,\dots\}$ we have $I \cap (S-\mathbf{e})^{odd} \ne \emptyset$, a contradiction. Thus, $I = \mathbf{e}$. We see that $\mathbf{e} \cap \mathrm{supp}(B) = \emptyset$ that is $\mathrm{supp}(B) \subset S - \mathbf{e}$. Moreover, $\{I_{i_1}, I_{i_2}, \dots, I_{i_l}\}$ consists of a single object namely $\mathbf{e}$. It remains to use 3.4(d).

Conversely,

(b) *If* $\mathrm{supp}(B) \cap \mathbf{e} = \emptyset$ *and* $|B| = (|S|-1)/2$*, then* $[\mathbf{e}] \in L_B$.
Note that L_B is an isotropic subspace of $\zeta_{\mathbf{e}}^{-1}(0)$ and in fact a maximal one since $\dim(L_B) = (\dim(\zeta_{\mathbf{e}}^{-1}(0))+1)/2$. But any maximal isotropic subspace of $\zeta_{\mathbf{e}}^{-1}(0)$ must contain the radical $F[\mathbf{e}]$. Thus, (b) holds.

4 Complements

4.1. In this subsection we assume that $(V, <>, e : S \to V)$ is as in 1.3(a), but the condition that $N \in \{3,5,7,\dots\}$ is replaced by the condition that $N \in \{4,6,8,\dots\}$. From the results in [1] one can deduce that $(V, <>, e : S \to V)$ is still perfect with V_0 having the same description as in 1.5. Let S' be a subset of S such that $S' \in \mathcal{I}$, $|S'| = |S|-1$. Let V' be the subspace of V spanned by $\{e_s; s \in S'\}$. Then V' with the restriction of $<,>$ to V' and with $S' \to V'$, $s \mapsto e_s$ is as in 1.3(a) so that $\phi(V')$ and the image V'_0 of $\epsilon : \phi(V') \to V'$ is defined. Let $S^{odd} \subset S$ be as in 1.1. (This is defined since $S \in \mathcal{I}^1$.) Note that the radical of $<,>$ on V is $Fe_{S^{odd}}$. One can show that

(a) $V_0 = V'_0 \sqcup (V'_0 + e_{S^{odd}})$.
Hence there is a unique fixed point free involution $B \mapsto B'$ of $\phi(V)$ such that $\epsilon(B') = \epsilon(B) + e_{S^{odd}}$ for all $B \in \phi(V)$.

4.2. In this subsection we assume that $(V, <>, e : S \to V)$ is as in 1.3(b); we preserve the notation of Section 3.

Let $\mathcal{F}(V)$ (resp. $\mathcal{F}^1(V)$) be the collection of subspaces of V of the form M_B (resp. $M_B \oplus Fe_S$) for various $B \in \phi(V)$. Let $\tilde{\mathcal{F}}(V) = \mathcal{F}(V) \sqcup \mathcal{F}^1(V)$. We show that $\tilde{\mathcal{F}}(V)$ has properties similar to those of $\mathcal{F}(V)$. We define $\tilde{\epsilon} : \tilde{\mathcal{F}}(V) \to V$ by $\tilde{\epsilon}(M_B) = \epsilon(B)$, $\tilde{\epsilon}(M_B \oplus Fe_S) = \epsilon(B) + e_S$. Note for any $X \in \tilde{\mathcal{F}}(V)$ we have $\tilde{\epsilon}(X) \in X$. (This is similar to 1.2(ii).)

Now $\tilde{\epsilon}$ restricts to the bijection $\mathcal{F}(V) \xrightarrow{\sim} V_0$, $M_B \mapsto \epsilon(B)$ and to the bijection $\mathcal{F}^1(V) \to V_1$, $M_B \oplus Fe_S \mapsto \epsilon(B) + e_S$ (recall the bijection $x \mapsto x + e_S$, $V_0 \xrightarrow{\sim} V_1$). Hence $\tilde{\epsilon}$ is a bijection. (This is similar to 1.2(iii).)

For X, X' in $\tilde{\mathcal{F}}(V)$ we say that $X' \le X$ if one of the following holds:

$X = M_B, X' = M_{B'}$ and $B' \le B$ in the partial order 1.2(a) on $\phi(V)$;

$X = M_B \oplus Fe_S, X' = M_{B'} \oplus Fe_S$ and $B' \le B$ in the partial order 1.2(a) on $\phi(V)$;

$X = M_B \oplus Fe_S, X' = M_{B'}$ and $B' \le B$ in the partial order 1.2(a) on $\phi(V)$.

This is a partial order on $\tilde{\mathcal{F}}(V)$. (This is similar to 1.2(iv).)

4.3. In this subsection we assume that $(V, <>, e : S \to V)$ (as in 1.1) is perfect. Let $B \in \phi(V)$. We will give an alternative formula for $\bar{\epsilon}(B)$.

We define a partition $B = B_1 \sqcup B_2 \sqcup B_3 \sqcup \dots$ as follows.

B_1 is the set of all $I \in B$ such that I is not properly contained in any $I' \in B$. Now B_2 is the set of all $I \in B - B_1$ such that I is not properly contained in any $I' \in B - B_1$. Now B_3 is the set of all $I \in B - (B_1 \cup B_2)$ such that I is not properly contained in any $I' \in B - (B_1 \cup B_2)$, etc.

For $k \ge 1$ we set

$v_k(B) = \sum_{I \in B_k} e_I \in V$.

We have

(a) $\bar{\epsilon}(B) = v_1(B) + v_3(B) + v_5(B) + \dots$.

Let $s \in S$. There is a unique sequence $I_1 \in B_1, I_2 \in B_2, \dots, I_l \in B_l$ such that $s \in I_l \subset I_{l-1} \subset \dots \subset I_1$ and $s \notin \cup_{I \in B_{l+1}} I$. The coefficient of e_s in $v_1(B) + v_3(B) + v_5(B) + \dots$ is 0 if $l = 0 \mod 4$; is 1 if $l = 1 \mod 4$; is 1 if $l = 2 \mod 4$; is 0 if $l = 3 \mod 4$. We have $g_s(B) = l$. Note that $(1/2)l(l+1) \mod 2$ is 0 if $l = 0 \mod 4$; is 1 if $l = 1 \mod 4$; is 1 if $l = 2 \mod 4$; is 0 if $l = 3 \mod 4$. This proves (a).

4.4. In this subsection we are in the setup of 2.1. Let $\bar{V}^{\mathbf{C}}$ be the $\mathbf{C}$-vector space of functions $\bar{V} \to \mathbf{C}$. For any $x \in \bar{V}$ let $f_x \in \bar{V}^{\mathbf{C}}$ be the function which takes value 1 on the subspace $L_{\bar{\epsilon}^{-1}(x)}$ of $\bar{V}$ and the value 0 on the complement of that subspace; let $f'_x \in \bar{V}^{\mathbf{C}}$ be the function which takes value 1 on the subspace $\{x' \in \bar{V}; < x', L_{\bar{\epsilon}^{-1}(x)} >= 0\}$ of $\bar{V}$ and the value 0 on the complement of that subspace. From Theorem 1.4 we see that for $x \in \bar{V}$ we have $f'_x = \sum_{y \in \bar{V}} c_{y,x} f_y$ where $c_{y,x} \in \mathbf{Z}$. Moreover, from the triangularity of Fourier transform [3] we see that $c_{y,x} = 0$ unless $x = y$ or $\dim L_{\bar{\epsilon}^{-1}(x)} < \dim L_{\bar{\epsilon}^{-1}(y)}$ and that $c_{x,x} = \pm 2^k$ for some $k \in \mathbf{N}$. We conjecture that

(a) for any x, y in $\bar{V}$, we have either $c_{y,x} = 0$ or $c_{y,x} = \pm 2^k$ for some $k \in \mathbf{N}$.

The dihedral group Di_{2N} of order $2N$ acts naturally on $\bar{V}$; see 1.3. Let Z_N be a set of representatives for the Di_{2N}-orbits. Assume for example that $x = 0$. Then $y \mapsto c_{y,0}$ is constant on each Di_N-orbit. We describe this function assuming that $S = S_N$ (see 2.8) and $N = 7$. We can take

(b) $\{1245\}, \{12345\}, \{1235\}, \{135\}, \{123\}, \{14\}, \{13\}, \{1\}, \{\emptyset\}$

where we write $i_1 i_2 \dots i_m$ instead of $\bar{e}_{i_1} + \bar{e}_{i_2} + \dots + \bar{e}_{i_m}$. The value of $y \mapsto c_{y,0}$ at the 9 elements in (b) (in the order written) is

$1, 0, 1, -1, -1, 0, 1, -2, 8$.

5 The set $\omega(\bar{V})$

5.1. In this section we assume that $(\bar{V}, <>, \pi e : S \to \bar{V})$ is as in 1.3(c). We fix a two element subset $\mathbf{e}$ of S such that $\mathbf{e} \in \mathfrak{E}$.

5.2. For $B \in R$ we set

$$n_B = |\{I \in B; \mathbf{e} \subset I\}| \in \mathbf{N}.$$

Let $\phi(\bar{V})^{\mathbf{e}} = \{B \in \phi(\bar{V}); \mathrm{supp}(B) \cap \mathbf{e} \neq \emptyset\}$.

If $B \in \phi(\bar{V})^{\mathbf{e}}$ (in particular if $n_B > 0$), then using $(P_0), (P_1)$, we see that there is a unique $I_B \in B$ such that $|I_B \cap \mathbf{e}| = 1$.

We have $\phi(\bar{V})^{\mathbf{e}} = \sqcup_{\tau \in \mathbf{e}} \phi(\bar{V})^{\tau}$ where $\phi(\bar{V})^{\tau} = \{B \in \phi(\bar{V})^{\mathbf{e}}; \tau \in I_B\}$.

For $B \in \phi(\bar{V})$ we define $B^! \in R$ by
$B^! = B - \{I_B\}$ if $n_B \in \{1, 3, 5, \dots\}$
$B^! = B$ if $n_B \in \{0, 2, 4, \}$.
Note that for $B \in \phi(\bar{V})$ we have $n_{B^!} = n_B$. We show:
(a) *If* $B \in \phi(\bar{V})$, $B' \in \phi(\bar{V})$ *satisfy* $B^! = B'^!$, *then* $B = B'$.
If n_B is odd, then from the definition we see that $B^!$ does not satisfy (P_1). Hence to prove (a) we can assume that both n_B and $n_{B'}$ are odd.

There is a unique $I \in B^! = B'^!$ such that $\mathbf{e} \subset I$ and such that any $I' \in B^! = B'^!$ with $I' \prec I$ satisfies $\mathbf{e} \cap I = \emptyset$. We have $I \in B, I \in B'$. Let $I_1, I_2, \dots, I_k$ (resp. $I'_1, I'_2, \dots, I'_l$) be defined in terms of I as in (P_1) for B (resp. B'). We can assume that $I_B = I_1$ (resp. $I_{B'} = I'_1$) and $I_2, I_3, \dots, I_k$ (resp. $I'_2, I'_3, \dots, I'_l$) are the maximal objects of $B^!$ (resp. $B'^!$) that are strictly contained in I. Hence $\{I_2, I_3, \dots, I_k\} = \{I'_2, I'_3, \dots, I'_l\}$. Note that I_1 is the unique object of $\mathcal{I}^1$ such that $I_1 \spadesuit I_j$ for $j > 1$ and $I^{ev} \subset I_1 \sqcup I_2 \sqcup \dots \sqcup I_k$; similarly I'_1 is the unique object of $\mathcal{I}^1$ such that $I'_1 \spadesuit I'_j$ for $j > 1$ (that is $I'_1 \spadesuit I_j$ for $j > 1$) and $I^{ev} \subset I'_1 \sqcup I'_2 \sqcup \dots \sqcup I'_l$ (that is $I^{ev} \subset I'_1 \sqcup I_2 \sqcup \dots \sqcup I_k$). It follows that $I_1 = I'_1$ so that $B = B'$. This proves (a).

Let
$$\omega(\bar{V}) = \{B^!; B \in \phi(\bar{V})\} \subset R.$$
From (a) we see that
(b) $B \mapsto B^!$ *defines a bijection* $\phi(\bar{V}) \xrightarrow{\sim} \omega(\bar{V})$.
For any $B \in \omega(\bar{V})$ we define $\tilde{B} \in \phi(\bar{V})$ by $B = \tilde{B}^!$.
There is a unique bijection $'\epsilon : \omega(\bar{V}) \xrightarrow{\sim} \bar{V}$ such that $'\epsilon(B) = \bar{\epsilon}(\tilde{B})$ for any $B \in \omega(\bar{V})$.

There is a unique involution $\iota : S \to S$ preserving the graph structure and interchanging the two elements of $\mathbf{e}$. It induces an involution on R denoted again by ι which leaves stable $\phi(\bar{V})$ and $\omega(\bar{V})$.

5.3. We now assume that instead of specifying an element $\mathbf{e}$ of $\mathfrak{E}$ we specify an element $\mathbf{e}' \in \mathfrak{E}'$ (see 2.8) that is a pair $\{s_1, s\}, \{s_2, s\}$ of two distict two edges of S whose intersection is $\{s\}$ for some $s \in S$. In terms of $\mathbf{e}'$ we have a function $(X_1, X_2, \dots, X_k) \mapsto n_{X_1, X_2, \dots, X_k}$ from $\phi(\bar{V}')$ (see 2.8) to $\mathbf{N}$ defined in a way analogous to the way $B \mapsto n_B$ from $\phi(\bar{V})$ to $\mathbf{N}$ was defined in terms of $\mathbf{e}$. We have
$$n_{X_1, X_2, \dots, X_k} = |\{i \in \{1, 2, \dots, k\}, s \subset \underline{X_i} - X_i\}|.$$
The analogue of the assignment $B \mapsto I_B$ for $B \in \phi(\bar{V})$ such that $n_B > 0$ is the assignment
$$\{X_1, X_2, \dots, X_k\} \mapsto I_{\{X_1, X_2, \dots, X_k\}} = X$$
for any $\{X_1, X_2, \dots, X_k\} \in \phi(\bar{V}')$ such that $n_{X_1, X_2, \dots, X_k} > 0$; here X is the unique X_i such that $s \in X_i$. Then $\omega(\bar{V}')$ is defined in terms of s in the same way as $\omega(\bar{V})$ was defined in terms in terms of $\mathbf{e}$. Namely $\omega(\bar{V}')$ consists of the sequences obtained from various sequences $\{X_1, X_2, \dots, X_k\} \in \phi(\bar{V}')$ by removing $X = I_{\{X_1, X_2, \dots, X_k\}}$ whenever X is defined and by not removing anything whenever X is not defined.

This approach appears in [4] (in a less symmetric and more complicated way) where $S = S_N$ as in 2.8. The set $\mathcal{X}_{N-2}$ defined in [4, 1.3] is the same as $\omega(\bar{V})$ if $\bar{V}, \bar{V}'$ are identified as in 2.8 and if $\mathbf{e}$ is taken to be $\{N-1, N\}$ so that $s = N$.

Hence $\omega(\bar{V})$ is closely related to the theory of unipotent representations of even orthogonal groups over a finite field in the same way as $\phi(\bar{V})$ is closely related to the theory of unipotent representations of symplectic groups over a finite field.

5.4. For $B \in \omega(\bar{V})$ we denote by $< B >$ the subspace of $\bar{V}$ spanned by $\{\bar{e}_I; I \in B\}$.

For B', B in $\omega(\bar{V})$ we write $B' \preceq B$ if there exists a sequence

$$B' = B_0, B_1, B_2, \dots, B_k = B$$

such that

(a) $$'\epsilon(B_0) \in < B_1 >, '\epsilon(B_1) \in < B_2 >, \dots, '\epsilon(B_{k-1}) \in < B_k > .$$

We show:

(b) $\preceq$ *is a partial order on* $\omega(\bar{V})$.

In the setup of (a), for $i = 0, 1, \dots, k$ we have $< B_i > \subset L_{\tilde{B}_i}$ hence $\bar{\epsilon}(\tilde{B}_i) = '\epsilon(B_i) \in L_{\tilde{B}_i}$. We see that if $B' \preceq B$ then $\tilde{B}' \le \tilde{B}$ in $\phi(\bar{V})$. It is enough to prove that if $B' \preceq B$ in $\omega(\bar{V})$ and $B \preceq B'$ in $\omega(\bar{V})$ then $B' = B$. We have $\tilde{B}' \le \tilde{B}$ in $\phi(\bar{V})$ and $\tilde{B} \le \tilde{B}'$ in $\phi(\bar{V})$. Since $\le$ is a partial order on $\phi(\bar{V})$ we have $\tilde{B}' = \tilde{B}$. It follows that $B = B'$. This proves (a). (See also [4, 2.10(a)]).

6 The subsets $\omega^+(bV), \omega^-(\bar{V})$ of $\omega(\bar{V})$

6.1. In this section we preserve the setup of 5.1. Let $z_{\mathbf{e}} : \bar{V} \to F$ be as in 3.5. Let $\bar{V}^+ = z_{\mathbf{e}}^{-1}(0), \bar{V}^- = z_{\mathbf{e}}^{-1}(1)$. We set $\omega^+(\bar{V}) = '\epsilon^{-1}(\bar{V}^+), \omega^-(\bar{V}) = '\epsilon^{-1}(\bar{V}^-)$. We have $\omega(\bar{V}) = \omega^+(\bar{V}) \sqcup \omega^-(\bar{V})$ and $'\epsilon$ restricts to bijections $\omega^+(\bar{V}) \to \bar{V}^+, \omega^-(\bar{V}) \to \bar{V}^-$. We show:

(a) *If* $B \in \phi(\bar{V}), n_B = 2k + 1$, *then* $\bar{\epsilon}(B) \in \bar{V}^+$ *so that* $B^! \in \omega^+(\bar{V})$.

By (P_1) we can find $I' \in B$ such that $I' \cap \mathbf{e} = \{\sigma\}$ for some $\sigma \in \mathbf{e}$; let $\sigma' \in \mathbf{e}, \sigma' \ne \sigma$. We then have $g_\sigma(B) = 2k + 2$, $g_{\sigma'}(B) = 2k + 1$. We have

$$\begin{aligned} \bar{\epsilon}_\sigma(B) + \bar{\epsilon}_{\sigma'}(B) &= (1/2)(2k+2)(2k+3) + (1/2)(2k+1)(2k+2) \\ &= (1/2)(2k+2)(4k+4) = 0 \mod 2 \end{aligned}$$

so that $z_{\mathbf{e}}(\bar{\epsilon}(B)) = 0$ that is $\bar{\epsilon}(B) \in \bar{V}^+$.

We show:

(b) *If* $B \in \phi(\bar{V}), n_B = 2k$, $k \ge 1$, *then* $\bar{\epsilon}(B) \in \bar{V}^-$ *so that* $B^! \in \omega^-(\bar{V})$.

By (P_1) we can find $I' \in B$ such that $I' \cap \mathbf{e} = \{\sigma\}$ for some $\sigma \in \mathbf{e}$; let $\sigma' \in \mathbf{e}, \sigma' \ne \sigma$. We then have $g_\sigma(B) = 2k + 1$, $g_{\sigma'}(B) = 2k$. We have

$$\begin{aligned} \bar{\epsilon}_\sigma(B) + \bar{\epsilon}_{\sigma'}(B) &= (1/2)(2k+1)(2k+2) + (1/2)2k(2k+1) \\ &= (1/2)(2k+1)(4k+2) = (2k+1)^2 = 1 \mod 2 \end{aligned}$$

so that $z_{\mathbf{e}}(\bar{\epsilon}(B)) = 1$ that is $\bar{\epsilon}(B) \in \bar{V}^-$. Note that
$\{B \in \omega^+(\bar{V}); n_B = 0\} = \{B \in \phi(\bar{V}); \text{supp}(B) \cap \mathbf{e} = \emptyset\}$,
$\{B \in \omega^-(\bar{V}); n_B = 0\} = \{B \in \phi(\bar{V}); |\text{supp}(B) \cap \mathbf{e}| = 1\}$.

6.2. Let $B' \in \omega(\bar{V})$. We write $B' = B^!$ where $B \in \phi(\bar{V})$.

Assume first that B is as in 6.1(a). Then $B' \in \omega^+(\bar{V})$ and I_B is the only $I \in B$ such that $|I \cap \mathbf{e}| = 1$; since $B^! = B - I_B$ we see that for any $I \in B'$ we have $|I \cap \mathbf{e}| \in \{0, 2\}$.

Assume next that B is as in 6.1(b). Then $B' = B \in \omega^-(\bar{V})$ and I_B satisfies $|I_B \cap \mathbf{e}| = 1$; thus, for some $I \in B'$ we have $|I \cap \mathbf{e}| = 1$,

We now assume that $n_B = 0$. If $\text{supp}(B) \cap \mathbf{e} = \emptyset$, then clearly we have $|I \cap \mathbf{e}| = 0$ for any $I \in B$. If $|\text{supp}(B) \cap \mathbf{e}| = 1$, then clearly we have $|I \cap \mathbf{e}| = 1$ for some $I \in B$.

We see that for $B \in \omega(\bar{V})$ the following holds:

(a) $B \in \omega^+(\bar{V})$ *if and only if* $|I \cap \mathbf{e}| \in \{0, 2\}$ *for any* $I \in B$.

6.3. We show:

(a) *Let* B', B *in* $\omega(\bar{V})$ *be such that* $B' \preceq B$. *If* $B \in \omega^+(\bar{V})$, *then* $B' \in \omega^+(\bar{V})$.

We can assume that $'\epsilon(B') \subset < B >$. (The general case would follow by using several times this special case.) By 6.2(a) we have $|I \cap \mathbf{e}| \in \{0, 2\}$ for any $I \in B$. It follows that any $x \in < B >$ satisfies $z_{\mathbf{e}}(x) = 0$. In particular we have $z_{\mathbf{e}}('\epsilon(B')) = 0$ so that $'\epsilon(B') \in \bar{V}^+) = 0$ and $B' \in \omega^+(\bar{V})$. This proves (a).

7 The sets $\mathcal{F}^+(\bar{\bar{V}})^\tau, \mathcal{F}^-(\bar{\bar{V}})^\tau$

7.1. In this section we preserve the setup of 5.1. For $\tau \in \mathbf{e}$ let $\omega(\bar{V})^\tau = \{B \in \omega(\bar{V}); \tilde{B} \in \phi(\bar{V})^\tau\}$. We have $\omega(\bar{V})^\tau = \omega^+(\bar{V})^\tau \sqcup \omega^-(\bar{V})^\tau$ where for $\delta \in \{+.-\}$ we set$\omega^\delta(\bar{V})\tau = \omega(\bar{V})^\tau \cap \omega^\delta(\bar{V})$.

Under the identification $\omega(\bar{V}) = \omega(\bar{V}')$ in 2.8, 5.3 and with notation of [4, 1.4], the following holds:

If $n \in \{1, 3, 5, \dots\}$, then
$\{B \in \omega^+(\bar{V})^{N-1}, n_B = n\}$ becomes $\mathcal{X}^{t,+}_{N-2}$, $t = -n - 1$;
$\{B \in \omega^+(bV)^N, n_B = n\}$ becomes $\mathcal{X}^{t,+}_{N-2}$, $t = n + 1$;
if $n \in \{0, 2, 4, 6, \dots\}$, then
$\{B \in \omega^-(\bar{V})^{N-1}, n_B = n\}$ becomes $\mathcal{X}^{t,-}_{N-2}$, $t = n$;
$\{B \in \omega^-(\bar{V})^N, n_B = n\}$ becomes $\mathcal{X}^{t,-}_{N-2}$, $t = -n - 2$.

7.2. Let $\tau \in \mathbf{e}$.

(a) *Assume that* $B' \in \omega^+(\bar{V}), B \in \omega^+(\bar{V})^\tau$ *satisfy* $B' \preceq B$ *and* $n_B > 0$. *Then we have either* $n_{B'} = n_B$ *and* $B' \in \omega^+(\bar{V})^\tau$, *or else* $n_{B'} < n_B$.

(b) *Assume that* $B' \in \omega^-(\bar{V}), B \in \omega^-(\bar{V})^\tau$ *satisfy* $B' \preceq B$ *and* $n_B \geq 0$. *Then we have either* $n_{B'} = n_B$ *and* $B' \in \omega^-(\bar{V})^\tau$, *or else* $n_{B'} < n_B$.

Using the identification $\omega(\bar{V}) = \omega(\bar{V}')$ in 2.8, 5.3 and the results in 7.1 we see that when $\tau = N - 1$, (a) follows from [4, 3.2] and (b) follows from [4, 3.4]. Using the symmetry ι, we see that (a) and (b) for $\tau = N$ follow from (a) and (b) for $\tau = N - 1$.

7.3. We choose a subset J of $S - \mathbf{e}$ such that $|J| = N - 3$ and such that when $N > 3$ we have $J \subset \mathcal{I}$.

Let $\omega(\bar{V})_J = \{B \in \omega(\bar{V}); \text{supp}B \subset J\}$. Then $'\epsilon$ defines a bijection of $\omega(\bar{V})_J$ onto a subset $\bar{V}_{J,0}$ of $\bar{V}$. We set

$$\bar{V}_{J,1} = {'\epsilon}(\{B \in \omega(\bar{V}); \text{supp}(B) \cap \mathbf{e} = \emptyset\}) - \bar{V}_{J,0} \subset \bar{V}.$$

Assume now that $B' \in \omega(\bar{V}), B \in \omega(\bar{V})_J$ satisfy $B' \preceq B$. From [4, 3.3] we deduce:

(a) *We have* $B'\in\omega(\bar V)_J$.

7.4. Let $\tau\in\mathbf{e}$. We set $\tilde\omega^+(\bar V)^\tau=\omega^+(\bar V)^\tau\cup\omega(\bar V_J)$ $\tilde\omega^-(\bar V)^\tau=\omega^-(\bar V)^\tau$.

Assume now that $B'\in\omega^\delta(\bar V),B\in\tilde\omega^\delta(\bar V)^\tau$ satisfy $B'\preceq B$. From 7.2(a),(b) and 7.3(a) we deduce:

(a) *We have either* $B'\in\tilde\omega^\delta(\bar V)^\tau$ *and* $n_{B'}=n_B$, *or else* $n_{B'}<n_B$.

7.5. Let $\bar{\bar V}=\bar V/F[\mathbf{e}]$ and let $\bar p:\bar V\to\bar{\bar V}$ be the obvious quotient map. Let $\bar{\bar V}^+=\bar p(\bar V^+)$, $\bar{\bar V}^-=\bar p(\bar V^-)$. We have $[\mathbf{e}]\in\bar V^+$ hence $\bar{\bar V}=\bar{\bar V}^+\sqcup\bar{\bar V}^-$ and $|\bar{\bar V}^+|=(1/2)|\bar V^+|=|\bar{\bar V}^-|$.

Let $\delta\in\{+,\}$. For $n\ge0$, $\tau\in\mathbf{e}$ we set

$$\bar V_n^{\delta,\tau}={}'\epsilon(\{B\in\omega^\delta(\bar V)^\tau;n_B=n\})\subset\bar V^\delta.$$

From the results in [4, 2.7, 3.5] we see that

(a) *the two subsets* $\bar V_n^{\delta,\tau}$ *(with* $\tau\in\mathbf{e}$*) are interchanged by the involution* $x\mapsto x+[\mathbf{e}]$ *of* $\bar V^\delta$*;*

(b) $\bar V_{J,0},\bar V_{J,1}$ *are interchanged by the involution* $x\mapsto x+[\mathbf{e}]$ *of* $\bar V$.

(For (b) see also 4.1(a).)

For $\tau\in\mathbf{e}$ we set

$$H^{\delta,\tau}={}'\epsilon(\tilde\omega^\delta(\bar V)^\tau)\subset\bar V^\delta.$$

We have

$$H^{+,\tau}=\bar V_{J,0}\cup\cup_{n\ge0}\bar V_n^{+,\tau},$$
$$H^{-,\tau}=\cup_{n\ge0}\bar V_n^{-,\tau}$$

From (a),(b) we see that $\bar p$ restricts to bijections $H^{\delta,\tau}\xrightarrow{\sim}\bar{\bar V}^\delta$.

For $y\in\bar{\bar V}^\delta$ we denote by $\tilde y^\tau\in H^{\delta,\tau}$ the inverse image of y under this bijection and we define $\nu_y\in\mathbf{N}$ by:

$\nu_y=n$ if $\tilde y^\tau\in\bar V_n^{\delta,\tau}$,

$\nu_y=0$ if $\delta=+$ and $\tilde y^\tau\in\bar V_{J,0}$.

7.6. Let $\delta\in\{+.-\},\tau\in\mathbf{e}$. For y',y in $\bar{\bar V}^\delta$ we say that $y'\le_\tau y$ if there exists

(a) *a sequence* $y'=y_0,y_1,y_2,\dots,y_k=y$ *in* $\bar{\bar V}^\delta$ *such that for* $i\in\{0,1,\dots,k-1\}$ *we have* $\tilde y_i^\tau\in<{}'\epsilon^{-1}(\tilde y_{i+1}^\tau)>$ *or* $\tilde y_i^\delta+[\mathbf{e}]\in<{}'\epsilon^{-1}(\tilde y_{i+1}^\delta)>$.

We show that in this situation, for any $i\in\{0,1,\dots,k-1\}$ we have

(b) $\nu_{y_i}\le\nu_{y_{i+1}}$.

We set $B_i={}'\epsilon^{-1}(\tilde y_i^\tau)$, $B'_i={}'\epsilon^{-1}(\tilde y_i^\tau+[\mathbf{e}])$, $B_{i+1}={}'\epsilon^{-1}(\tilde y_{i+1}^\tau)$.

If $\tilde y_i^\tau\in<{}'\epsilon^{-1}(\tilde y_{i+1}^\tau)>$, then $B_i\preceq B_{i+1}$ so that by 7.4(a) we have $n_{B_i}\le n_{B_{i+1}}$. But $n_{B_i}=\nu_{y_i}$, $n_{B_{i+1}}=\nu_{y_{i+1}}$, so that (b) holds.

If $\tilde y_i^\tau+[\mathbf{e}]\in<{}'\epsilon^{-1}(\tilde y_{i+1}^\tau)>$, then $B'_i\preceq B_{i+1}$, so that by 7.4(a) we have $n_{B'_i}\le n_{B_{i+1}}$. But $n_{B'_i}=\nu_{y_i}$, $n_{B_{i+1}}=\nu_{y_{i+1}}$, so that (b) holds.

We now see:

(c) *If* $y'\le_\tau y$, *then* $\nu_{y'}\le\nu_y$.

We show:

(d) $\le_\tau$ *is a partial order on* $\bar{\bar V}^\delta$.

For $y\in\bar{\bar V}^\delta$ we have $\tilde y^\tau\in<{}'\epsilon^{-1}(\tilde y^\tau)>$ so that $y\le_\tau y$. It remains to show that

(e) *if* y, y' *in* $\bar{\bar{V}}^{\delta}$ *satisfy* $y \le_\tau y'$ *and* $y' \le_\tau y$, *then* $y = y'$.
Using (c) we have $\nu_{y'} \le \nu_y$ and $\nu_y \le \nu_{y'}$, hence $\nu_y = \nu_{y'}$. Consider now a sequence $y' = y_0, y_1, y_2, \dots, y_k = y$ as in (a). Using (b) and $\nu_y = \nu_{y'}$ we see that for $i \in \{0, 1, \dots, k-1\}$ we have $\nu_{y_i} = \nu_{y_{i+1}}$. Recall that we have either

(i) $B_i \preceq B_{i+1}$, or

(ii) $B'_i \preceq B_{i+1}$,

where as before we set $B_i = {}'\epsilon^{-1}(\tilde{y}_i^\tau)$, $B'_i = {}'\epsilon^{-1}(\tilde{y}_i^\tau + [\mathbf{e}])$, $B_{i+1} = {}'\epsilon^{-1}(\tilde{y}_{i+1}^\tau)$. Note that $n_{B_i} = n_{B'_i} = n_{B_{i+1}}$.

We have $B_i \in \tilde{\omega}^\delta(\bar{V})^\tau$, $B'_i \in \tilde{\omega}^\delta(\bar{V})^{\tau'}$, $B_{i+1} \in \tilde{\omega}^\delta(\bar{V})^\tau$, where $\tau' \in \mathbf{e}$ and $\tau \ne \tau'$. Using 7.4(a), we see that if (ii) holds, then (since $n_{B'_i} = n_{B_{i+1}}$) we would have $\tau = \tau'$, a contradiction. Thus, (i) holds. Using this for $i = 0, 1, \dots, k-1$ we see that

$$B_0 \preceq B_1 \preceq B_2 \le \dots \preceq B_k.$$

In particular we have $B' \preceq B$. Reversing the roles of y, y' we have similarly $B \preceq B'$. Since $\preceq$ is a partial order on $\omega(\bar{V})$, it follows that $B = B'$. Applying ${}'\epsilon$, we obtain $\tilde{y}^\tau = \tilde{y}'^t$ hence $y = y'$. This proves (e) and hence (d).

7.7. Let $\delta \in \{+.-\}, \tau \in \mathbf{e}$. For any $y \in \bar{\bar{V}}^{\delta}$ we set $< y >_\tau := \bar{p}(< {}'\epsilon^{-1}(\tilde{y}^\tau) >)$ (a subspace of $\bar{\bar{V}}$) and $< y >_{\tau,\delta} = < y >_\tau \cap \bar{\bar{V}}^{\delta}$. Note that if $\delta = +$ then $< y >_{\tau,\delta} = < y >_\tau$; if $\delta = -$ then $< y >_{\tau,\delta}$ is the complement in $< y >_\tau$ of a hyperplane of $< y >_\tau$. Now, the condition that

$\tilde{y}_i^\delta \in < {}'\epsilon^{-1}(\tilde{y}_{i+1}^\tau) >$ or $\tilde{y}_i^\delta + [\mathbf{e}] \in < {}'\epsilon^{-1}(\tilde{y}_{i+1}^\delta) >$

(in 7.6(a)) is equivalent to the condition that $y_i \in \bar{p}(< {}'\epsilon^{-1}(\tilde{y}_{i+1}^\tau) >)$. Thus, the condition that y, y' in $\bar{\bar{V}}^{\delta}$ satisfy $y' \le_\tau y$ is equivalent to the following condition:

there exists a sequence $y' = y_0, y_1, y_2, \dots, y_k = y$ in $\bar{\bar{V}}^{\delta}$ such that for $i \in \{0, 1, \dots, k-1\}$ we have $y_i \in < y_{i+1} >_{\tau,\delta}$.

Let $\mathcal{F}^\delta(\bar{\bar{V}})^\tau$ be the collection of subsets of $\bar{\bar{V}}^{\delta}$ of the form $< y >_{\tau,\delta}$ for various $y \in \bar{\bar{V}}^{\delta}$. We show:

(a) *If* y', y *in* $\bar{\bar{V}}^{\delta}$ *satisfy* $< y' >_{\tau,\delta} = < y >_{\tau,\delta}$, *then* $y = y'$.
Indeed, we have $y \in < y >_{\tau,\delta}$, $y' \in < y' >_{\tau,\delta}$, hence $y \in < y' >_{\tau,\delta}$, $y' \in < y >_{\tau,\delta}$, so that $y \le_\tau y'$, $y' \le_\tau y$. Since $\le_\tau$ is a partial order, it follows that $y = y'$, proving (a).

We show:

(b) *The map* $\tilde{\omega}^\delta(\bar{V})^\tau \to \mathcal{F}^\delta(\bar{\bar{V}})^\tau$, ${}'\epsilon^{-1}(\tilde{y}^\tau) \mapsto < y >_{\tau,\delta}$ *(for* $y \in \bar{\bar{V}}^{\delta}$*) is bijective.*

This map is obviously surjective. Moreover we have $|\tilde{\omega}^\delta(\bar{V})^\tau \to \mathcal{F}^\delta(\bar{\bar{V}})^\tau| = |\bar{\bar{V}}^{\delta}|$. It is then enough to show that $|\mathcal{F}^\delta(\bar{\bar{V}})^\tau| = |\bar{\bar{V}}^{\delta}|$. This follows from (a).

We show:

(c) *If* $y \in \bar{\bar{V}}^{\delta}$ *and* $B = {}'\epsilon^{-1}(\tilde{y}^\tau)$ *so that* $< y >_{\tau,\delta} = \pi(< B >)$ *then* $\bar{p}$ *restricts to an isomorphism* $< B > \xrightarrow{\sim} < y >_{\tau,\delta}$.
Indeed it is enough to show that $[\mathbf{e}] \notin < B >$. But in fact we have even $[\mathbf{e}] \notin L_B$ as a consequence of 3.5(a).

7.8. Now the two sets $\mathcal{F}^-(\bar{\bar{V}})^\tau$ (for the two values of $\tau \in \mathbf{e}$) are interchanged by the involution induced by ι; they do not depend on the choice of J in 7.3. This is not so for the two sets $\mathcal{F}^+(\bar{\bar{V}})^\tau$ (for the two values of $\tau \in \mathbf{e}$), at least if $N > 3$; these sets do depend on the choice of J in 7.3. But we prefer one of them over the other; namely we prefer the value of τ such that τ is not joined in our graph to any element of J. (This determines τ uniquely if $N > 3$.) This is the choice made in [4].

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