Bull. Math. Soc. Sci. Math. Roumanie
Tome 67 (115), No. 3, 2024, 305–319

Inclusions of C*-algebras

by
Cornel Pasnicu

Abstract

We introduce three notions of inclusions of C*-algebras: with the ideal property, with the weak ideal property, and with topological dimension zero. We characterize these notions and we show that for an inclusion of C*-algebras, the ideal property $\Rightarrow$ the weak ideal property $\Rightarrow$ topological dimension zero. We prove that any two of these three notions do not coincide in general, but they are all equivalent in many interesting cases. We show some permanence properties for these notions, and we prove that they behave well with respect to tensor products and crossed products by discrete (finite) groups, in many interesting cases. For example, we prove that if $A \subseteq B$ is an inclusion of C*-algebras which has topological dimension zero and $\alpha\colon G \to \mathrm{Aut}(B)$ is a strongly pointwise outer action of a finite group G on B and if A is α-invariant, then the inclusion of crossed products $C^*(G, A, \alpha) \subseteq C^*(G, B, \alpha)$ has topological dimension zero. We show that for an inclusion of C*-algebras, the real rank zero (in the sense of Gabe and Neagu [5]) $\Rightarrow$ the ideal property, and that these two notions do not coincide in general.

Key Words: Inclusions of C*-algebras, ideal property, weak ideal property, topological dimension zero, tensor products of C*-algebras, crossed products, primitive spectrum.
2020 Mathematics Subject Classification: Primary 46L05; Secondary 46L06, 46L55.

1 Introduction

The weak ideal property was introduced in [19]. A C*-algebra has the weak ideal property if every nonzero quotient of ideals in its stabilization has a nonzero projection (see Definition 8.1 of [19]). The weak ideal property is closely related to two other important properties: the ideal property and topological dimension zero. A C*-algebra A has the ideal property if any ideal of A is generated, as an ideal, by its projections. A C*-algebra A has topological dimension zero if its primitive spectrum $\mathrm{Prim}(A)$ has a basis for its topology consisting of compact open sets (Remark 2.5(vi) of [3]), or, equivalently, if and only if every ideal of A is the closure of the union of an increasing net of compact ideals (see [24]). Note that a separable purely infinite C*-algebra A has real rank zero if and only if A has topological dimension zero and it satisfies a certain K-theoretical condition (see Theorem 4.2 of [24]). The real rank zero was introduced and studied by Brown and Pedersen in [2].

Note that the AH algebras with the ideal property and slow dimension growth have been classified up to a shape equivalence by a K-theoretical invariant in [12], important reduction theorems for AH algebras with the ideal property have been proved in [7] and [8], and the remarkable classification up to isomorphism of the AH algebras with the ideal property and no dimension growth was obtained in [6], using, among other things, techniques from [12], [7] and [8]. It is worth to point out that in the case of AH algebras, the ideal property $\Leftrightarrow$ the weak ideal property $\Leftrightarrow$ topological dimension zero (as proved in [20]).

The weak ideal property and topological dimension zero have good permanence properties (see, e.g., [18], [19], [20], and [3]). It is known that the ideal property $\Rightarrow$ the weak ideal property $\Rightarrow$ topological dimension zero (the first implication is obvious, the second one is Theorem 2.8 of [20]). These three properties are not identical (see [20]). However, it was shown in [20] that, in many interesting cases, these three concepts are equivalent. It is known that for a C*-algebra, real rank zero $\Rightarrow$ the ideal property. A good understanding of the ideal property, the weak ideal property, and topological dimension zero is important in identifying and studying regularity properties for nonsimple C*-algebras, in an attempt to extend Elliott's Classification Program beyond the class of simple C*-algebras. These three properties have been studied in the recent years in [21], [13], [14], [15], [16], [17], and [22].

On the other hand, the study of inclusions in Operator Algebras was very successful in the last decades. We can point out Jones' celebrated theory of subfactors ([9]), Rørdam's C*-irreducible inclusions ([25]), Gabe and Neagu's inclusions with real rank zero ([5]), and [4], to mention just a few.

In this paper we introduce the notions of the ideal property, the weak ideal property, and topological dimension zero for an inclusion of C*-algebras (see Definitions 1, 3, and 4), we characterize them (see Propositions 2, 6, and 7), we prove that for an inclusion of C*-algebras, the real rank zero (in the sense of Gabe and Neagu [5]) $\Rightarrow$ the ideal property (see Definition 2, Proposition 3, and the paragraph before Definition 3), and we show that these four notions are different (see Remark 1).

We prove that for an inclusion of C*-algebras, the ideal property $\Rightarrow$ the weak ideal property $\Rightarrow$ topological dimension zero (see Proposition 5 and Theorem 1). We show that for an inclusion of C*-algebras $A \subseteq B$ these three notions are equivalent in many interesting cases, e.g., when A is a type I C*-algebra or when A is purely infinite (see Theorems 2, 3, and 9).

We give some sufficient conditions for an inclusion of C*-algebras to have the ideal property, the weak ideal property or topological dimension zero (see Propositions 8, 9, and 10), and we prove some permanence properties of these notions (see Propositions 11 and 12).

We prove that these three notions for inclusions of C*-algebras behave well with respect to tensor products (see Theorems 4, 6, 7, and 8). We show, e.g., that the tensor product of two inclusions of C*-algebras with the ideal property has also the ideal property, provided that at least one of the largest C*-algebras in these two inclusions is exact (see Theorem 4). We also prove, that under some mild and natural conditions, the tensor product of two inclusions of C*-algebras has topological dimension zero if and only if each of the two inclusions has topological dimension zero, where at least one of the largest C*-algebras in these two inclusions is exact (see Theorem 7).

Let $A \subseteq B$ be an inclusion of C*-algebras. Let $\alpha\colon G \to \mathrm{Aut}(B)$ be a spectrally free action of a discrete group G on the C*-algebra B (see Definition 1.3 of [19]). Assume that α is exact (see Definition 0.2 of [19]) and that A is α-invariant. We prove that if the inclusion $A \subseteq B$ has the ideal property (respectively, the weak ideal property), then the inclusion of crossed products $C_r^*(G, A, \alpha) \subseteq C_r^*(G, B, \alpha)$ has the ideal property (respectively, the weak ideal property). If the inclusion $A \subseteq B$ has topological dimension zero and G is finite, then we show that the inclusion $C^*(G, A, \alpha) \subseteq C^*(G, B, \alpha)$ has topological dimension zero (see Theorem 10 and Remark 2).

Ideals in C*-algebras are assumed to be closed and two sided. If A is a C*-algebra, then A_+ will denote the set of all positive elements of A, $\mathrm{Prim}(A)$ will denote the primitive spectrum of A, and $I \lhd A$ will denote the fact that I is an ideal of A. If A is a C*-algebra and $I \lhd A$, then I is compact if and only if $\mathrm{Prim}(I)$ is a compact open subset of $\mathrm{Prim}(A)$ (see Lemma 3.10 of [18]). A Kirchberg algebra is a simple, separable, nuclear, purely infinite C*-algebra. We recall that exactness passes to C*-subalgebras. The C*-algebra of all compact linear bounded operators acting on a separable infinite dimensional Hilbert space is denoted by $\mathcal{K}$. If A and B are C*-algebras, then $A \otimes B$ denotes the minimal tensor product of A with B.

2 The results

Proposition 1. *Let $A \subseteq B$ be an inclusion of C*-algebras. Then there exists an ideal in B which is the largest ideal of B included in A.*

Proof. Since the family $\mathcal{F}$ of all the ideals of B included in A (which is nonempty, since 0 is such an ideal) is directed (the sum of two ideals of B included in A is an ideal of B included in A), the closure of the union of all the ideals belonging to $\mathcal{F}$ is an ideal I of B. Clearly, $I \subseteq A$, and each ideal of B included in A is contained in I. Hence I is the largest ideal of B included in A.

□

Notation 1. *If $A \subseteq B$ is an inclusion of C*-algebras, we shall denote by $\mathcal{I}(A, B)$ the largest ideal of B included in A (whose existence was proved in Proposition 1).*

It may be helpful to point out the intuition behind this object, by briefly specializing for unital inclusions of unital commutative C*-algebras: Specifically, if $A = C(X)$ and $B = C(Y)$ (where X and Y are compact Hausdorff spaces) with a unital inclusion map $i \colon A \hookrightarrow B$, one obtains by duality the continous surjective map $\pi := i_* \colon Y \to X$. There clearly exists the largest open subset $V \subseteq X$ such that $\pi^{-1}(x)$ is a singleton for every $x \in V$. Then $\pi^{-1}(V) \subseteq Y$ is an open subset that corresponds to the ideal $\mathcal{I}(A, B) \subseteq B$. Alternatively, V is the largest open subset of X for which $\pi|_{\pi^{-1}(V)} \colon \pi^{-1}(V) \to V$ is a homeomorphism.

Definition 1. *An inclusion of C*-algebras $A \subseteq B$ is said to have the* ideal property *if every $I \lhd B$ with the property that $I \subseteq A$ is generated, as an ideal of B, by its projections.*

Proposition 2. *Let $A \subseteq B$ be an inclusion of C*-algebras. The following are equivalent:*

(1) The inclusion $A \subseteq B$ has the ideal property.

(2) $\mathcal{I}(A, B)$ has the ideal property.

(3) For every $I, J \lhd B$, $I, J \subseteq A$, $I \subsetneq J$ there is a projection in $J \setminus I$.

Proof. We start with the observation that if $A \subseteq B$ is an inclusion of C*-algebras, then the ideals of $\mathcal{I}(A, B)$ are exactly the ideals of B included in A.

Indeed, let $I \lhd \mathcal{I}(A, B)$. Then, since $\mathcal{I}(A, B) \lhd B$, it follows that $I \lhd B$. Also, we have that $I \subseteq \mathcal{I}(A, B) \subseteq A$, and hence $I \subseteq A$.

Assume now that $I \lhd B$ and $I \subseteq A$. Then, using the definition of $\mathcal{I}(A, B)$ we get that $I \subseteq \mathcal{I}(A, B)$, which implies that $I \lhd \mathcal{I}(A, B)$.

This ends the proof of the above observation.

We first prove that (1) $\Leftrightarrow$ (2).

The proof of (1) $\Leftrightarrow$ (2) follows using the above observation, the definition of an inclusion of C*-algebras with the ideal property, and the definition of a C*-algebra with the ideal property.

We now prove that (1) $\Rightarrow$ (3).

This follows immediately from the fact that every ideal of B and included in A is generated by its projections.

We now prove that (3) $\Rightarrow$ (1).

Assume that (3) holds. Let $J \lhd B$ with the property that $J \subseteq A$. Let I be the ideal of J generated by the projections of J. Then clearly $I \subseteq J \subseteq A$ and $I \lhd B$. Assume, by contradiction, that $I \neq J$. Then (3) implies that there is a projection in $J \setminus I$, which is a contradiction. Hence $J = I$, which means that J is generated by its projections. In conclusion, (1) holds.

□

We recall the following definition from [5].

Definition 2. *(Definition 1.3 of [5]) We say that an inclusion of C*-algebras $A \subseteq B$ has* real rank zero *if for any nonzero positive element $a \in A$, the hereditary C*-subalgebra $\overline{aBa}$ of B has an approximate unit of projections.*

Proposition 3. *If an inclusion of C*-algebras has real rank zero, then it has the ideal property.*

Proof. Consider an inclusion of C*-algebras $A \subseteq B$, and let $I \lhd B, I \subseteq A$. Let $0 \neq a \in I_+$. Then:

$$a \in \overline{aBa} \subseteq \overline{BaB} \subseteq I$$

and $\overline{aBa}$ has an approximate unit of projections (by hypothesis). Hence, given an arbitrary $\varepsilon > 0$, there is a projection $p \in \overline{aBa}$ (and hence $p \in I$), such that:

$$||a - app|| = ||a - ap|| < \varepsilon$$

Hence, I is generated, as an ideal of B, by its projections. In conclusion, the inclusion of C*-algebras $A \subseteq B$ has the ideal property.

□

Proposition 4. *If the inclusion of C*-algebras $A \subseteq B$ has real rank zero, then $\mathcal{I}(A, B)$ has real rank zero.*

Proof. Let $0 \neq a \in \mathcal{I}(A,B)_+$. Then, if we denote $I := \mathcal{I}(A,B)$, we have:

$$\overline{aIa} = \overline{aBa} \tag{2.1}$$

Indeed, observe first that $\overline{aIa} \subseteq \overline{aBa}$, since $I \subseteq B$. To prove that $\overline{aBa} \subseteq \overline{aIa}$, note that $\overline{aIa}$ is a hereditary C*-subalgebra of I, I is a hereditary C*-subalgebra of B, and therefore $\overline{aIa}$ is a hereditary C*-subalgebra of B. Since also $a \in \overline{aIa}$, it follows that $\overline{aBa} \subseteq \overline{aIa}$. This ends the proof of (2.1).

Since the inclusion $A \subseteq B$ has real rank zero, it follows that $\overline{aBa}$ has an approximate unit of projections, which implies, using also (2.1), that $\overline{aIa}$ has an approximate unit of projections. Since this is true for every $a \in I_+ = \mathcal{I}(A,B)_+$, Theorem 2.6 of [2] implies that $I = \mathcal{I}(A,B)$ has real rank zero.

□

Note that Proposition 4, Proposition 2, and the fact that every C*-algebra with real rank zero has the ideal property, give a new and natural proof of Proposition 3.

Definition 3. *An inclusion of C*-algebras $A \subseteq B$ is said to have the* weak ideal property *if every $I, J \lhd B$, $I, J \subseteq A$, $I \subsetneq J$, there is a nonzero projection in $(J/I) \otimes \mathcal{K}$.*

Proposition 5. *If an inclusion of C*-algebras has the ideal property, then it has the weak ideal property.*

Proof. Assume that an inclusion of C*-algebras $A \subseteq B$ has the ideal property, and let $I, J \lhd B$, $I, J \subseteq A$, $I \subsetneq J$ be arbitrary. Then, Proposition 2 implies that there is a projection $p \in J \setminus I$. Let q be an arbitrary nonzero projection of $\mathcal{K}$. Then $(p+I) \otimes q$ is a nonzero projection of $(J/I) \otimes \mathcal{K}$. Hence the inclusion of C*-algebras $A \subseteq B$ has the weak ideal property.

□

.

Proposition 6. *Let $A \subseteq B$ be an inclusion of C*-algebras. The following are equivalent:*

(1) The inclusion $A \subseteq B$ has the weak ideal property.

(2) $\mathcal{I}(A,B)$ has the weak ideal property.

Proof. We observed in the proof of Proposition 2 that if $A \subseteq B$ is an inclusion of C*-algebras, then the ideals of $\mathcal{I}(A,B)$ are exactly the ideals of B included in A.

Now the proof of the proposition follows using this observation, the definition of an inclusion of C*-algebras with the weak ideal property, and the definition of a C*-algebra with the weak ideal property.

□

Definition 4. *An inclusion of C*-algebras $A \subseteq B$ is said to have* topological dimension zero *if every ideal of B included in A is the closure of the union of an increasing net of compact ideals of B.*

Observe that since the sum of finitely many compact ideals is again a compact ideal (see Lemma 2.3 of [24]), we clearly have that an inclusion of C*-algebras $A \subseteq B$ has topological dimension zero if for every ideal I of B contained in A there exists a family of compact ideals of B whose union is dense in I.

Proposition 7. *Let $A \subseteq B$ be an inclusion of C*-algebras. The following are equivalent:*

(1) The inclusion $A \subseteq B$ has topological dimension zero.

(2) $\mathcal{I}(A, B)$ has topological dimension zero.

Proof. We observed in the proof of Proposition 2 that if $A \subseteq B$ is an inclusion of C*-algebras, then the ideals of $\mathcal{I}(A, B)$ are exactly the ideals of B included in A. Note also that if D is C*-algebra and $K \lhd I \lhd D$, then: K is a compact ideal of $I \Leftrightarrow K$ is a compact ideal of D (since $\mathrm{Prim}(I)$ is an open subset of $\mathrm{Prim}(A)$).

Now the proof of the proposition follows using these observations, the definition of an inclusion of C*-algebras with topological dimension zero, and the equivalent definition of topological dimension zero for a C*-algebra mentioned in the first paragraph of the Introduction (see also [24]).

□

Theorem 1. *If an inclusion of C*-algebras has the weak ideal property, then it has topological dimension zero.*

Proof. Let $A \subseteq B$ be an inclusion of C*-algebras. Assume that the inclusion $A \subseteq B$ has the weak ideal property. Then, Proposition 6 implies that $\mathcal{I}(A, B)$ has the weak ideal property. Using Theorem 2.8 of [20] it follows that $\mathcal{I}(A, B)$ has topological dimension zero. This implies, using Proposition 7, that the inclusion $A \subseteq B$ has topological dimension zero.

□

Proposition 8. *Let $A \subseteq B$ be an inclusion of C*-algebras. If there exists an intermediate C*-algebra with the ideal property, then the inclusion $A \subseteq B$ has the ideal property. In particular, if either A or B has the ideal property, then the inclusion $A \subseteq B$ has the ideal property. Moreover, a C*-algebra C has the ideal property if and only if the inclusion $C \subseteq C$ has the ideal property.*

Proof. Suppose that there exists a C*-algebra D with the ideal property such that $A \subseteq D \subseteq B$. Let $I \lhd B$ with the property that $I \subseteq A$. Then $I \lhd D$, and since D has the ideal property it follows that I, as an ideal of D, is generated by its projections, and hence I, as an ideal of B, is generated by its projections. Therefore, the inclusion $A \subseteq B$ has the ideal property. In particular, it follows that if either A or B has the ideal property, then the inclusion $A \subseteq B$ has the ideal property. The fact that a C*-algebra C has the ideal property if and only if the inclusion $C \subseteq C$ has the ideal property is obvious.

□

Proposition 9. *Let $A \subseteq B$ be an inclusion of C*-algebras. If there exists an intermediate C*-algebra with the weak ideal property, then the inclusion $A \subseteq B$ has the weak ideal property. In particular, if either A or B has the weak ideal property, then the inclusion $A \subseteq B$ has the weak ideal property. Moreover, a C*-algebra C has the weak ideal property if and only if the inclusion $C \subseteq C$ has the weak ideal property.*

Proof. Suppose that there exists a C*-algebra D with the weak ideal property such that $A \subseteq D \subseteq B$. Let $I, J \lhd B$, $I, J \subseteq A$, $I \subsetneq J$. Then $I, J \lhd D$, $I \subsetneq J$, and since D has the weak ideal property, it follows that in $(J/I) \otimes \mathcal{K}$ has a nonzero projection. This proves that the inclusion $A \subseteq B$ has the weak ideal property. In particular, it follows that if either A or B has the weak ideal property, then the inclusion $A \subseteq B$ has the weak ideal property. The fact that a C*-algebra C has the weak ideal property if and only if the inclusion $C \subseteq C$ has the weak ideal property is obvious.

□

Proposition 10. *Let $A \subseteq B$ be an inclusion of C*-algebras and D be a hereditary C*-subalgebra of B such that $A \subseteq D$. If D has topological dimension zero, then the inclusion $A \subseteq B$ has topological dimension zero. In particular, if B has topological dimension zero, then the inclusion $A \subseteq B$ has topological dimension zero. Moreover, a C*-algebra C has topological dimension zero if and only if the inclusion $C \subseteq C$ has topological dimension zero.*

Proof. Let $I \lhd B$, $I \subseteq A$. Then, the hypothesis implies that $I \lhd D$. Since also D has topological dimension zero, it follows that I is the closure of the union of an increasing net of compact ideals of D (see the equivalent definition of topological dimension zero for a C*-algebra mentioned in the first paragraph of the Introduction and [24]). But since any ideal of B included in D which is compact in D is also compact in B (since $\mathrm{Prim}(D)$ is an open subset of $\mathrm{Prim}(B)$, because D is a hereditary C*-subalgebra of B), we deduce that I is the closure of the union of an increasing net of compact ideals of B. Hence, the inclusion $A \subseteq B$ has topological dimension zero. In particular, it follows that if B has topological dimension zero, then the inclusion $A \subseteq B$ has topological dimension zero. The fact that a C*-algebra C has topological dimension zero if and only if the inclusion $C \subseteq C$ has topological dimension zero is obvious.

□

Remark 1. *(1) Let B be a separable C*-algebra such that B has the ideal property and B does not have real rank zero. Then, by Theorem 2.6 of [2] there exists a hereditary C*-subalgebra A of B such that A does not have an approximate unit of projections. But since A is separable, there exists $a \in A_+$ such that $A = \overline{aBa}$. Then the inclusion $A \subseteq B$ does not have real rank zero, but it has the ideal property since B has the ideal property (see Proposition 8).*

(2) Let B be a C-algebra with the weak ideal property and which does not have the ideal property (see, e.g., Example 8.4 of [19]). Then there exists $A \lhd B$ such that A is not generated, as an ideal of B, by its projections. Hence the inclusion $A \subseteq B$ does not have the ideal property, but it has the weak ideal property, since B has the weak ideal property (see Proposition 9).*

(3) Let B be a C-algebra with topological dimension zero and which does not have the weak ideal property (e.g., B could be a simple, projectionless C*-algebra). Then there are $I, J \lhd B$, $I \subsetneq J$ such that $(J/I) \otimes \mathcal{K}$ is projectionless. It follows that the inclusion $A := J \subseteq B$ has topological dimension zero (since B has topological dimension zero, see, e.g., Proposition 10), but it does not have the weak ideal property (since $(J/I) \otimes \mathcal{K}$ is projectionless).*

(4) Let B be a C-algebra which does not have topological dimension zero (e.g., B could be $C([0, 1])$). Then there exists $I \lhd B$ such that I is not the closure of the union of an increasing net of compact ideals of B. It follows that the inclusion $A := I \subseteq B$ does not have topological dimension zero (since $I = A \lhd B$, $I = A \subseteq A$ and $I = A$ is not the closure of the union of an increasing net of compact ideals of B).*

The next two results show that the notions of the ideal property, the weak ideal property, and topological dimension zero for inclusions of C*-algebras have some nice permanence properties.

Proposition 11. *Let $A \subseteq B$ be an inclusion of C*-algebras, and let $A \subseteq I \subseteq B$, where $I \lhd B$.*

(1) If the inclusion $A \subseteq B$ has the ideal property, then the inclusion $A \subseteq I$ has the ideal property.

(2) If the inclusion $A \subseteq B$ has the weak ideal property, then the inclusion $A \subseteq I$ has the weak ideal property.

(3 If the inclusion $A \subseteq B$ has topological dimension zero, then the inclusion $A \subseteq I$ has topological dimension zero.

Proof. Note that (1) and (2) follow from the fact that if $J \lhd I$ (and $J \subseteq A$), then $J \lhd B$.

We now prove (3).

Assume that the inclusion $A \subseteq B$ has topological dimension zero. Let $J \lhd I$, $J \subseteq A$. Then also $J \lhd B$, and since the inclusion $A \subseteq B$ has topological dimension zero, it follows that J is the closure of the union of an increasing net of compact ideals of B. But since any ideal of B included in I which is compact in B is also compact in I (since $\mathrm{Prim}(I)$ is an open subset of $\mathrm{Prim}(B)$, because $I \lhd B$), we deduce that J is the closure of the union of an increasing net of compact ideals of I. Hence, the inclusion $A \subseteq I$ has topological dimension zero.

□

Proposition 12. *Let $A \subseteq B$ be an inclusion of C*-algebras, let $I \lhd B$, $I \subseteq A$, and let $A/I \subseteq B/I$ be the canonically induced inclusion.*

(1) If the inclusion $A \subseteq B$ has the ideal property, then the inclusion $A/I \subseteq B/I$ has the ideal property.

(2) If the inclusion $A \subseteq B$ has the weak ideal property, then the inclusion $A/I \subseteq B/I$ has the weak ideal property.

(3 If the inclusion $A \subseteq B$ has topological dimension zero, then the inclusion $A/I \subseteq B/I$ has topological dimension zero.

Proof. We first prove (1).

Assume that the inclusion $A \subseteq B$ has the ideal property. Let $K \triangleleft B/I$, $K \subseteq A/I$. Then $K = J/I$, for some $J \triangleleft B$, $I \subseteq J$. Since $K = J/I \subseteq A/I$, it follows that $J \subseteq A$. Using now that the inclusion $A \subseteq B$ has the ideal property, we obtain that J is generated, as an ideal of B, by its projections, and therefore $K = J/I$ is generated, as an ideal of B/I, by its projections (in fact, by the images in J/I of the projections of J). Hence, the inclusion $A/I \subseteq B/I$ has the ideal property.

We now prove (2).

Assume that the inclusion $A \subseteq B$ has the weak ideal property. Let $K_1, K_2 \triangleleft B/I$, $K_1, K_2 \subseteq A/I$, $K_1 \subsetneq K_2$. Then, it easily follows that $K_i = J_i/I$, where $J_i \triangleleft B$, $I \subseteq J_i \subseteq A$ for $i = 1, 2$, and $J_1 \subsetneq J_2$. Since the inclusion $A \subseteq B$ has the weak ideal property, we deduce that $(J_2/J_1) \otimes \mathcal{K}$ has a nonzero projection, and therefore $(K_2/K_1) \otimes \mathcal{K} \cong (J_2/J_1) \otimes \mathcal{K}$ has a nonzero projection. Hence, the inclusion $A/I \subseteq B/I$ has the weak ideal property.

We now prove (3).

Assume that the inclusion $A \subseteq B$ has topological dimension zero. Let $K \triangleleft B/I$, $K \subseteq A/I$. Then, it easily follows that $K = J/I$ for some $J \triangleleft B$, and $I \subseteq J \subseteq A$. Since also the inclusion $A \subseteq B$ has topological dimension zero, it follows that J is the closure of the union of an increasing net of compact ideals in B, and hence compact in the ideal J ($\mathrm{Prim}(J)$ is an open subset of $\mathrm{Prim}(B)$, since $J \triangleleft B$). Now Lemma 3.8 of [18] implies that J/I is the closure of the union of an increasing net of compact ideals in J/I, and hence compact in B/I ($\mathrm{Prim}(J/I)$ is an open subset of $\mathrm{Prim}(B/I)$, since $J/I \triangleleft B/I$). Hence, the inclusion $A/I \subseteq B/I$ has topological dimension zero.

□

In some interesting cases, the notions of the ideal property, the weak ideal property, and topological dimension zero for inclusions of C*-algebras are equivalent.

Theorem 2. *Let $A \subseteq B$ be an inclusion of C*-algebras. Assume that A is a type I C*-algebra. The following are equivalent:*

(1) The inclusion $A \subseteq B$ has the ideal property.

(2) The inclusion $A \subseteq B$ has the weak ideal property.

(3) The inclusion $A \subseteq B$ has topological dimension zero.

(4) $\mathcal{I}(A, B)$ is an AF algebra.

Proof. A C*-subalgebra of a type I C*-algebra is of type I. Hence $\mathcal{I}(A, B)$ is of type I. Now use Proposition 2, Proposition 6, and Proposition 7 above, and also Proposition 4 of [15].

□

We can illustrate the above theorem in the case of unital inclusions of unital commutative C*-algebras (commutative C*-algebras are of type I). Indeed, if $A = C(X)$ and $B = C(Y)$

(where X and Y are compact Hausdorff spaces) with a unital inclusion map $i\colon A \hookrightarrow B$, then, using the notation from the paragraph before Definition 1, we have that for the above inclusion of C*-algebras the ideal property $\Leftrightarrow$ the weak ideal property $\Leftrightarrow$ topological dimension zero $\Leftrightarrow \mathcal{I}(A,B)$ is an AF algebra $\Leftrightarrow \pi^{-1}(V) \subseteq Y$ is a totally disconnected subset. The first three equivalences above follow from Theorem 2. Using now the comments from the paragraph before Definition 1, Proposition 2, and the known result which says that a commutative C*-algebra D has the ideal property $\Leftrightarrow$ Prim(D) is totally disconnected, it follows that our inclusion of C*-algebras has the ideal property $\Leftrightarrow \pi^{-1}(V) \subseteq Y$ is a totally disconnected subset.

Theorem 3. *Let $A \subseteq B$ be an inclusion of C*-algebras. Assume that A is purely infinite or that $\mathcal{I}(A,B)$ is a separable LS algebra (in particular, a separable locally AH algebra [20]). The following are equivalent:*

(1) The inclusion $A \subseteq B$ has the ideal property.

(2) The inclusion $A \subseteq B$ has the weak ideal property.

(3) The inclusion $A \subseteq B$ has topological dimension zero.

Proof. Combine Proposition 2, Proposition 6, and Proposition 7 above, with Theorem 3.13 of [22] (see also [24]) and Theorem 7.15 of [20]. We also used that $\mathcal{I}(A,B)$ is purely infinite if A is purely infinite, since pure infiniteness passes to ideals (see Proposition 4.17 of [11]). □

Several of the next results show that the notions of the ideal property, the weak ideal property, and topological dimension zero for inclusions of C*-algebras behave well with respect to the operation of taking tensor products.

Theorem 4. *Let $A \subseteq B$ and $C \subseteq D$ be two inclusions of C*-algebras with the ideal property. Assume that B or D is exact. Then the inclusion $A \otimes C \subseteq B \otimes D$ has the ideal property.*

Proof. Let $I \lhd B \otimes D$, $I \subseteq A \otimes C$. We may assume that I is nonzero. Since $I \lhd B \otimes D$ and B or D is exact, a theorem of Kirchberg (see Proposition 2.13 of [10]; see also Theorem 1.3 of [23] and [1]) implies that I is generated (as an ideal of $B \otimes D$) by the family of rectangular ideals $K \otimes L$ (i.e., of course, that $K \lhd B$ and $L \lhd D$) contained in I. Let $K \otimes L$ be a nonzero such ideal. Then, since also $K \otimes L \subseteq I \subseteq A \otimes C$, it follows that $K \subseteq A$ and $L \subseteq C$. Since the inclusion $A \subseteq B$ has the ideal property it follows that K is generated, as an ideal of B, by its projections, and since the inclusion $C \subseteq D$ has the ideal property it follows that L is generated, as an ideal of D, by its projections. Therefore, $K \otimes L$ is generated, as an ideal of $B \otimes D$, by its projections. This implies that I, as an ideal of $B \otimes D$, is generated by its projections. In conclusion, we proved that the inclusion $A \otimes C \subseteq B \otimes D$ has the ideal property.

□

Theorem 5. *Let $A \subseteq B$ and $C \subseteq D$ be two inclusions of C*-algebras. Assume that B or D is exact. Then we have that $\mathcal{I}(A \otimes C, B \otimes D) = \mathcal{I}(A,B) \otimes \mathcal{I}(C,D)$.*

Proof. First note that $\mathcal{I}(A,B) \lhd B$, $\mathcal{I}(A,B) \subseteq A$, and $\mathcal{I}(C,D) \lhd D$, $\mathcal{I}(C,D) \subseteq C$. This implies that $\mathcal{I}(A,B) \otimes \mathcal{I}(C,D) \subseteq A \otimes C$ and that $\mathcal{I}(A,B) \otimes \mathcal{I}(C,D) \lhd B \otimes D$, and hence $\mathcal{I}(A,B) \otimes \mathcal{I}(C,D) \subseteq \mathcal{I}(A \otimes C, B \otimes D)$.

We now prove that $\mathcal{I}(A \otimes C, B \otimes D) \subseteq \mathcal{I}(A,B) \otimes \mathcal{I}(C,D)$.

Since $\mathcal{I}(A \otimes C, B \otimes D) \lhd B \otimes D$ and B or D is exact, a theorem of Kirchberg (see Proposition 2.13 of [10]; see also Theorem 1.3 of [23] and [1]) implies that $\mathcal{I}(A \otimes C, B \otimes D)$ is generated (as an ideal of $B \otimes D$) by the family of rectangular ideals $K \otimes L$ (i.e., of course, that $K \lhd B$ and $L \lhd D$) contained in $\mathcal{I}(A \otimes C, B \otimes D)$. Let $K \otimes L$ be a nonzero such ideal. Then, since also $K \otimes L \subseteq \mathcal{I}(A \otimes C, B \otimes D) \subseteq A \otimes C$, it follows that $K \subseteq A$ and $L \subseteq C$. Hence $K \subseteq \mathcal{I}(A,B)$ and $L \subseteq \mathcal{I}(C,D)$. This implies (taking into account that this type of ideals $K \otimes L$ generate, as an ideal, $\mathcal{I}(A \otimes C, B \otimes D)$), that $\mathcal{I}(A \otimes C, B \otimes D) \subseteq \mathcal{I}(A,B) \otimes \mathcal{I}(C,D)$.

The proof of this theorem is over.

□

Theorem 6. *Let $A \subseteq B$ and $C \subseteq D$ be two inclusions of C*-algebras. Assume that the inclusion $A \subseteq B$ has the ideal property and that the inclusion $C \subseteq D$ has the weak ideal property. Suppose that $\mathcal{I}(A,B)$ and $\mathcal{I}(C,D)$ are separable C*-algebras (this happens, e.g., if A and C are separable), and that B or D is exact. Then the inclusion $A \otimes C \subseteq B \otimes D$ has the weak ideal property.*

Proof. Proposition 2 implies that $\mathcal{I}(A,B)$ has the ideal property, and Proposition 6 implies that $\mathcal{I}(C,D)$ has the weak ideal property. Since also $\mathcal{I}(A,B)$ and $\mathcal{I}(C,D)$ are separable and $\mathcal{I}(A,B)$ or $\mathcal{I}(C,D)$ is exact (exactness passes to C*-subalgebras), Theorem 4.8 of [20] implies that $\mathcal{I}(A,B) \otimes \mathcal{I}(C,D)$ has the weak ideal property. Finally, using also Theorem 5 we get that $\mathcal{I}(A \otimes C, B \otimes D)(= \mathcal{I}(A,B) \otimes \mathcal{I}(C,D))$ has the weak ideal property, which means, by Proposition 6 that the inclusion $A \otimes C \subseteq B \otimes D$ has the weak ideal property.

□

Theorem 7. *Let $A \subseteq B$ and $C \subseteq D$ be two inclusions of C*-algebras. Suppose that $\mathcal{I}(A,B)$ and $\mathcal{I}(C,D)$ are nonzero and separable, and assume that B or D is exact. The following are equivalent:*

(1) The inclusion $A \otimes C \subseteq B \otimes D$ has topological dimension zero.

(2) The inclusions $A \subseteq B$ and $C \subseteq D$ have topological dimension zero.

Proof. The proof combines Proposition 7 and Theorem 5 above, and Theorem 4.4 of [20] (we also used that $\mathcal{I}(A,B)$ or $\mathcal{I}(C,D)$ is exact, since exactness passes to C*-subalgebras).

□

Theorem 8. *Let $A \subseteq B$ and $C \subseteq D$ be two inclusions of C*-algebras. Suppose that B and D are C*-algebras of type I and that $\mathcal{I}(A,B)$ and $\mathcal{I}(C,D)$ are nonzero. Let (P) be any of the following properties: the ideal property, the weak ideal property, and topological dimension zero. The following are equivalent:*

(1) The inclusion $A \otimes C \subseteq B \otimes D$ has (P).

(2) The inclusions $A \subseteq B$ and $C \subseteq D$ have (P).

Proof. A C*-subalgebra of a type I C*-algebra is of type I. Hence $\mathcal{I}(A, B)$ and $\mathcal{I}(C, D)$ are C*-algebras of type I. Now use Proposition 2, Proposition 6, Proposition 7, and Theorem 5 above, and also Theorem 3 of [15]. We also used that since B and D are C*-algebras of type I, they are nuclear and hence exact.

∎

Theorem 9. *Let $A \subseteq B$ be an inclusion of C*-algebras. Suppose that A is separable. Let D be a unital Kirchberg algebra. The following are equivalent:*

(1) The inclusion $A \subseteq B$ has topological dimension zero.

(2) The inclusion $A \otimes D \subseteq B \otimes D$ has topological dimension zero.

(3) The inclusion $A \otimes D \subseteq B \otimes D$ has the weak ideal property.

(4) The inclusion $A \otimes D \subseteq B \otimes D$ has the ideal property.

Proof. By Proposition 7, the inclusion $A \subseteq B$ has topological dimension zero if and only if $\mathcal{I}(A, B)$ has topological dimension zero, the inclusion $A \otimes D \subseteq B \otimes D$ has topological dimension zero if and only if $\mathcal{I}(A \otimes D, B \otimes D)$ has topological dimension zero, by Proposition 6, the inclusion $A \otimes D \subseteq B \otimes D$ has the weak ideal property if and only if $\mathcal{I}(A \otimes D, B \otimes D)$ has the weak ideal property, and by Proposition 2, the inclusion $A \otimes D \subseteq B \otimes D$ has the ideal property if and only if $\mathcal{I}(A \otimes D, B \otimes D)$ has the ideal property. Since D is exact (being nuclear) and simple, Theorem 5 implies that $\mathcal{I}(A \otimes D, B \otimes D) = \mathcal{I}(A, B) \otimes D$.

We first prove that (1) $\Leftrightarrow$ (2)

Note that the inclusion of C*-algebras $D \subseteq D$ has topological dimension zero, since D being simple, Prim(D) is finite (in fact, it cosists of only one element), and hence D has topological dimension zero (see also Proposition 10). Since $\mathcal{I}(D, D) = D$ ($\neq 0$) is separable and D is exact (being nuclear), apply now Theorem 7 to the inclusions $A \subseteq B$ and $D \subseteq D$. (Note that $\mathcal{I}(A, B) = 0$ if and only if $\mathcal{I}(A \otimes D, B \otimes D)(= \mathcal{I}(A, B) \otimes D) = 0$).

We now prove that (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

Since $\mathcal{I}(A, B) \otimes D$ is separable and purely infinite (see Proposition 4.5 of [11]), Theorem 7.15 of [20] implies that $\mathcal{I}(A, B) \otimes D$ has topological dimension zero if and only if it has the weak ideal property, if and only if it has the ideal property. This ends the proof.

∎

The next result shows that the notions of the ideal property, the weak ideal property, and topological dimension zero for an inclusion of C*-algebras behave well with respect to crossed products by discrete (finite) groups, in many interesting cases.

Theorem 10. *Let $A \subseteq B$ be an inclusion of C*-algebras. Let $\alpha\colon G \to Aut(B)$ be a spectrally free action of a discrete group G on the C*-algebra B (see Definition 1.3 of [19]). Assume that α is exact (see Definition 0.2 of [19]) and that A is α-invariant. The inclusion $A \subseteq B$ induces a natural inclusion of C*-algebras $C^*_r(G, A, \alpha) \subseteq C^*_r(G, B, \alpha)$.*

(1) If the inclusion $A \subseteq B$ has the ideal property, then the inclusion $C^*_r(G, A, \alpha) \subseteq C^*_r(G, B, \alpha)$ has the ideal property.

(2) If the inclusion $A \subseteq B$ has the weak ideal property, then the inclusion $C^*_r(G, A, \alpha) \subseteq C^*_r(G, B, \alpha)$ has the weak ideal property.

(3) If the inclusion $A \subseteq B$ has topological dimension zero and G is finite, then the inclusion $C^*(G, A, \alpha) \subseteq C^*(G, B, \alpha)$ has topological dimension zero.

Proof. We first prove (1).

Assume that the inclusion $A \subseteq B$ has the ideal property. Let $J \lhd C^*_r(G, B, \alpha)$, $J \subseteq C^*_r(G, A, \alpha)$. Then, by Proposition 3.5 of [19] B separates the ideals of $C^*_r(G, B, \alpha)$ (in the sense of Definition 0.5 of [19]). This implies that $J = C^*_r(G, I, \alpha)$, for some α-invariant ideal I of B. Let $E\colon C^*_r(G, B, \alpha) \to B$ be the canonical conditional expectation. It follows that $I = E(C^*_r(G, I, \alpha)) = E(J) \subseteq E(C^*_r(G, A, \alpha)) = A$. Now, since the inclusion $A \subseteq B$ has the ideal property, it follows that I is generated, as an ideal of B, by its projections. This easily implies that $J = C^*_r(G, I, \alpha)$ is generated, as an ideal of $C^*_r(G, B, \alpha)$, by its projections. This proves that the inclusion $C^*_r(G, A, \alpha) \subseteq C^*_r(G, B, \alpha)$ has the ideal property.

We now prove (2).

Assume that the inclusion $A \subseteq B$ has the weak ideal property. Let $I, J \lhd C^*_r(G, B, \alpha)$, $I, J \subseteq C^*_r(G, A, \alpha)$, $I \subsetneq J$. Using again Proposition 3.5 of [19], we deduce that $I = C^*_r(G, I_0, \alpha)$ and $J = C^*_r(G, J_0, \alpha)$, for some α-invariant ideals I_0 and J_0 of B. Using again the canonical conditional expectation $E\colon C^*_r(G, B, \alpha) \to B$, we deduce that $E(I) = E(C^*_r(G, I_0, \alpha)) = I_0 \subseteq E(C^*_r(G, A, \alpha)) = A$ and $E(J) = E(C^*_r(G, J_0, \alpha)) = J_0 \subseteq E(C^*_r(G, A, \alpha)) = A$. Since $I \subsetneq J$, we deduce in a similar way that $I_0 \subsetneq J_0$. Since the inclusion $A \subseteq B$ has the weak ideal property, it follows that $(J_0/I_0) \otimes \mathcal{K}$ contains a nonzero projection p. Now, since α is exact, it follows that $J/I \cong C^*_r(G, J_0/I_0, \alpha)$. Let $\gamma\colon G \to \mathrm{Aut}(A \otimes \mathcal{K})$ be the action $\gamma_g := \alpha_g \otimes id_{\mathcal{K}}$, for $g \in G$. Then, we have that $C^*_r(G, J_0/I_0, \alpha) \otimes \mathcal{K} \cong C^*_r(G, (J_0/I_0) \otimes \mathcal{K}, \gamma)$. Since $p \in (J_0/I_0) \otimes \mathcal{K} \subseteq C^*_r(G, (J_0/I_0) \otimes \mathcal{K}, \gamma)$ and p is a nonzero projection, it follows that $C^*_r(G, J_0/I_0, \alpha) \otimes \mathcal{K}$ contains a nonzero projection. In conclusion, the inclusion $C^*_r(G, A, \alpha) \subseteq C^*_r(G, B, \alpha)$ has the weak ideal property.

We now prove (3).

Assume that the inclusion $A \subseteq B$ has topological dimension zero and that the group G is finite. Let $I \lhd C^*_r(G, B, \alpha)$, $I \subseteq C^*_r(G, A, \alpha)$. Using again Proposition 3.5 of [19], we deduce that $I = C^*_r(G, I_0, \alpha)$, for some α-invariant ideal I_0 of B. Since $I \subseteq C^*_r(G, A, \alpha)$, using the canonical conditional expectation $E\colon C^*_r(G, B, \alpha) \to B$ as above, we obtain that $I_0 \subseteq A$. Then, since also the inclusion $A \subseteq B$ has topological dimension zero, it follows that I_0 is the closure of the union of an increasing net (J_i) of compact ideals of B. Replacing each J_i by $\sum_{g \in G} \alpha_g(J_i)$, we may assume that each ideal J_i is α-invariant and compact (note that a finite sum of compact ideals is compact, see Lemma 2.3(i) of [24]). Then I is the closure of the union of the increasing net $(C^*_r(G, J_i, \alpha))$ of compact ideals of $C^*_r(G, B, \alpha)$ (since by Proposition 3.5 of [19] B separates the ideals of $C^*_r(G, B, \alpha)$ (in the sense of Definition 0.5 of [19]), and then we can use Lemma 3.19 of [18] to deduce that each $C^*_r(G, J_i, \alpha)$ is a compact ideal of $C^*_r(G, B, \alpha)$). This proves that the inclusion $C^*_r(G, A, \alpha) = C^*(G, A, \alpha) \subseteq C^*_r(G, B, \alpha) = C^*(G, B, \alpha)$ has topological dimension zero (we also used that the group G is amenable, since it is finite).

∎

Remark 2. *In Theorem 10, if G is finite, then G is exact (e.g., since it is compact), and then the action* $\alpha\colon G \to Aut(B)$ *is automatically exact. In this case, the action* α *is spectrally free if and only if* α *is strongly pointwise outer (see Definition 1.1 of [19] and Theorem 1.16 of [19]).*

References

[1] E. Blanchard, E. Kirchberg, Non-simple purely infinite C*-algebras: the Hausdorff case, *J. Funct. Anal.* **207** (2004), 461-513.

[2] L. G. Brown, G. K. Pedersen, C*-algebras of real rank zero, *J. Funct. Anal.* **99** (1991), 131-149.

[3] L. G. Brown, G. K. Pedersen, Limits and C*-algebras of low rank or dimension, *J. Operator Theory* **61** (2009), 381-417.

[4] J. R. Carrión, J. Gabe, C. Schafhauser, A. Tikuisis, S. White, Classifying *-homomorphisms I: Unital simple nuclear C*-algebras, arXiv:2307.06480v3 (2023).

[5] J. Gabe, R. Neagu, Inclusions of real rank zero, arXiv:2312.03622v2 (2024).

[6] G. Gong, C. Jiang, L. Li, A classification of inductive limit C*-algebras with ideal property, *Trans. London Math. Soc.* **9** (2022), 158-236.

[7] G. Gong, C. Jiang, L. Li, C. Pasnicu, A$\mathbb{T}$ structure of AH algebras with the ideal property and torsion free K-theory, *J. Funct. Anal.* **258** (2010), 2119-2143.

[8] G. Gong, C. Jiang, L. Li, C. Pasnicu, A reduction theorem for AH algebras with the ideal property, *Int. Math. Res. Not.* **24** (2018), 7606-7641.

[9] V. F. R. Jones, Index for subfactors, *Invent. Math.* **72 (1)** (1983), 1-25.

[10] E. Kirchberg, The classification of purely infinite C*-algebras using Kasparov's theory, *Fields Inst. Commun., Amer. Math. Soc.*, Providence, in preparation (Preprint 1994).

[11] E. Kirchberg, M. Rørdam, Non-simple purely infinite C*-algebras, *Amer. J. Math.* **122** (2000), 637-666.

[12] C. Pasnicu, Shape equivalence, nonstable K-theory and AH algebras, *Pacific J. Math.* **192** (2000), 159-182.

[13] C. Pasnicu, On and around the weak ideal property, *Bull. Math. Soc. Sci. Math. Roumanie* **63 (111)** (2020), 199-204.

[14] C. Pasnicu, Ideals in tensor products, topological dimension zero and the weak ideal property, *J. Math. Anal. Appl.* **501** (2021), 125210.

[15] C. Pasnicu, On the weak ideal property and some related properties, *Bull. Math. Soc. Sci. Math. Roumanie* **64 (112)** (2021), 147-157.

[16] C. Pasnicu, The weak ideal property in tensor products, *Bull. Math. Soc. Sci. Math. Roumanie* **65 (113)** (2022), 119-125.

[17] C. Pasnicu, Crossed products, the weak ideal property, and topological dimension zero, *J. Math. Anal. Appl.* **525** (2023), 127152.

[18] C. Pasnicu, N. C. Phillips, Permanence properties for crossed products and fixed point algebras of finite groups, *Trans. Amer. Math. Soc.* **366** (2014), 4625-4648.

[19] C. Pasnicu, N. C. Phillips, Crossed products by spectrally free actions, *J. Funct. Anal.* **269** (2015), 915-967.

[20] C. Pasnicu, N. C. Phillips, The weak ideal property and topological dimension zero, *Canad. J. Math.* **69** (2017), 1385-1421.

[21] C. Pasnicu, N. C. Phillips, Relating properties of crossed products to those of fixed point algebras, *Indiana Univ. Math. J.* **68** (2019), 1885-1901.

[22] C. Pasnicu, M. Rouzbehani, Topological dimension zero and some related properties, preprint (2023).

[23] C. Pasnicu, M. Rørdam, Tensor products of C*-algebras with the ideal property, *J. Funct. Anal.* **177** (2000), 130-137.

[24] C. Pasnicu, M. Rørdam, Purely infinite C*-algebras of real rank zero, *J. reine angew. Math.* **613** (2007), 51-73.

[25] M. Rørdam, Irreducible inclusions of simple C*-algebras, *Enseign. Math.* **69** (2023), 275-314.

Received: 03.04.2024
Revised: 06.05.2024
Accepted: 06.05.2024

Department of Mathematics, The University of Texas at San Antonio,
San Antonio TX 78249, USA
E-mail: `Cornel.Pasnicu@utsa.edu`