Bull. Math. Soc. Sci. Math. Roumanie
Tome 67 (115), No. 3, 2024, 321–332

Twists for weak Turaev π-coalgebras
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Abstract

The main purpose of the present paper is to introduce the twists for the weak Turaev π-coalgebras. We mainly show that a new weak Turaev π-coalgebra could be constructed from the given one through the twists. The relationship between their representations is also discussed.

Key Words: Twist, crossed π-category, weak Turaev π-coalgebra, quasitriangular structure.
2020 Mathematics Subject Classification: Primary 16T05; Secondary 16W50.

1 Introduction

For a group π, Turaev ([9]) introduced the notion of a braided π-monoidal category, called *braided crossed π-category*, and showed that such a category gives rise to a 3-dimensional homotopy quantum field theory. A purely algebraic study of Hopf group-coalgebras was initiated by Virelizier ([12]), and then continued by Wang ([11], [15]-[18]) and Zunino ([21], [22]). It turns out that many of the classical results in Hopf algebra theory can be generalized to the π-(co)algebra setting. Recently, several new results are reported in the construction of a braided crossed π-category, see [2]-[4], [6]-[7], [14]-[18], [20]-[22].

The gauge transformations or twists were first introduced by Drinfeld [5] on quasi-Hopf algebras, in order to twist the coproduct without changing its product. Indeed, a Drinfeld twist for a Hopf algebra H is an invertible element $\sigma \in H \otimes H$, satisfying the 2-cocycle condition

$$(\sigma \otimes 1)(\Delta \otimes id)(\sigma) = (1 \otimes \sigma)(id \otimes \Delta)(\sigma).$$

They have become an important tool in the classification of finite-dimensional Hopf algebras. The twisting elements for the generalized Hopf-type algebra have been discussed in [1], [6], [19] and so on.

It is now very natural to ask several questions: can we get another weak Turaev π-coalgebra from the given one? What kind of relationship should be between their representations? How to describe the twists under the crossed structures? In order to investigate this question, in this article, we essentially construct a class of new braided π-crossed category (in the setting of weak Turaev π-coalgebras) by Drinfeld twists. This is the purpose of the present article.

This paper is organized as follows. In Section 2, we first review some basic definitions. In Section 3, we give the definition of the twists of a weak Turaev π-coalgebra. Further, we use these twists to obtain a new weak Turaev π-coalgebra. We show that this construction is quasitriangular-preserving. In Section 4, we mainly show that their representation categories are monoidal crossed isomorphic.

2 Preliminaries

2.1 Braided crossed π-categories

Throughout the paper, we let k be a fixed field and all algebras are supposed to be over k. For the comultiplication Δ of a coalgebra C, we use the Sweedler-Heyneman's notation:

$$\Delta(c) = c_1 \otimes c_2,$$

for any $c \in C$. In this section, we will review several definitions and notations related to Turaev crossed braided category.

Let π be a group with the unit e. A *π-graded monoidal category* $\mathcal{C}$ (or shortly *π-category*) is given by the following datum:

• a monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$;

• a family of subcategories $\{\mathcal{C}_\alpha\}_{\alpha\in\pi}$ such that $\mathcal{C}$ is a disjoint union of this family and such that $U \otimes V \in \mathcal{C}_{\alpha\beta}$, for any $\alpha, \beta \in \pi$, if the $U \in \mathcal{C}_\alpha$ and $V \in \mathcal{C}_\beta$. $I \in \mathcal{C}_e$. Here the subcategory $\mathcal{C}_\alpha$ is called the αth component of $\mathcal{C}$.

We recall that a *crossed π-category* (see [9]) is a π-category $\mathcal{C} = \{\mathcal{C}_\alpha\}$ endowed with a group homomorphism $\varphi : \pi \longrightarrow \mathrm{aut}(\mathcal{C}), \beta \mapsto \varphi_\beta$, (where $\mathrm{aut}(\mathcal{C})$ is the group of invertible strict tensor functors from $\mathcal{C}$ to itself) such that $\varphi_\beta(\mathcal{C}_\alpha) = \mathcal{C}_{\beta\alpha\beta^{-1}}$ for any $\alpha, \beta \in \pi$. Here the functors φ_β are called *conjugation isomorphisms*.

We will use the left index notation in [8] or in [10]. Given $\beta \in \pi$ and an object $V \in \mathcal{C}_\beta$, the functor φ_β will be denoted by ${}^V(\cdot)$ or ${}^\beta(\cdot)$. We use the notation ${}^{\overline{V}}(\cdot)$ for ${}^{\beta^{-1}}(\cdot)$. Then we have ${}^V id_U = id_{{}^V U}$ and ${}^V(g \circ f) = {}^V g \circ {}^V f$. We remark that since the conjugation $\varphi : \pi \longrightarrow \mathrm{aut}(\mathcal{C})$ is a group homomorphism, for any $V, W \in \mathcal{C}$, we have ${}^{V\otimes W}(\cdot) = {}^V({}^W(\cdot))$ and ${}^e(\cdot) = {}^V({}^{\overline{V}}(\cdot)) = {}^{\overline{V}}({}^V(\cdot)) = \mathrm{id}_\mathcal{C}$ and that since, for any $V \in \mathcal{C}$, the functor ${}^V(\cdot)$ is strict, we have ${}^V(f \otimes g) = {}^V f \otimes {}^V g$, for any $f, g \in \mathcal{C}$, and ${}^V\mathrm{id} = \mathrm{id}$. And we will use $\mathcal{C}(U, V)$ for a set of morphisms (or arrows) from U to V in $\mathcal{C}$.

Recall from [9] that a *braided crossed π-category* (or shortly *braided π-category*) is a crossed π-category $\mathcal{C}$ endowed with a braiding, i.e., with a family of isomorphisms

$$\tau = \{\tau_{U,V} \in \mathcal{C}(U \otimes V, ({}^U V) \otimes U)\}_{U,V\in\mathcal{C}},$$

satisfying the following conditions:

- for any arrow $f \in \mathcal{C}_\alpha(U, U')$ with $\alpha \in \pi$, $g \in \mathcal{C}(V, V')$, we have
$$(({}^\alpha g) \otimes f) \circ \tau_{U,V} = \tau_{U',V'} \circ (f \otimes g); \tag{2.1}$$
- for all $U, V, W \in \mathcal{C}$, we have
$$\tau_{U\otimes V,W} = a_{{}^{U\otimes V}W,U,V} \circ (\tau_{U,{}^V W} \otimes \mathrm{id}_V) \circ a^{-1}_{U,{}^V W,V} \circ (\mathrm{id}_U \otimes \tau_{V,W}) \circ a_{U,V,W}, \tag{2.2}$$
$$\tau_{U,V\otimes W} = a^{-1}_{{}^U V,{}^U W,U} \circ (\mathrm{id}_{({}^U V)} \otimes \tau_{U,W}) \circ a_{{}^U V,U,W} \circ (\tau_{U,V} \otimes \mathrm{id}_W) \circ a^{-1}_{U,V,W}, \tag{2.3}$$
- for any $U, V \in \mathcal{C}, \alpha \in \pi, \varphi_\alpha(\tau_{U,V}) = \tau_{\varphi_\alpha(U),\varphi_\alpha(V)}$. (2.4)

Definition 2.1. *(1) Let π and π' be two groups, $\mathcal{C}$ be a π-category, $\mathcal{D}$ be a π'-category. A group-graded functor is a couple (f, G), where $f : \pi \to \pi'$ is a group homomorphism and $G = (G, G_2, G_0) : \mathcal{C} \to \mathcal{D}$ is a monoidal functor, satisfy*

$$G(U) \in \mathcal{D}_{f(\alpha)}, \quad \text{for any } U \in \mathcal{C}_\alpha,\ \alpha \in \pi. \tag{2.5}$$

(2) Let $(\mathcal{C},\varphi)$ be a crossed π-category, $(\mathcal{D},\varphi')$ be a crossed π'-category. A group-graded functor $(f,G):\mathcal{C}\to\mathcal{D}$ is called a crossed functor, *if the following condition is satisfied*

$$G\circ\varphi_\alpha={\varphi'}_{f(\alpha)}\circ G,\qquad \text{for any } \alpha\in\pi. \tag{2.6}$$

(3) Let $(\mathcal{C},\varphi,C)$ be a braided π-category, $(\mathcal{D},\varphi',C')$ be a braided π'-category. A crossed functor $(f,G):\mathcal{C}\to\mathcal{D}$ is called a braided crossed functor, *if the following condition is satisfied*

$$G_2({}^UV,U)\circ {C'}_{GU,GV}=GC_{U,V}\circ G_2(U,V),\qquad \text{for any } U\in\mathcal{C}_\alpha,\ V\in\mathcal{C}_\beta,\ \alpha,\beta\in\pi. \tag{2.7}$$

2.2 Weak Turaev π-coalgebras

Recall from [12] that a *π-coalgebra* is a family of k-spaces $C=\{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta=\{\Delta_{\alpha,\beta}:C_{\alpha\beta}\longrightarrow C_\alpha\otimes C_\beta\}_{\alpha,\beta\in\pi}$ (called the *comultiplication*) and a k-linear map $\varepsilon:C_e\longrightarrow k$ (called the *counit*), such that Δ is coassociative in the sense that,

- $(\Delta_{\alpha,\beta}\otimes \mathrm{id}_{C_\lambda})\Delta_{\alpha\beta,\lambda}=(\mathrm{id}_{C_\alpha}\otimes\Delta_{\beta,\lambda})\Delta_{\alpha,\beta\lambda},$ for any $\alpha,\beta,\lambda\in\pi$.
- $(\mathrm{id}_{C_\alpha}\otimes\varepsilon)\Delta_{\alpha,e}=\mathrm{id}_{C_\alpha}=(\varepsilon\otimes\mathrm{id}_{C_\alpha})\Delta_{e,\alpha},$ for all $\alpha\in\pi$.

We use the Sweedler's notation (see [16]) for a comultiplication in the following way: for any $\alpha,\beta\in\pi$ and $c\in C_{\alpha\beta}$, we write

$$\Delta_{\alpha,\beta}(c)=c_{(1,a)}\otimes c_{(2,\beta)}.$$

Recall from [11] that a *weak Turaev π-coalgebra* is a π-coalgebra $H=(\{H_\alpha\},\Delta,\varepsilon)$ together with a family of k-linear maps $S=\{S_\alpha:H_\alpha\to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ (the *antipode*), and a family of algebra isomorphisms $\varphi=\{\varphi_\beta:H_\alpha\to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$ (the *crossing*) such that for all $\alpha,\beta,\lambda\in\pi$, $g,h,x\in H_e$, $a\in H_\alpha$, we have:

(WTGC1) The comultiplication $\Delta_{\alpha,\beta}:C_{\alpha\beta}\longrightarrow C_\alpha\otimes C_\beta$ is a (not necessary unit-preserving) homomorphism of algebras such that

$$(\Delta_{\alpha,\beta}\otimes\mathrm{id}_{H_\lambda})\Delta_{\alpha\beta,\lambda}(1_{\alpha\beta\lambda})=(\Delta_{\alpha,\beta}(1_{\alpha\beta})\otimes 1_\lambda)(1_\alpha\otimes\Delta_{\beta,\lambda}(1_{\beta\lambda})),$$
$$(\Delta_{\alpha,\beta}\otimes\mathrm{id}_{H_\lambda})\Delta_{\alpha\beta,\lambda}(1_{\alpha\beta\lambda})=(1_\alpha\otimes\Delta_{\beta,\lambda}(1_{\beta\lambda}))(\Delta_{\alpha,\beta}(1_{\alpha\beta})\otimes 1_\lambda).$$

(WTGC2) The counit $\varepsilon:H_e\longrightarrow k$ is a k-linear map satisfying the identity:

$$\varepsilon(gxh)=\varepsilon(gx_{(2,e)})\varepsilon(x_{(1,e)}h)=\varepsilon(gx_{(1,e)})\varepsilon(x_{(2,e)}h).$$

(WTGC3) The properties of the antipode:

$$m_\alpha(S_{\alpha^{-1}}\otimes\mathrm{id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha}(h)=1_{(1,\alpha)}\varepsilon(h1_{(2,e)}),$$
$$m_\alpha(\mathrm{id}_{H_\alpha}\otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}(h)=\varepsilon(1_{(1,e)}h)1_{(2,\alpha)},$$
$$S_\alpha(a_{(1,\alpha)})a_{(2,\alpha^{-1})}S_\alpha(a_{(3,\alpha)})=S_\alpha(a).$$

(WTGC4) The properties of the crossing:

$$(\varphi_\alpha\otimes\varphi_\alpha)\Delta_{\beta,\lambda}=\Delta_{\alpha\beta\alpha^{-1},\alpha\lambda\alpha^{-1}}\varphi_\alpha,$$
$$\varepsilon\circ\varphi_\alpha=\varepsilon,\quad \varphi_{\alpha\beta}=\varphi_\alpha\varphi_\beta.$$

Remark 2.2. *If the π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)$ only satisfies (WTGC1)-(WTGC2), then we call H a* weak semi-Hopf π-coalgebra. *If H only satisfies (WTGC1)-(WTGC3), then we call H a* weak Hopf π-coalgebra. *Note that a weak Hopf π-coalgebra H is said to be of finite type if, for all $\alpha \in \pi$, H_α is finite-dimensional as a k-vector space. The antipode $S = \{S_\alpha\}_{\alpha\in\pi}$ of the weak Hopf π-coalgebra H is said to be bijective if each S_α is bijective. Note that if H is of finite type, then the antipode S is bijective.*

Remark 2.3. *It is easy to get the following identities:*

(a) $\varphi_e|H_\alpha = id_{H_\alpha}$ and $\varphi_\alpha^{-1} = \varphi_{\alpha^{-1}}$ for all $\alpha \in \pi$;

(b) φ preserves the antipode, i.e., $\varphi_\beta S_\alpha = S_{\beta\alpha\beta^{-1}}\varphi_\beta$ for all $\alpha, \beta \in \pi$;

(c) S_α is an anti-algebra morphism, and satisfies

$$\Delta_{\beta^{-1},\alpha^{-1}} \circ S_{\alpha\beta} = (S_\beta \otimes S_\alpha) \circ \Delta^{coH}_{\alpha,\beta}, \quad \varepsilon \circ S_e = \varepsilon.$$

Example 2.4. *Let $H = (\{H_\alpha\}, \Delta, \varepsilon, S, \varphi)$ be a weak Turaev π-coalgebra. For any $\alpha \in \pi$, set $\overline{H}_\alpha = H_{\alpha^{-1}}$ as an algebra, $\overline{\Delta}_{\alpha,\beta} = (\varphi_\beta \otimes id_{\beta^{-1}}) \circ \Delta_{\beta^{-1}\alpha^{-1}\beta,\beta^{-1}}$, $\overline{\varepsilon} = \varepsilon$, $\overline{S}_\alpha = \varphi_\alpha \circ S_{\alpha^{-1}}$ and $\overline{\varphi}_\beta \mid_{\overline{H}_\alpha} = \varphi_\beta \mid_{H_{\alpha^{-1}}}$. It is easy to check that this is also a weak Turaev π-coalgebra. We call it the* mirror *of H, denoted by $\overline{H}$.*

Example 2.5. *Let π be a group acting on a weak Hopf algebra $(H, m, \eta, \Delta, \varepsilon, S)$ by endomorphisms. Set $H^\pi = \{H_\alpha\}_{\alpha\in\pi}$ where for each $\alpha \in \pi$, the algebra H_α is a copy of H. Fix an identification isomorphism of algebras $i_\alpha : H \to H_\alpha$, $h \mapsto \alpha(h)$. For $\alpha, \beta \in \pi$, we define a comultiplication*

$$\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta, \quad \Delta_{\alpha\beta}(i_{\alpha\beta}(h)) = i_\alpha(h_1) \otimes i_\beta(h_2),$$

where $h \in H$ and $\Delta(h) = h_1 \otimes h_2$. The counit $\varepsilon : H_e \to k$ is defined by $\varepsilon(i_e(h)) = \varepsilon(h)$ for all $i_e(h) \in H_e$. For $\alpha \in \pi$, the antipode $S_\alpha : H_\alpha \to H_{\alpha^{-1}}$ is given by

$$S_\alpha(i_\alpha(h)) = i_{\alpha^{-1}}(S(h)),$$

for all $i_\alpha(h) \in H_\alpha$. Then it is easy to check that H^π is a weak Hopf π-coalgebra, and is crossed with the homomorphism $\varphi_\alpha : H_\beta \to H_{\alpha\beta\alpha^{-1}}$ defined by $\varphi_\alpha(i_\beta(h)) = i_{\alpha\beta\alpha^{-1}}(h)$ for all $\alpha, \beta \in \pi$.

Let H be a weak Hopf π-coalgebra. Define the family of linear maps $\varepsilon^t = \{\varepsilon^t_\alpha : H_e \longrightarrow H_\alpha\}_{\alpha\in\pi}$ and $\varepsilon^s = \{\varepsilon^s_\alpha : H_e \longrightarrow H_\alpha\}_{\alpha\in\pi}$ by the formulae

$$\varepsilon^t_\alpha(h) := \varepsilon(1_{(1,e)}h)1_{(2,\alpha)} = m_\alpha(\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}(h),$$
$$\varepsilon^s_\alpha(h) := 1_{(1,\alpha)}\varepsilon(h1_{(2,e)}) = m_\alpha(S_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha}(h),$$

for any $h \in H_e$, where $\varepsilon^t, \varepsilon^s$ are called the *π-target* and *π-source counital maps*. Introduce the notations $H^t := \varepsilon^t(H) = \{H^t_\alpha := \varepsilon^t_\alpha(H_e)\}_{\alpha\in\pi}$ and $H^s := \varepsilon^s(H) = \{H^s_\alpha := \varepsilon^s_\alpha(H_e)\}_{\alpha\in\pi}$ for their images. Further, we have the following identities:

$$\Delta_{\alpha,\beta}(1_{\alpha\beta}) \in H^s_\alpha \otimes H^t_\beta, \;\; \varepsilon^t \circ \varepsilon^t = \varepsilon^t, \;\; \varepsilon^s \circ \varepsilon^s = \varepsilon^s, \;\; \varepsilon^t \circ \varepsilon^s = \varepsilon^s \circ \varepsilon^t.$$

Similarly, we can define a family of linear maps $\widehat{\varepsilon^t} = \{\widehat{\varepsilon^t}_\alpha : H_e \to H_\alpha\}_{\alpha\in G}$ and $\widehat{\varepsilon^s} = \{\widehat{\varepsilon^s}_\alpha : H_e \to H_\alpha\}_{\alpha\in G}$ by the formula

$$\widehat{\varepsilon^t}_\alpha(h) = \varepsilon(h1_{(1,e)})1_{(2,\alpha)}, \quad \widehat{\varepsilon^s}_\alpha(h) = 1_{(1,\alpha)}\varepsilon(1_{(2,e)}h),$$

for any $h \in H_e$. Their images are denoted by $\widehat{H^t} = \{\widehat{H^t}_\alpha = \widehat{\varepsilon^t}_\alpha(H_e)\}_{\alpha\in G}$ and $\widehat{H^s} = \{\widehat{H^s}_\alpha = \widehat{\varepsilon^s}_\alpha(H_e)\}_{\alpha\in G}$. And then we obtain the following identities

$$\Delta_{\alpha,\beta}(1_{\alpha\beta}) \in \widehat{H^s}_\alpha \otimes \widehat{H^t}_\beta, \quad \widehat{\varepsilon^t}\circ\widehat{\varepsilon^t} = \widehat{\varepsilon^t}, \quad \widehat{\varepsilon^s}\circ\widehat{\varepsilon^s} = \widehat{\varepsilon^s}, \quad \widehat{\varepsilon^t}\circ\widehat{\varepsilon^s} = \widehat{\varepsilon^s}\circ\widehat{\varepsilon^t}.$$

Now we will list some formulas frequently used in our computation for any $\alpha, \beta \in \pi$:

$$\begin{cases}
(W1)\ x_{(1,\alpha)}\otimes\varepsilon^t_\beta(x_{(2,e)}) = 1_{(1,\alpha)}x\otimes 1_{(2,\beta)}, \quad \varepsilon^s_\beta(x_{(1,e)})\otimes x_{(2,\alpha)} = 1_{(1,\beta)}\otimes x1_{(2,\alpha)};\\
(W2)\ \Delta_{\alpha,\beta}(\varepsilon^s_{\alpha\beta}(h)) = 1_{(1,\alpha)}\otimes 1_{(2,\beta)}\varepsilon^s_\beta(h) = 1_{(1,\alpha)}\otimes\varepsilon^s_\beta(h)1_{(2,\beta)};\\
\qquad\ \Delta_{\alpha,\beta}(\varepsilon^t_{\alpha\beta}(h)) = 1_{(1,\alpha)}\varepsilon^\alpha_t(h)\otimes 1_{(2,\beta)} = \varepsilon^\alpha_t(h)1_{(1,\alpha)}\otimes 1_{(2,\beta)};\\
(W3)\ S_\alpha(x) = S_\alpha(x_{(1,\alpha)})\varepsilon^t_{\alpha^{-1}}(x_{(2,e)}) = \varepsilon^s_{\alpha^{-1}}(x_{(1,e)})S_\alpha(x_{(2,\alpha)});\\
(W4)\ \varepsilon(gh) = \varepsilon(g\varepsilon^t_e(h)), \quad \varepsilon(gh) = \varepsilon(\varepsilon^s_e(g)h);\\
(W5)\ x_{(1,\alpha)}\otimes\varepsilon^s_\beta(x_{(2,e)}) = x1_{(1,\alpha)}\otimes S_{\beta^{-1}}(1_{(2,\beta^{-1})}),\\
\qquad\ \varepsilon^t_\beta(x_{(1,e)})\otimes x_{(2,\alpha)} = S_{\beta^{-1}}(1_{(1,\beta^{-1})})\otimes 1_{(2,\alpha)}x;\\
(W6)\ \varepsilon(x_{(1,e)}h)x_{(2,\alpha)} = x\varepsilon^t_\alpha(h), \quad x_{(1,\alpha)}\varepsilon(x_{(2,e)}h) = x\widehat{\varepsilon^s}_\alpha(h);\\
(W7)\ x_{(1,\alpha)}\varepsilon(hx_{(2,e)}) = \varepsilon^s_\alpha(h)x, \quad \varepsilon(hx_{(1,e)})x_{(2,\alpha)} = \widehat{\varepsilon^t}_\alpha(h)x;\\
(W8)\ \varepsilon^t_\alpha(x\varepsilon^t_e(y)) = \varepsilon^t_\alpha(xy) = x_{(1,\alpha)}\varepsilon^t_\alpha(y)S_{\alpha^{-1}}(x_{(2,\alpha^{-1})}),\\
\qquad\ \varepsilon^s_\alpha(\varepsilon^s_e(x)y) = \varepsilon^s_\alpha(xy) = S_{\alpha^{-1}}(y_{(1,\alpha^{-1})})\varepsilon^s_\alpha(x)y_{(2,\alpha)}.
\end{cases}$$

Let H be a weak Turaev π-coalgebra with bijective antipode and $\overline{H}$ its mirror with the comultiplication $\overline{\Delta}_{\alpha,\beta} = (\varphi_\beta\otimes \mathrm{id}_{\beta^{-1}})\circ\Delta_{\beta^{-1}\alpha^{-1}\beta,\beta^{-1}}$ for all $\alpha,\beta\in\pi$. We write $\overline{\Delta}^{cop}_{\alpha,\beta}(h) = h_{(2,\beta^{-1})}\otimes\varphi_\beta(h_{(1,\beta^{-1}\alpha^{-1}\beta)}) \in H_{\beta^{-1}}\otimes H_{\alpha^{-1}}$ for any $h\in H_{\beta^{-1}\alpha^{-1}}$.

A weak Turaev π-coalgebra $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S, \varphi)$ is called a *quasitriangular weak Turaev π-coalgebra* if H endowed with a family of elements

$$R = \{R_{\alpha,\beta} \in \overline{\Delta}^{cop}_{\beta^{-1},\alpha^{-1}}(1_{\alpha\beta})(H_\alpha\otimes_k H_\beta)\Delta_{\alpha,\beta}(1_{\alpha\beta})\}_{\alpha,\beta\in\pi}$$

(*the R-matrix*) such that the following identities hold:

$$\begin{cases}
(Q1)\ R_{\alpha,\beta}\Delta_{\alpha,\beta}(h) = \overline{\Delta}^{cop}_{\beta^{-1},\alpha^{-1}}(h)R_{\alpha,\beta};\\
(Q2)\ \sum R^\alpha_{\alpha,\lambda}R^\alpha_{\alpha,\beta}\otimes R^\beta_{\alpha,\beta}\otimes R^\lambda_{\alpha,\lambda} = \sum R^\alpha_{\alpha,\beta\lambda}\otimes R^{\beta\lambda}_{\alpha,\beta\lambda(1,\beta)}\otimes R^{\beta\lambda}_{\alpha,\beta\lambda(2,\lambda)};\\
(Q3)\ \sum R^\alpha_{\alpha,\beta\lambda\beta^{-1}}\otimes R^\beta_{\beta,\lambda}\otimes\varphi_{\beta^{-1}}(R^{\beta\lambda\beta^{-1}}_{\alpha,\beta\lambda\beta^{-1}})R^\lambda_{\beta,\lambda} = \sum R^{\alpha\beta}_{\alpha\beta,\lambda(1,\alpha)}\otimes R^{\alpha\beta}_{\alpha\beta,\lambda(2,\beta)}\otimes R^\lambda_{\alpha\beta,\lambda};\\
(Q4)\ (\varphi_\lambda\otimes\varphi_\lambda)(R_{\alpha,\beta}) = R_{\lambda\alpha\lambda^{-1},\lambda\beta\lambda^{-1}};\\
(Q5)\ \text{there exists } \overline{R} = \{\overline{R}_{\alpha,\beta}\in\Delta_{\alpha,\beta}(1_{\alpha\beta})(H_\alpha\otimes H_\beta)\overline{\Delta}^{cop}_{\beta^{-1},\alpha^{-1}}(1_{\alpha\beta})\}, \text{ such that}\\
\qquad\qquad \overline{R}_{\alpha,\beta}R_{\alpha\beta} = \Delta_{\alpha,\beta}(1_{\alpha\beta}), \qquad R_{\alpha\beta}\overline{R}_{\alpha,\beta} = \overline{\Delta}^{cop}_{\beta^{-1},\alpha^{-1}}(1_{\alpha\beta}).
\end{cases}$$

3 Twists for crossed structures

This section will extend Drinfeld's twisting construction to the (weak) crossed case, and construct a new quasitriangular weak Turaev π-coalgebra by a twist.

Now let H be a weak Turaev π-coalgebra.

Definition 3.1. *A twist for H is a pair (F, f), where $F = \{F_{\alpha,\beta} = \sum F^\alpha_{\alpha,\beta}\otimes F^\beta_{\alpha,\beta} \in \Delta_{\alpha,\beta}(1_{\alpha\beta})(H_\alpha\otimes H_\beta)\}_{\alpha,\beta\in\pi}$, $f = \{f_{\alpha,\beta} = \sum f^\alpha_{\alpha,\beta}\otimes f^\beta_{\alpha,\beta} \in (H_\alpha\otimes H_\beta)\Delta_{\alpha,\beta}(1_{\alpha\beta})\}_{\alpha,\beta\in\pi}$, satisfying the following axioms*

$$\begin{cases} (T1)\ F_{\alpha,\beta}f_{\alpha,\beta}=\Delta_{\alpha,\beta}(1_{\alpha\beta});\\ (T2)\ (\varphi_\gamma\otimes\varphi_\gamma)(F_{\alpha,\beta})=F_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}},\ (\varphi_\gamma\otimes\varphi_\gamma)(f_{\alpha,\beta})=f_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}};\\ (T3)\ \sum\varepsilon(F^e_{e,\alpha})F^\alpha_{e,\alpha}=\sum F^\alpha_{e,\alpha}\varepsilon(F^e_{e,\alpha})=\sum\varepsilon(f^e_{e,\alpha})f^\alpha_{e,\alpha}=\sum f^\alpha_{e,\alpha}\varepsilon(f^e_{e,\alpha})=1_\alpha;\\ (T4)\ \sum \underline{F^{\alpha\beta}_{\alpha\beta,\gamma}}_{(1,\alpha)}F^\alpha_{\alpha,\beta}\otimes\underline{F^{\alpha\beta}_{\alpha\beta,\gamma}}_{(2,\beta)}F^\beta_{\alpha,\beta}\otimes F^\gamma_{\alpha\beta,\gamma}\\ \qquad =\sum F^\alpha_{\alpha,\beta\gamma}\otimes\underline{F^{\beta\gamma}_{\alpha,\beta\gamma}}_{(1,\beta)}F^\beta_{\beta,\gamma}\otimes\underline{F^{\beta\gamma}_{\alpha,\beta\gamma}}_{(2,\gamma)}F^\gamma_{\beta,\gamma}. \end{cases}$$

for any $\alpha,\beta,\gamma\in\pi$.

Corollary 3.2. *Let (F,f) be a twist for a weak Turaev π-coalgebra H. We have the following identities*

$$\sum\varepsilon^t_\alpha(F^e_{e,\alpha})F^\alpha_{e,\alpha}=1_\alpha,\qquad \sum f^\alpha_{\alpha,e}\varepsilon^s_\alpha(f^e_{\alpha,e})=1_\alpha, \tag{3.1}$$

$$\sum f^\alpha_{e,\alpha}\widehat{\varepsilon^t}_\alpha(f^e_{e,\alpha})=1_\alpha,\qquad \sum\widehat{\varepsilon^s}_\alpha(F^e_{\alpha,e})F^\alpha_{\alpha,e}=1_\alpha. \tag{3.2}$$

Proof. Straightforward. □

Corollary 3.3. *If H is a Turaev π-coalgebra, then each of the four conditions Eq.(T4) - Eq.(T7) implies the other three, where*

$$\begin{cases} (T5)\ \sum f^\alpha_{\alpha,\beta\gamma}\otimes f^\beta_{\beta,\gamma}\underline{f^{\beta\gamma}_{\alpha,\beta\gamma}}_{(1,\beta)}\otimes f^\gamma_{\beta,\gamma}\underline{f^{\beta\gamma}_{\alpha,\beta\gamma}}_{(2,\gamma)}\\ \qquad=\sum f^\alpha_{\alpha,\beta}\underline{f^{\alpha\beta}_{\alpha\beta,\gamma}}_{(1,\alpha)}\otimes f^\beta_{\alpha,\beta}\underline{f^{\alpha\beta}_{\alpha\beta,\gamma}}_{(2,\beta)}\otimes f^\gamma_{\alpha\beta,\gamma};\\ (T6)\ \sum\underline{f^{\alpha\beta}_{\alpha\beta,\gamma}}_{(1,\alpha)}F^\alpha_{\alpha,\beta\gamma}\otimes\underline{f^{\alpha\beta}_{\alpha\beta,\gamma}}_{(2,\beta)}\underline{F^{\beta\gamma}_{\alpha,\beta\gamma}}_{(1,\beta)}\otimes f^\gamma_{\alpha\beta,\gamma}\underline{F^{\beta\gamma}_{\alpha,\beta\gamma}}_{(2,\gamma)}\\ \qquad=\sum F^\alpha_{\alpha,\beta}\otimes F^\beta_{\alpha,\beta}f^\beta_{\beta,\gamma}\otimes f^\gamma_{\beta,\gamma};\\ (T7)\ \sum f^\alpha_{\alpha,\beta\gamma}\underline{F^{\alpha\beta}_{\alpha\beta,\gamma}}_{(1,\alpha)}\otimes\underline{f^{\beta\gamma}_{\alpha,\beta\gamma}}_{(1,\beta)}\underline{F^{\alpha\beta}_{\alpha\beta,\gamma}}_{(2,\beta)}\otimes\underline{f^{\beta\gamma}_{\alpha,\beta\gamma}}_{(2,\gamma)}F^\gamma_{\alpha\beta,\gamma}\\ \qquad=\sum f^\alpha_{\alpha,\beta}\otimes F^\beta_{\beta,\gamma}f^\beta_{\alpha,\beta}\otimes F^\gamma_{\beta,\gamma}. \end{cases}$$

Proof. For any $\alpha,\beta,\gamma\in\pi$, we set

$$\dot F_{\alpha,\beta,\gamma}=\sum\underline{F^{\alpha\beta}_{\alpha\beta,\gamma}}_{(1,\alpha)}F^\alpha_{\alpha,\beta}\otimes\underline{F^{\alpha\beta}_{\alpha\beta,\gamma}}_{(2,\beta)}F^\beta_{\alpha,\beta}\otimes F^\gamma_{\alpha\beta,\gamma},$$

$$\ddot F_{\alpha,\beta,\gamma}=\sum F^\alpha_{\alpha,\beta\gamma}\otimes\underline{F^{\beta\gamma}_{\alpha,\beta\gamma}}_{(1,\beta)}F^\beta_{\beta,\gamma}\otimes\underline{F^{\beta\gamma}_{\alpha,\beta\gamma}}_{(2,\gamma)}F^\gamma_{\beta,\gamma},$$

$$\dot f_{\alpha,\beta,\gamma}=\sum f^\alpha_{\alpha,\beta}\underline{f^{\alpha\beta}_{\alpha\beta,\gamma}}_{(1,\alpha)}\otimes f^\beta_{\alpha,\beta}\underline{f^{\alpha\beta}_{\alpha\beta,\gamma}}_{(2,\beta)}\otimes f^\gamma_{\alpha\beta,\gamma},$$

$$\ddot f_{\alpha,\beta,\gamma}=\sum f^\alpha_{\alpha,\beta\gamma}\otimes f^\beta_{\beta,\gamma}\underline{f^{\beta\gamma}_{\alpha,\beta\gamma}}_{(1,\beta)}\otimes f^\gamma_{\beta,\gamma}\underline{f^{\beta\gamma}_{\alpha,\beta\gamma}}_{(2,\gamma)}.$$

Since H is a Turaev π-coalgebra, we immediately get $F_{\alpha,\beta}f_{\alpha\beta}=f_{\alpha,\beta}F_{\alpha\beta}=1_\alpha\otimes 1_\beta$. Further, it is clear that $\dot F$ and $\dot f$, $\ddot F$ and $\ddot f$ are inverses with each other, respectively. Hence Eq.(T4) is equivalent to Eq.(T5). Similarly, Eq.(T6) and Eq.(T7) are also equivalent.

Let us now prove Eq.(T5) is equivalent to Eq.(T6). Since $F_{\alpha,\beta}\otimes 1_\gamma$ and $1_\alpha\otimes f_{\beta,\gamma}$ are all invertible, we obtain

$$\begin{aligned} &(F_{\alpha,\beta}\otimes 1_\gamma)\dot f\ddot F(1_\alpha\otimes f_{\beta,\gamma})=(F_{\alpha,\beta}\otimes 1_\gamma)(1_\alpha\otimes f_{\beta,\gamma})\\ \Leftrightarrow\quad &\dot f\ddot F=1_\alpha\otimes 1_\beta\otimes 1_\gamma\Leftrightarrow \dot f=\ddot f, \end{aligned}$$

which implies the conclusion. □

Lemma 3.4. *Let (F, f) be a twist for a weak Turaev π-coalgebra H. Define a comultiplication Δ^F by*

$$\Delta^F_{\alpha,\beta}(h) = h_{[1,\alpha]} \otimes h_{[2,\beta]} = f_{\alpha,\beta}\Delta_{\alpha,\beta}(h)F_{\alpha,\beta}, \quad \textit{where } h \in H_{\alpha\beta}.$$

Then $H^F = (\{H_\alpha\}, \Delta^F, \varepsilon)$ is a weak semi-Hopf π-coalgebra.

Proof. Assume that $\alpha, \beta, \gamma \in \pi$, $a \in H_\alpha$, $b, c, d \in H_e$, $h, g \in H_{\alpha\beta}$, $x \in H_{\alpha\beta\gamma}$. Firstly, since

$$\begin{aligned} &\Delta^F_{\alpha,\beta}(hg) = f_{\alpha,\beta}\Delta_{\alpha,\beta}(h1_{\alpha\beta}g)F_{\alpha,\beta} \\ \overset{(T1)}{=}\ & f_{\alpha,\beta}\Delta_{\alpha,\beta}(h)F'_{\alpha,\beta}f'_{\alpha,\beta}\Delta_{\alpha,\beta}(g)F_{\alpha,\beta} = \Delta^F_{\alpha,\beta}(h)\Delta^F_{\alpha,\beta}(g), \end{aligned}$$

Δ^F is an algebra homomorphism. We also have

$$\begin{aligned} (\varepsilon \otimes \mathrm{id}_\alpha)\Delta^F_{e,\alpha}(a) &= \sum \varepsilon(f^e_{e,\alpha}a_{(1,e)})f^\alpha_{e,\alpha}\varepsilon(a_{(2,e)}F^e_{e,\alpha})a_{(3,\alpha)}F^\alpha_{e,\alpha} \\ &\overset{(W6,\ 3.1)}{=} \sum \varepsilon(f^e_{e,\alpha}a_{(1,e)})f^\alpha_{e,\alpha}a_{(2,e)} \overset{(W7,\ 5.2)}{=} a, \end{aligned}$$

and similarly we can prove $(id_\alpha \otimes \varepsilon)\Delta^F_{\alpha,e}(a) = a$, thus ε is still a counit.

Secondly, we compute

$$\begin{aligned} &(\Delta^F_{\alpha,\beta} \otimes \mathrm{id}_\gamma)\Delta^F_{\alpha\beta,\gamma}(x) = \dot{f}_{\alpha,\beta,\gamma}(\Delta_{\alpha,\beta} \otimes \mathrm{id}_\gamma)\Delta_{\alpha\beta,\gamma}(x)\dot{F}_{\alpha,\beta,\gamma} \\ \overset{(T4,\ T5)}{=}\ & \ddot{f}_{\alpha,\beta,\gamma}(\mathrm{id}_\alpha \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}(x)\ddot{F}_{\alpha,\beta,\gamma} = (\mathrm{id}_\alpha \otimes \Delta^F_{\beta,\gamma})\Delta^F_{\alpha,\beta\gamma}(x), \end{aligned}$$

for the coassociativity law. We also have

$$\begin{aligned} &\varepsilon(bc_{[1,e]})\varepsilon(c_{[2,e]}d) = \sum \varepsilon(bf^{(1,e)}_{e,e}c_{(1,e)})\varepsilon(c_{(2,e)}F^{(1,e)}_{e,e})\varepsilon(f^{(2,e)}_{e,e}c_{(3,e)})\varepsilon(c_{(4,e)}F^{(2,e)}_{e,e}d) \\ =\ & \sum \varepsilon(bf^{(1,e)}_{e,e}c_{(1,e)})\varepsilon(f^{(2,e)}_{e,e}c_{(2,e)})\varepsilon(c_{(3,e)}F^{(1,e)}_{e,e})\varepsilon(c_{(4,e)}F^{(2,e)}_{e,e}d) \\ \overset{(W6,\ W7)}{=}\ & \sum \varepsilon(bf^{(1,e)}_{e,e}\varepsilon^s_e(f^{(2,e)}_{e,e})c_{(1,e)})\varepsilon(c_{(2,e)}\varepsilon^t_e(F^{(1,e)}_{e,e})F^{(2,e)}_{e,e}d) \\ \overset{(3.1,\ 3.2)}{=}\ & \varepsilon(bc_{(1,e)})\varepsilon(c_{(2,e)}d) = \varepsilon(bcd), \end{aligned}$$

and similarly we can prove

$$\varepsilon(bc_{[2,e]})\varepsilon(c_{[1,e]}d) = \varepsilon(bcd).$$

At last, we compute

$$\begin{aligned} & 1_{[1,\alpha]} \otimes 1_{[2,\beta]} \otimes 1_{[3,\gamma]} \\ =\ & \ddot{f}_{\alpha,\beta,\gamma}(1_{(1,\alpha)} \otimes 1_{(2,\beta)} \otimes 1_{(3,\gamma)})\ddot{F}_{\alpha,\beta,\gamma} = \ddot{f}_{\alpha,\beta,\gamma}\ddot{F}_{\alpha,\beta,\gamma} \overset{(T4)}{=} \ddot{f}_{\alpha,\beta,\gamma}\dot{F}_{\alpha,\beta,\gamma} \\ \overset{(T7)}{=}\ & (1_\alpha \otimes f_{\beta,\gamma})(1_\alpha \otimes F_{\beta,\gamma})(f_{\alpha,\beta} \otimes 1_\gamma)(F_{\alpha,\beta} \otimes 1_\gamma) \\ =\ & 1_{[1,\alpha]} \otimes 1'_{[1,\beta]}1_{[2,\beta]} \otimes 1'_{[2,\gamma]}, \end{aligned}$$

and similarly we can get

$$1_{[1,\alpha]} \otimes 1_{[2,\beta]} \otimes 1_{[3,\gamma]} = 1_{[1,\alpha]} \otimes 1'_{[1,\beta]}1_{[2,\beta]} \otimes 1'_{[2,\gamma]},$$

which implies the conclusion. ∎

Theorem 3.5. *Let (F, f) be a twist for a weak Turaev π-coalgebra H. Define a new antipode S^F by*

$$S^F_\alpha(h) = \sum f^{\alpha^{-1}}_{\alpha^{-1},\alpha} S_\alpha(f^{\alpha}_{\alpha^{-1},\alpha}) S_\alpha(h) S_\alpha(F^{\alpha}_{\alpha,\alpha^{-1}}) F^{\alpha^{-1}}_{\alpha,\alpha^{-1}}, \quad \textit{where } h \in H_\alpha.$$

Then $H^F = (\{H_\alpha\}, \Delta^F, \varepsilon, S^F, \varphi)$ is also a weak Turaev π-coalgebra.

Proof. We only prove (WTGC3). For any $\alpha \in \pi$, $h \in H_e$, we compute

$$\begin{aligned}
& S^F_{\alpha^{-1}}(h_{[1,\alpha^{-1}]})h_{[2,\alpha]} \\
= \quad & \sum f^{\alpha}_{\alpha,\alpha^{-1}} S_{\alpha^{-1}}(f^{\alpha^{-1}}_{\alpha,\alpha^{-1}}) S_{\alpha^{-1}}({F'}^{\alpha^{-1}}_{\alpha^{-1},\alpha}) S_{\alpha^{-1}}(h_{(1,\alpha^{-1})}) S_{\alpha^{-1}}(f^{\alpha^{-1}}_{\alpha^{-1},\alpha}) S_{\alpha^{-1}}(F^{\alpha^{-1}}_{\alpha^{-1},\alpha}) \\
& \quad F^{\alpha}_{\alpha^{-1},\alpha} f^{\alpha}_{\alpha^{-1},\alpha} h_{(2,\alpha)} {F'}^{\alpha}_{\alpha^{-1},\alpha} \\
\overset{(T1)}{=} \quad & \sum f^{\alpha}_{\alpha,\alpha^{-1}} S_{\alpha^{-1}}({F'}^{\alpha^{-1}}_{\alpha^{-1},\alpha} f^{\alpha^{-1}}_{\alpha,\alpha^{-1}}) \varepsilon^s_\alpha(h) {F'}^{\alpha}_{\alpha^{-1},\alpha} \\
\overset{(T7)}{=} \quad & \sum f^{\alpha}_{\alpha,e} \underline{F^e_{e,\alpha}}_{(1,\alpha)} S_{\alpha^{-1}}(\underline{f^e_{\alpha,e}}_{(1,\alpha^{-1})} \underline{F^e_{e,\alpha}}_{(2,\alpha^{-1})}) \varepsilon^s_\alpha(h) \underline{f^e_{\alpha,e}}_{(2,\alpha)} F^{\alpha}_{e,\alpha} \\
\overset{(W8)}{=} \quad & \sum f^{\alpha}_{\alpha,e} \varepsilon^t_\alpha(F^e_{e,\alpha}) \varepsilon^s_\alpha(h f^e_{\alpha,e}) F^{\alpha}_{e,\alpha} \overset{(5.1)}{=} \sum f^{\alpha}_{\alpha,e} \varepsilon^s_\alpha(h f^e_{\alpha,e}) \\
\overset{(3.2)}{=} \quad & \sum f^{\alpha}_{\alpha,e} 1_{(1,\alpha)} \widehat{\varepsilon^s}_\alpha(F^e_{\alpha,e}) F^{\alpha}_{\alpha,e} \varepsilon(h f^e_{\alpha,e} 1_{(2,e)}) \\
= \quad & \sum f^{\alpha}_{\alpha,e} 1_{(1,\alpha)} F^{\alpha}_{\alpha,e} \varepsilon(h f^e_{\alpha,e} 1_{(2,e)} F^e_{\alpha,e}) = 1_{[1,\alpha]} \varepsilon(h 1_{[2,e]}).
\end{aligned}$$

Similarly, we have

$$h_{[1,\alpha]} S^F_{\alpha^{-1}}(h_{[2,\alpha^{-1}]}) = \varepsilon_\alpha(1_{[1,e]}h) 1_{[2,\alpha]}.$$

Then we get

$$S^F_\alpha(a_{[1,\alpha]}) a_{[2,\alpha^{-1}]} S^F_\alpha(a_{[3,\alpha]}) = S^F_\alpha(a), \quad \forall a \in H_\alpha,$$

which implies (WTGC3). □

Proposition 3.6. *Let (F, f) be a twist for a quasitriangular weak Turaev π-coalgebra (H, R). For any $\alpha, \beta \in \pi$, define R^F by*

$$(R^F)_{\alpha,\beta} = (id_\alpha \otimes \varphi_{\alpha^{-1}})(\underline{f_{\alpha\beta\alpha^{-1},\alpha}}_{21}) R_{\alpha,\beta} F_{\alpha,\beta} \in \overline{\Delta}^{cop}_{\beta^{-1},\alpha^{-1}}(1_{\alpha\beta})(H_\alpha \otimes_k H_\beta)\Delta_{\alpha,\beta}(1_{\alpha\beta})\}_{\alpha,\beta\in\pi},$$

here $\underline{f_{\alpha\beta\alpha^{-1},\alpha}}_{21} = \sum f^{\alpha}_{\alpha\beta\alpha^{-1},\alpha} \otimes f^{\alpha\beta\alpha^{-1}}_{\alpha\beta\alpha^{-1},\alpha}$, then $H^F = (\{H_\alpha\}, \Delta^F, \varepsilon, S^F, \varphi, R^F)$ is also a quasitriangular weak Turaev π-coalgebra.

Proof. Firstly, for any $\alpha, \beta, \lambda \in \pi$, we compute

$$\begin{aligned}
& (\varphi_\lambda \otimes \varphi_\lambda)((R^F)_{\alpha,\beta}) \\
= \quad & (\varphi_\lambda \otimes \varphi_\lambda)(\mathrm{id}_\alpha \otimes \varphi_{\alpha^{-1}})(\underline{f_{\alpha\beta\alpha^{-1},\alpha}}_{21})(\varphi_\lambda \otimes \varphi_\lambda)(R_{\alpha,\beta})(\varphi_\lambda \otimes \varphi_\lambda)(F_{\alpha,\beta}) \\
= \quad & (\mathrm{id}_{\lambda\alpha\lambda^{-1}} \otimes \varphi_{\lambda\alpha^{-1}\lambda^{-1}})(\underline{f_{\lambda\alpha\beta\alpha^{-1}\lambda^{-1},\lambda\alpha\lambda^{-1}}}_{21}) R_{\lambda\alpha\lambda^{-1},\lambda\beta\lambda^{-1}} F_{\lambda\alpha\lambda^{-1},\lambda\beta\lambda^{-1}} \\
= \quad & (R^F)_{\lambda\alpha\lambda^{-1},\lambda\beta\lambda^{-1}},
\end{aligned}$$

thus Eq.(Q4) holds.

Secondly, for any $h \in H_{\alpha\beta}$, we have

$$\begin{aligned}
& \overline{\Delta^F}^{cop}_{\beta^{-1},\alpha^{-1}}(h)(R^F)_{\alpha,\beta} \\
= \; & ((\mathrm{id}_\alpha \otimes \varphi_{\alpha^{-1}}) \circ (\Delta^F)^{cop}_{\alpha\beta\alpha^{-1},\alpha})(h)(\mathrm{id}_\alpha \otimes \varphi_{\alpha^{-1}})(F_{\alpha,\alpha\beta\alpha^{-1}})R_{\alpha,\beta}f_{\alpha,\beta} \\
= \; & (\mathrm{id}_\alpha \otimes \varphi_{\alpha^{-1}})(\underline{f_{\alpha\beta\alpha^{-1},\alpha}}_{21})R_{\alpha,\beta}\Delta_{\alpha,\beta}(h)F_{\alpha,\beta} \\
= \; & (R^F)_{\alpha,\beta}\Delta^F_{\alpha,\beta}(h)
\end{aligned}$$

which implies Eq.(Q1).

Thirdly, since

$$\begin{aligned}
& (\mathrm{id}_\alpha \otimes \Delta^F_{\beta,\gamma})((R^F)_{\alpha,\beta\gamma}) \\
= \; & \sum f^\alpha_{\alpha,\alpha\beta\gamma\alpha^{-1}} R^\alpha_{\alpha,\beta\gamma} F^\alpha_{\alpha,\beta\gamma} \otimes \varphi_{\alpha^{-1}}(f^{\alpha\beta\alpha^{-1}}_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}} \underline{f^{\alpha\beta\gamma\alpha^{-1}}_{\alpha\beta\gamma\alpha^{-1},\alpha}}_{(1,\alpha\beta\alpha^{-1})}) \underline{R^{\beta\gamma}_{\alpha,\beta\gamma}}_{(1,\beta)} \\
& \underline{F^{\beta\gamma}_{\alpha,\beta\gamma}}_{(1,\beta)} F^\beta_{\beta,\gamma} \otimes \varphi_{\alpha^{-1}}(f^{\alpha\gamma\alpha^{-1}}_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}} \underline{f^{\alpha\beta\gamma\alpha^{-1}}_{\alpha\beta\gamma\alpha^{-1},\alpha}}_{(2,\alpha\gamma\alpha^{-1})}) \underline{R^{\beta\gamma}_{\alpha,\beta\gamma}}_{(2,\gamma)} \underline{F^{\beta\gamma}_{\alpha,\beta\gamma}}_{(2,\gamma)} F^\gamma_{\beta,\gamma} \\
\overset{(T4,\ T5)}{=} \; & \sum f^\alpha_{\alpha\gamma\alpha^{-1},\alpha} \underline{f^{\alpha\gamma}_{\alpha\beta\alpha^{-1},\alpha\gamma}}_{(2,\alpha)} R^\alpha_{\alpha,\gamma} R^\alpha_{\alpha,\beta} \underline{F^{\alpha\beta}_{\alpha\beta,\gamma}}_{(1,\alpha)} F^\alpha_{\alpha,\beta} \otimes \varphi_{\alpha^{-1}}(f^{\alpha\beta\alpha^{-1}}_{\alpha\beta\alpha^{-1},\alpha\gamma}) R^\beta_{\alpha,\beta} \\
& \underline{F^{\alpha\beta}_{\alpha\beta,\gamma}}_{(2,\beta)} F^\beta_{\alpha,\beta} \otimes \varphi_{\alpha^{-1}}(f^{\alpha\gamma\alpha^{-1}}_{\alpha\gamma\alpha^{-1},\alpha} \underline{f^{\alpha\gamma}_{\alpha\beta\alpha^{-1},\alpha\gamma}}_{(1,\alpha\gamma\alpha^{-1})}) R^\gamma_{\alpha,\gamma} F^\gamma_{\alpha\beta,\gamma} \\
\overset{(T7)}{=} \; & \sum f^\alpha_{\alpha\gamma\alpha^{-1},\alpha} R^\alpha_{\alpha,\gamma} F^\alpha_{\alpha,\gamma} f^\alpha_{\alpha\beta\alpha^{-1},\alpha} R^\alpha_{\alpha,\beta} F^\alpha_{\alpha,\beta} \otimes \varphi_{\alpha^{-1}}(f^{\alpha\beta\alpha^{-1}}_{\alpha\beta\alpha^{-1},\alpha}) R^\beta_{\alpha,\beta} F^\beta_{\alpha,\beta} \\
& \otimes \varphi_{\alpha^{-1}}(f^{\alpha\gamma\alpha^{-1}}_{\alpha\gamma\alpha^{-1},\alpha}) R^\gamma_{\alpha,\gamma} F^\gamma_{\alpha,\gamma} \\
= \; & ((R^F)_{\alpha,\gamma})_{1\beta3}((R^F)_{\alpha,\beta})_{12\gamma},
\end{aligned}$$

we obtain Eq.(Q2). And we can get Eq.(Q3) in a similar way.

At last, define $\overline{R^F} = f_{\alpha,\beta}\overline{R}_{\alpha,\beta}(\mathrm{id}_\alpha \otimes \varphi_{\alpha^{-1}})(F_{\alpha,\alpha\beta\alpha^{-1}})$, then we obtain Eq.(Q5). □

4 On the representations

Recall from [11], $\mathrm{Rep}(H) = \{\mathrm{Rep}_\alpha(H)\}_{\alpha \in G}$, the representation category of a weak Turaev π-coalgebra H is a crossed π-category with the following structures:

• For any $\alpha \in G$, the αth component of $\mathrm{Rep}(H)$, denoted $\mathrm{Rep}_\alpha(H)$, is the category of left representations of the algebra H_α.

• The tensor product $U \otimes V$ of $U \in \mathrm{Rep}_\alpha(H)$ and $V \in \mathrm{Rep}_\beta(H)$ is obtained by

$$U \otimes_t V = \Delta_{\alpha,\beta}(1_{\alpha\beta})(U \otimes_k V),$$

with the action of $H_{\alpha\beta}$ given by $h \cdot (u \otimes_t v) = h_{(1,\alpha)} \cdot u \otimes h_{(2,\beta)} \cdot v$, for any $h \in H_{\alpha\beta}$, $u \in U$, $v \in V$.

• The tensor product of two morphisms $f \in \mathrm{Rep}_\alpha(H)$ and $g \in \mathrm{Rep}_\beta(H)$ is given by the tensor product of k-linear morphisms, i.e., the forgetful functor from $\mathrm{Rep}(H)$ to the category of k-spaces is faithful.

• For any $h \in H_e$, $x \in H_e^t$, H_e^t is the unit object of $\mathrm{Rep}(H)$ with the action: $h{\cdot}x = \varepsilon_e^t(hx)$, and for any $x \in H_e^t,\ v \in V_\alpha$, the unity constraint are defined by

$$l_{V_\alpha}(x \otimes_t v) = \varepsilon_\alpha^t(x)\cdot v;\ l_{V_\alpha}^{-1}(v) = 1_e \otimes_t v = \varepsilon_t^e(1_{\alpha(1,e)}) \otimes 1_{\alpha(2,\alpha)}\cdot v = S_e(1_{\alpha(1,e)}) \otimes 1_{\alpha(2,\alpha)}\cdot v,$$

and

$$r_{V_\alpha}(v \otimes_t x) = \widehat{\varepsilon^s}_\alpha(x)\cdot v = S_\alpha^{-1}\varepsilon_{\alpha^{-1}}^t(x)\cdot v, \qquad r_{V_\alpha}^{-1}(v) = v \otimes_t 1_e = 1_{\alpha(1,\alpha)}\cdot v \otimes 1_{\alpha(2,e)}.$$

• The automorphism φ_α of H defines an automorphism, $\widetilde{\varphi}_\alpha$ of $\mathrm{Rep}(H)$. For $U \in \mathrm{Rep}_\beta(H)$, then ${}^\alpha U := \widetilde{\varphi}_\alpha(U)$ has the same underlying k-space as U and each $h \in H_{\alpha\beta\alpha^{-1}}$ acts by

$$H_{\alpha\beta\alpha^{-1}} \otimes {}^\alpha U \to {}^\alpha U, \qquad h \star {}^\alpha u = {}^\alpha(\varphi_{\alpha^{-1}}(h)\cdot u), \tag{4.1}$$

here we denote ${}^\alpha u$ the corresponding element for $u \in U$ in ${}^\alpha U$. For any morphism $f : M \to N$ in $\mathrm{Rep}(H)$, then ${}^\alpha f : {}^\alpha M \to {}^\alpha N$ satisfies

$${}^\alpha f({}^\alpha m) = {}^\alpha(f(m)), \qquad \text{for all } m \in M.$$

Note that if H is quasitriangular, then $\mathrm{Rep}(H)$ is a braided crossed π-category with the following braiding

$$\tau_{U,V} : U \otimes V \to ({}^U V) \otimes U, \qquad u \otimes v \mapsto {}^\alpha(R_{\alpha,\beta}^\beta \cdot v) \otimes R_{\alpha,\beta}^\alpha \cdot u,$$

where $U \in \mathrm{Rep}_\alpha(H)$ and $V \in \mathrm{Rep}_\beta(H)$.

Theorem 4.1. *Let (F, f) be a twist for a weak Turaev π-coalgebra H. Then Rep(H) and Rep(H^F) are isomorphic as crossed π-categories.*

Proof. For any $m \in M$, $n \in N$, $M \in \mathrm{Rep}_\alpha(H)$, $N \in \mathrm{Rep}_\beta(H)$, $\xi : M \to N \in \mathrm{Mor}(\mathrm{Rep}(H))$, define the functor

$$G = (G, G_2, G_0) : \mathrm{Rep}(H) \to \mathrm{Rep}(H^F)$$

by $G(M) := M,\ \ G(\xi) := \xi,\ \ G_0 = \mathrm{id}_{H_t}$, and

$$\begin{aligned} G_2(M,N) : G(M) \otimes_t G(N) &\to G(M \otimes_t N),\\ m \otimes_t n &\mapsto F_{\alpha,\beta}(m \otimes_t n). \end{aligned}$$

Now we have

$$\begin{aligned} &G_2(M,N)(h\cdot(m\otimes n)) = G_2(M,N)(\Delta_{\alpha,\beta}^F(h)(m\otimes n))\\ &= F_{\alpha,\beta}(f_{\alpha,\beta}\Delta_{\alpha,\beta}(h)F_{\alpha,\beta}(m\otimes n))\\ &= h\cdot(G_2(M,N)(m\otimes n)), \end{aligned}$$

where $h \in H_{\alpha\beta}$. Obviously G satisfies $G \circ \varphi_\alpha = \varphi_\alpha \circ G$ for any $\alpha \in \pi$, so (id_π, G) is a crossed isomorphic functor. ∎

Corollary 4.2. *Let (F, f) be a twist for a quasitriangular weak Turaev π-coalgebra (H, R). Then Rep(H) and Rep(H^F) are isomorphic as braided π-categories.*

Proof. We only need to show the crossed functor (id_π, G) defined above satisfies Eq.(2.7). Indeed, for any $u \in U \in \mathrm{Rep}^\alpha(H)$, $v \in V \in \mathrm{Rep}^\beta(H)$, $\alpha, \beta \in \pi$, we compute

$$
\begin{aligned}
& (G_2({}^U V, U) \circ C'_{GU,GV})(u \otimes v) \\
= \quad & G_2({}^U V, U)(\sum \varphi_{\alpha^{-1}}(f^{\alpha\beta\alpha^{-1}}_{\alpha\beta\alpha^{-1},\alpha}) R^\beta_{\alpha,\beta} F^\beta_{\alpha,\beta} \cdot v \otimes f^\alpha_{\alpha\beta\alpha^{-1},\alpha} R^\alpha_{\alpha,\beta} F^\alpha_{\alpha,\beta} \cdot u) \\
\overset{(4.1)}{=} \quad & \sum \varphi_{\alpha^{-1}}(F^{\alpha\beta\alpha^{-1}}_{\alpha\beta\alpha^{-1},\alpha}) \varphi_{\alpha^{-1}}(f^{\alpha\beta\alpha^{-1}}_{\alpha\beta\alpha^{-1},\alpha}) R^\beta_{\alpha,\beta} F^\beta_{\alpha,\beta} \cdot v \otimes F^\alpha_{\alpha\beta\alpha^{-1},\alpha} f^\alpha_{\alpha\beta\alpha^{-1},\alpha} R^\alpha_{\alpha,\beta} F^\alpha_{\alpha,\beta} \cdot u \\
= \quad & \sum R^\beta_{\alpha,\beta} F^\beta_{\alpha,\beta} \cdot v \otimes R^\alpha_{\alpha,\beta} F^\alpha_{\alpha,\beta} \cdot u = (GC_{U,V} \circ G_2(U,V))(u \otimes v),
\end{aligned}
$$

hence the conclusion holds. □

Acknowledgement *The work was partially supported by the National Natural Science Foundation of Shandong Province (No. ZR2023MA008), the National Natural Science Foundation of China (No. 12271292) and the Taishan Scholar Project of Shandong Province (No. tsqn202103060).*

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Received: 17.06.2022
Revised: 16.10.2022
Accepted: 19.01.2023

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