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Two new q-congruences involving double basic hypergeometric series

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Abstract

In this paper, we establish two new q-congruences involving double basic hypergeometric series. As conclusions, we give several congruences including the following one:

$$\sum_{k=0}^{p-1}\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{(1)_k^4 4^k}\sum_{j=1}^{k}\left\{\frac{1}{(4j)^2}-\frac{1}{(2j-1)^2}\right\}$$

$$\equiv\begin{cases} p\dfrac{(\frac{1}{2})_{(p-1)/4}}{(1)_{(p-1)/4}}\displaystyle\sum_{j=1}^{(p-1)/4}\frac{1}{(4j)^2} \pmod{p^2}, & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^2}, & \text{if } p\equiv 3 \pmod 4. \end{cases}$$

where p is an odd prime.

Key Words: q-Congruence, basic hypergeometric series, Watson's ${}_8\phi_7$ transformation formula, Gasper and Rahman's summation formula.
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1 Introduction

For a complex variable x, define the shifted-factorial to be

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{when} \quad n\in\mathbb{N}.$$

In 2011, Long [12] conjectured the following interesting congruence involving double series: for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}(-1)^k\frac{6k+1}{8^k}\frac{(\frac{1}{2})_k^3}{k!^3}\sum_{j=1}^{k}\left\{\frac{1}{(2j-1)^2}-\frac{1}{16j^2}\right\}\equiv 0 \pmod p. \tag{1.1}$$

This conjecture was proved by Swisher [14] in 2015.

For two complex numbers x and q, define the q-shifted factorial to be

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) \quad \text{when} \quad n\in\mathbb{N}.$$

For shortening many of the formulas in this paper, we also adopt the notation

$$(x_1, x_2, \dots, x_r; q)_n = (x_1; q)_n (x_2; q)_n \cdots (x_r; q)_n.$$

Recently, Gu and Guo [2] discovered the beautiful q-analogue of (1.1): for any positive odd integer,

$$\sum_{k=0}^{(n-1)/2} (-1)^k [6k+1] \frac{(q;q^2)_k^3}{(q^4;q^4)_k^3} \sum_{j=1}^{k} \left\{ \frac{q^{2j-1}}{[2j-1]^2} - \frac{q^{4j}}{[4j]^2} \right\} \equiv 0 \pmod{\Phi_n(q)}.$$

Here and throughout the paper, $[n] = 1 + q + \cdots + q^{n-1}$ denotes the q-integer and $\Phi_n(q)$ stands for the n-th cyclotomic polynomial in q:

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k \leqslant n \\ \gcd(k,n)=1}} (q - \zeta^k),$$

where ζ is an n-th primitive root of unity. Furthermore, Wand and Yu [16, Theorem 1] gave the following conclusion: for any integers n, $d > 1$ with $n \equiv 1 \pmod{d}$,

$$\sum_{k=0}^{(n-1)/d} (-1)^k [2dk+1] \frac{(q;q^d)_k^3}{(q^d;q^d)_k^3} q^{d\binom{k+1}{2}-k} \sum_{j=1}^{k} \left(\frac{q^{dj-d+1}}{[dj-d+1]^2} - \frac{q^{dj}}{[dj]^2} \right) \equiv 0 \pmod{\Phi_n(q)}. \tag{1.2}$$

For more q-analogues of supercongruences, the reader is referred to the papers [3, 4, 5, 7, 6, 8, 9, 11, 13, 15, 17].

Inspired by the work just mentioned, we shall establish the following two theorems.

Theorem 1. *Let $n > 1$, $d > 1$, $r > 0$ be integers with $\gcd(r, d) = 1$ and $n \equiv r \pmod{d}$ such that $n \geq r$. Then*

$$\sum_{k=0}^{(n-r)/d} [2dk+r] \frac{(q^r;q^d)_k^3 (xq^r, yq^r, zq^r; q^d)_k}{(q^d;q^d)_k^3 (q^d/x, q^d/y, q^d/z; q^d)_k} \frac{q^{(2d-3r)k}}{(xyz)^k} \times \sum_{j=1}^{k} \left(\frac{q^{dj-d+r}}{[dj-d+r]^2} - \frac{q^{dj}}{[dj]^2} \right) \equiv 0 \pmod{\Phi_n(q)}. \tag{1.3}$$

Theorem 2. *Let n be a positive odd integer. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{n-1} [6k+1] \frac{(q;q^2)_k^3 (q;q^4)_k}{(q^2;q^2)_k (q^4;q^4)_k^3} q^{k^2+k} \sum_{j=1}^{k} \left(\frac{q^{4j}}{[4j]^2} - \frac{q^{2j-1}}{[2j-1]^2} \right)$$

$$\equiv \begin{cases} [n] q^{(1-n)/4} \dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}} \displaystyle\sum_{j=1}^{(n-1)/4} \frac{q^{4j}}{[4j]^2}, & \text{if } n \equiv 1 \pmod{4}, \\ 0, & \text{if } n \equiv 3 \pmod{4}. \end{cases} \tag{1.4}$$

With the change of the parameters x, y and z, Theorem 1 can produce a lot of concrete q-congruences. Six ones of them are laid out as follows.

Choosing $(x,y,z)\to(q^{d-r},q^{d-r},\infty)$ in Theorem 1, we obtain the following q-congruence.

Corollary 1. *Let $n>1$, $d>1$, $r>0$ be integers with* $\gcd(r,d)=1$ *and* $n\equiv r \pmod d$ *such that $n\geq r$. Then*

$$\sum_{k=0}^{(n-r)/d}(-1)^k[2dk+r]\frac{(q^r;q^d)_k}{(q^d;q^d)_k}q^{d\binom{k}{2}}\sum_{j=1}^{k}\left(\frac{q^{dj-d+r}}{[dj-d+r]^2}-\frac{q^{dj}}{[dj]^2}\right)\equiv 0 \pmod{\Phi_n(q)}.$$

Fixing $(x,y,z)=(q^{d-r},q^{d-r},1)$ in Theorem 1, we get the following result.

Corollary 2. *Let $n>1$, $d>1$, $r>0$ be integers with* $\gcd(r,d)=1$ *and* $n\equiv r \pmod d$ *such that $n\geq r$. Then*

$$\sum_{k=0}^{(n-r)/d}[2dk+r]\frac{(q^r;q^d)_k^2}{(q^d;q^d)_k^2}q^{-rk}\sum_{j=1}^{k}\left(\frac{q^{dj-d+r}}{[dj-d+r]^2}-\frac{q^{dj}}{[dj]^2}\right)\equiv 0 \pmod{\Phi_n(q)}.$$

Taking $(x,y,z)\to(q^{d-r},1,\infty)$ in Theorem 1, we have the following formula.

Corollary 3. *Let $n>1$, $d>1$, $r>0$ be integers with* $\gcd(r,d)=1$ *and* $n\equiv r \pmod d$ *such that $n\geq r$. Then*

$$\sum_{k=0}^{(n-r)/d}(-1)^k[2dk+r]\frac{(q^r;q^d)_k^3}{(q^d;q^d)_k^3}q^{d\binom{k}{2}+(d-r)k}\sum_{j=1}^{k}\left(\frac{q^{dj-d+r}}{[dj-d+r]^2}-\frac{q^{dj}}{[dj]^2}\right)$$
$$\equiv 0 \pmod{\Phi_n(q)}.$$

When $r=1$, Corollary 3 reduces to (1.2). So the former can be regarded as a generalization of the latter.

Choosing $(x,y,z)=(q^{d-r},1,1)$ in Theorem 1, we obtain the following q-congruence.

Corollary 4. *Let $n>1$, $d>1$, $r>0$ be integers with* $\gcd(r,d)=1$ *and* $n\equiv r \pmod d$ *such that $n\geq r$. Then*

$$\sum_{k=0}^{(n-r)/d}[2dk+r]\frac{(q^r;q^d)_k^4}{(q^d;q^d)_k^4}q^{(d-2r)k}\sum_{j=1}^{k}\left(\frac{q^{dj-d+r}}{[dj-d+r]^2}-\frac{q^{dj}}{[dj]^2}\right)\equiv 0 \pmod{\Phi_n(q)}.$$

Taking $(x,y,z)\to(1,1,\infty)$ in Theorem 1, we get the following result.

Corollary 5. *Let $n>1$, $d>1$, $r>0$ be integers with* $\gcd(r,d)=1$ *and* $n\equiv r \pmod d$ *such that $n\geq r$. Then*

$$\sum_{k=0}^{(n-r)/d}(-1)^k[2dk+r]\frac{(q^r;q^d)_k^5}{(q^d;q^d)_k^5}q^{d\binom{k}{2}+2(d-r)k}\sum_{j=1}^{k}\left(\frac{q^{dj-d+r}}{[dj-d+r]^2}-\frac{q^{dj}}{[dj]^2}\right)$$
$$\equiv 0 \pmod{\Phi_n(q)}.$$

Setting $(x,y,z)=(1,1,1)$ in Theorem 1, we have the following formula.

Corollary 6. *Let $n>1$, $d>1$, $r>0$ be integers with $\gcd(r,d)=1$ and $n\equiv r \pmod d$ such that $n\geq r$. Then*

$$\sum_{k=0}^{(n-r)/d}[2dk+r]\frac{(q^r;q^d)_k^6}{(q^d;q^d)_k^6}q^{(2d-3r)k}\sum_{j=1}^{k}\left(\frac{q^{dj-d+r}}{[dj-d+r]^2}-\frac{q^{dj}}{[dj]^2}\right)\equiv 0 \pmod{\Phi_n(q)}.$$

Letting $n=p$ be an odd prime with $p\equiv r \pmod d$ and $\gcd(r,d)=1$ such that $p\geq r$ and then letting $q\to 1$ in Corollaries 1-6, we arrive at the following congruences:

$$\sum_{k=0}^{(p-r)/d}(-1)^k(2dk+r)\frac{(\frac{r}{d})_k}{(1)_k}\sum_{j=1}^{k}\left\{\frac{1}{(dj-d+r)^2}-\frac{1}{(dj)^2}\right\}\equiv 0 \pmod p,$$
$$\sum_{k=0}^{(p-r)/d}(2dk+r)\frac{(\frac{r}{d})_k^2}{(1)_k^2}\sum_{j=1}^{k}\left\{\frac{1}{(dj-d+r)^2}-\frac{1}{(dj)^2}\right\}\equiv 0 \pmod p,$$
$$\sum_{k=0}^{(p-r)/d}(-1)^k(2dk+r)\frac{(\frac{r}{d})_k^3}{(1)_k^3}\sum_{j=1}^{k}\left\{\frac{1}{(dj-d+r)^2}-\frac{1}{(dj)^2}\right\}\equiv 0 \pmod p,$$
$$\sum_{k=0}^{(p-r)/d}(2dk+r)\frac{(\frac{r}{d})_k^4}{(1)_k^4}\sum_{j=1}^{k}\left\{\frac{1}{(dj-d+r)^2}-\frac{1}{(dj)^2}\right\}\equiv 0 \pmod p,$$
$$\sum_{k=0}^{(p-r)/d}(-1)^k(2dk+r)\frac{(\frac{r}{d})_k^5}{(1)_k^5}\sum_{j=1}^{k}\left\{\frac{1}{(dj-d+r)^2}-\frac{1}{(dj)^2}\right\}\equiv 0 \pmod p,$$
$$\sum_{k=0}^{(p-r)/d}(2dk+r)\frac{(\frac{r}{d})_k^6}{(1)_k^6}\sum_{j=1}^{k}\left\{\frac{1}{(dj-d+r)^2}-\frac{1}{(dj)^2}\right\}\equiv 0 \pmod p.$$

Letting $n=p$ be an odd prime and then letting $q\to 1$ in Theorem 2, we are led to the following conclusion:

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac12)_k^3(\frac14)_k}{(1)_k^4 4^k}\sum_{j=1}^{k}\left\{\frac{1}{(4j)^2}-\frac{1}{(2j-1)^2}\right\}
\equiv\begin{cases} p\dfrac{(\frac12)_{(p-1)/4}}{(1)_{(p-1)/4}}\displaystyle\sum_{j=1}^{(p-1)/4}\frac{1}{(4j)^2} \pmod{p^2}, & \text{if } p\equiv 1 \pmod 4,\\ 0 \pmod{p^2}, & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

The rest of the paper is arranged as follows. We shall display the proof of Theorem 1 in Section 2. The proof of Theorem 2 will be provided in Section 3. Two conjectures will be proposed in Section 4.

2 Proof of Theorem 1

Following Gasper and Rahman [1], define the basic hypergeometric series to be

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix} ; q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(q, b_1, b_2, \ldots, b_r; q)_k} z^k. $$

Then Watson's ${}_8\phi_7$ transformation formula (cf. [1, Appendix (III.17)]) can be expressed as

$$ {}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix} ; q,\ \frac{a^2q^{n+2}}{bcde}\right] $$

$$ = \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n}\, {}_4\phi_3\left[\begin{matrix} aq/bc,\ d,\ e,\ q^{-n} \\ aq/b,\ aq/c,\ deq^{-n}/a \end{matrix} ; q,\ q\right]. \tag{2.1} $$

Now we begin to prove Theorem 1.

Proof. In terms of (2.1), we can catch hold of

$$ \sum_{k=0}^{(n-r)/d} [2dk+r] \frac{(q^r, xq^r, yq^r, zq^r, q^{r+n}, q^{r-n}; q^d)_k}{(q^d, q^d/x, q^d/y, q^d/z, q^{d-n}, q^{d+n}; q^d)_k} \frac{q^{(2d-3r)k}}{(xyz)^k} $$

$$ = [n](zq^r)^{(r-n)/d} \frac{(zq^{2r}; q^d)_{(n-r)/d}}{(q^d/z; q^d)_{(n-r)/d}} \sum_{k=0}^{(n-r)/d} \frac{(q^{d-r}/xy, zq^r, q^{r+n}, q^{r-n}; q^d)_k}{(q^d, q^d/x, q^d/y, zq^{2r}; q^d)_k} q^{dk}. $$

Then we can proceed as follows:

$$ \sum_{k=0}^{(n-r)/d} [2dk+r] \frac{(q^r; q^d)_k^3 (xq^r, yq^r, zq^r; q^d)_k}{(q^d; q^d)_k^3 (q^d/x, q^d/y, q^d/z; q^d)_k} \frac{q^{(2d-3r)k}}{(xyz)^k} $$

$$ - [n](zq^r)^{(r-n)/d} \frac{(zq^{2r}; q^d)_{(n-r)/d}}{(q^d/z; q^d)_{(n-r)/d}} \sum_{k=0}^{(n-r)/d} \frac{(q^{d-r}/xy, zq^r, q^{r+n}, q^{r-n}; q^d)_k}{(q^d, q^d/x, q^d/y, zq^{2r}; q^d)_k} q^{dk} $$

$$ = \sum_{k=0}^{(n-r)/d} [2dk+r] \frac{(q^r; q^d)_k^3 (xq^r, yq^r, zq^r; q^d)_k}{(q^d; q^d)_k^3 (q^d/x, q^d/y, q^d/z; q^d)_k} \frac{q^{(2d-3r)k}}{(xyz)^k} $$

$$ - \sum_{k=0}^{(n-r)/d} [2dk+r] \frac{(q^r, xq^r, yq^r, zq^r, q^{r+n}, q^{r-n}; q^d)_k}{(q^d, q^d/x, q^d/y, q^d/z, q^{d-n}, q^{d+n}; q^d)_k} \frac{q^{(2d-3r)k}}{(xyz)^k} $$

$$ = \sum_{k=0}^{(n-r)/d} [2dk+r] \frac{(q^r, xq^r, yq^r, zq^r; q^d)_k}{(q^d, q^d/x, q^d/y, q^d/z; q^d)_k} \frac{q^{(2d-3r)k}}{(xyz)^k} $$

$$ \times \frac{(q^r; q^d)_k^2 (q^{d+n}, q^{d-n}; q^d)_k - (q^d; q^d)_k^2 (q^{r+n}, q^{r-n}; q^d)_k}{(q^d; q^d)_k^2 (q^{d+n}, q^{d-n}; q^d)_k}. \tag{2.2} $$

Noticing that $1-q^n\equiv 0 \pmod{\Phi_n(q)}$, we find

$$\begin{aligned}(q^{d+n},q^{d-n};q^d)_k&=\prod_{j=1}^{k}(1-q^{dj+n})(1-q^{dj-n})\\&=\prod_{j=1}^{k}\Big\{(1-q^{dj})^2-(1-q^n)^2q^{dj-n}\Big\}\\&\equiv (q^d;q^d)_k^2-(q^d;q^d)_k^2\sum_{j=1}^{k}\frac{(1-q^n)^2}{(1-q^{dj})^2}q^{dj-n}\pmod{\Phi_n(q)^4}.\end{aligned}$$

Similarly, we can discover

$$(q^{r+n},q^{r-n};q^d)_k\equiv (q^r;q^d)_k^2-(q^r;q^d)_k^2\sum_{j=1}^{k}\frac{(1-q^n)^2}{(1-q^{dj-d+r})^2}q^{dj-d+r-n}\pmod{\Phi_n(q)^4}. \tag{2.3}$$

The combination of the last two equation engenders

$$\begin{aligned}&(q^r;q^d)_k^2(q^{d+n},q^{d-n};q^d)_k-(q^d;q^d)_k^2(q^{r+n},q^{r-n};q^d)_k\\&\equiv (q^r;q^d)_k^2(q^d;q^d)_k^2[n]^2\sum_{j=1}^{k}\left(\frac{q^{dj-d+r-n}}{[dj-d+r]^2}-\frac{q^{dj-n}}{[dj]^2}\right)\pmod{\Phi_n(q)^4}.\end{aligned}\tag{2.4}$$

The $(a,b,c,e)\to(1,1/z,x,y)$ case of Liu and Wang [10, Lemma 2]) reads

$$\begin{aligned}&\sum_{k=0}^{(n-r)/d}[2dk+r]\frac{(q^r;q^d)_k^3(xq^r,yq^r,zq^r;q^d)_k}{(q^d;q^d)_k^3(q^d/x,q^d/y,q^d/z;q^d)_k}\frac{q^{(2d-3r)k}}{(xyz)^k}\\&\equiv [n](zq^r)^{(r-n)/d}\frac{(zq^{2r};q^d)_{(n-r)/d}}{(q^d/z;q^d)_{(n-r)/d}}\\&\quad\times\sum_{k=0}^{(n-r)/d}\frac{(q^{d-r}/xy,zq^r,q^r,q^r;q^d)_k}{(q^d,q^d/x,q^d/y,zq^{2r};q^d)_k}q^{dk}\pmod{\Phi_n(q)^3}.\end{aligned}\tag{2.5}$$

Substituting (2.4) and (2.5) into (2.2), we obtain

$$\begin{aligned}&[n](zq^r)^{(r-n)/d}\frac{(zq^{2r};q^d)_{(n-r)/d}}{(q^d/z;q^d)_{(n-r)/d}}\sum_{k=0}^{(n-r)/d}\frac{(q^{d-r}/xy,zq^r,q^r,q^r;q^d)_k}{(q^d,q^d/x,q^d/y,zq^{2r};q^d)_k}q^{dk}\\&-[n](zq^r)^{(r-n)/d}\frac{(zq^{2r};q^d)_{(n-r)/d}}{(q^d/z;q^d)_{(n-r)/d}}\sum_{k=0}^{(n-r)/d}\frac{(q^{d-r}/xy,zq^r,q^{r+n},q^{r-n};q^d)_k}{(q^d,q^d/x,q^d/y,zq^{2r};q^d)_k}q^{dk}\end{aligned}$$

$$
\begin{aligned}
&= [n]^2 \sum_{k=0}^{(n-r)/d} [2dk+r] \frac{(q^r, xq^r, yq^r, zq^r; q^d)_k}{(q^d, q^d/x, q^d/y, q^d/z; q^d)_k} \frac{q^{(2d-3r)k}}{(xyz)^k} \\
&\quad \times \frac{(q^r; q^d)_k^2}{(q^{d+n}, q^{d-n}; q^d)_k} \sum_{j=1}^{k} \left(\frac{q^{dj-d+r-n}}{[dj-d+r]^2} - \frac{q^{dj-n}}{[dj]^2} \right) \pmod{\Phi_n(q)^3}.
\end{aligned}
$$

Multiplying both sides by q^n and using $(q^{r+n}, q^{r-n}; q^d)_k \equiv (q^r; q^d)_k^2 \pmod{\Phi_n(q)^2}$, we derive Theorem 1. ∎

3 Proof of Theorem 2

In order to prove Theorem 2, we require Gasper and Rahman's summation formula for basic hypergeometric series (cf. [1, Equation (3.8.12)]):

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1-aq^{3k}}{1-a} \frac{(a, b, q/b; q)_k (d, f, a^2q/df; q^2)_k}{(q^2, aq^2/b, abq; q^2)_k (aq/d, aq/f, df/a; q)_k} q^k \\
&\quad + \frac{(aq, f/a, b, q/b; q)_\infty (d, aq^2/df, fq^2/d, df^2q/a^2; q^2)_\infty}{(a/f, fq/a, aq/d, df/a; q)_\infty (aq^2/b, abq, fq/ab, bf/a; q^2)_\infty} \\
&\quad \times {}_3\phi_2 \left[\begin{matrix} f, bf/a, fq/ab \\ fq^2/d, df^2q/a^2 \end{matrix} ; q^2, q^2 \right] \\
&\quad = \frac{(aq, f/a; q)_\infty (aq^2/bd, abq/d, bdf/a, dfq/ab; q^2)_\infty}{(aq/d, df/a; q)_\infty (aq^2/b, abq, bf/a, fq/ab; q^2)_\infty}.
\end{aligned} \tag{3.1}
$$

Now we start to prove Theorem 2.

Proof. Setting $d = q^{-2n}$ and then letting $n \to \infty$ in (3.1), we get

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1-aq^{3k}}{1-a} \frac{(a, b, q/b; q)_k (f; q^2)_k}{(q^2, aq^2/b, abq; q^2)_k (aq/f; q)_k} q^{\frac{k^2+k}{2}} \left(\frac{a}{f}\right)^k \\
&\quad = \frac{(aq, aq^2, aq^2/bf, abq/f; q^2)_\infty}{(aq/f, aq^2/f, aq^2/b, abq; q^2)_\infty}.
\end{aligned} \tag{3.2}
$$

When $n \equiv 1 \pmod 4$, the last equation gives

$$
\sum_{k=0}^{n-1} [6k+1] \frac{(q, q^{1+n}, q^{1-n}; q^2)_k (q; q^4)_k}{(q^4, q^{4-n}, q^{4+n}; q^4)_k (q^2; q^2)_k} q^{k^2+k} = [n] q^{(1-n)/4} \frac{(q^2; q^4)_{(n-1)/4}}{(q^4; q^4)_{(n-1)/4}}.
$$

Then we have

$$\begin{aligned}&\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^2)_k^3(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k}-[n]q^{(1-n)/4}\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}\\&\quad=\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^2)_k^3(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k}-\sum_{k=0}^{n-1}[6k+1]\frac{(q,q^{1+n},q^{1-n};q^2)_k(q;q^4)_k}{(q^4,q^{4-n},q^{4+n};q^4)_k(q^2;q^2)_k}q^{k^2+k}\\&\quad=\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^2)_k(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_k}q^{k^2+k}\\&\qquad\times\frac{(q;q^2)_k^2(q^{4+n},q^{4-n};q^4)_k-(q^4;q^4)_k^2(q^{1+n},q^{1-n};q^2)_k}{(q^4;q^4)_k^2(q^{4+n},q^{4-n};q^4)_k}.\end{aligned}\tag{3.3}$$

Via (2.3), there hold

$$(q^{4+n},q^{4-n};q^4)_k\equiv(q^4;q^4)_k^2-(q^4;q^4)_k^2\sum_{j=1}^{k}\frac{(1-q^n)^2}{(1-q^{4j})^2}q^{4j-n}\pmod{\Phi_n(q)^4},$$

$$(q^{1+n},q^{1-n};q^2)_k\equiv(q;q^2)_k^2-(q;q^2)_k^2\sum_{j=1}^{k}\frac{(1-q^n)^2}{(1-q^{2j-1})^2}q^{2j-1-n}\pmod{\Phi_n(q)^4}.$$

Combing the last two equations, we conclude that

$$\begin{aligned}&(q;q^2)_k^2(q^{4+n},q^{4-n};q^4)_k-(q^4;q^4)_k^2(q^{1+n},q^{1-n};q^2)_k\\&\quad\equiv(q;q^2)_k^2(q^4;q^4)_k^2[n]^2\sum_{j=1}^{k}\left(\frac{q^{2j-1-n}}{[2j-1]^2}-\frac{q^{4j-n}}{[4j]^2}\right)\pmod{\Phi_n(q)^4}.\end{aligned}\tag{3.4}$$

A q-supercongruence due to Wei [17, Theorem 1.3]) can be expressed as

$$\begin{aligned}&\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^2)_k^3(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_k^3}q^{k^2+k}\\&\quad\equiv[n]q^{(1-n)/4}\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}\left\{1-[n]^2\sum_{j=1}^{(n-1)/4}\frac{q^{4j}}{[4j]^2}\right\}\pmod{[n]\Phi_n(q)^3}.\end{aligned}\tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we arrive at

$$\begin{aligned}&\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^2)_k(q;q^4)_k}{(q^2;q^2)_k(q^4;q^4)_k}q^{k^2+k}\frac{(q;q^2)_k^2}{(q^{4+n},q^{4-n};q^4)_k}\sum_{j=1}^{k}\left(\frac{q^{4j-n}}{[4j]^2}-\frac{q^{2j-1-n}}{[2j-1]^2}\right)\\&\quad\equiv[n]q^{(1-n)/4}\frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}\sum_{j=1}^{(n-1)/4}\frac{q^{4j}}{[4j]^2}\pmod{\Phi_n(q)^2}.\end{aligned}$$

Multiplying both sides by q^n and utilizing $(q^{4+n}, q^{4-n}; q^4)_k \equiv (q^4; q^4)_k^2 \pmod{\Phi_n(q)^2}$, we deduce the $n \equiv 1 \pmod 4$ case of Theorem 2.

When $n \equiv 3 \pmod 4$, equation (3.2) produces

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q, q^{1+n}, q^{1-n}; q^2)_k (q; q^4)_k}{(q^4, q^{4-n}, q^{4+n}; q^4)_k (q^2; q^2)_k} q^{k^2+k} = 0.$$

A q-supercongruence due to Wei [17, Theorem 1.1]) can be written as

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q; q^2)_k^3 (q; q^4)_k}{(q^2; q^2)_k (q^4; q^4)_k^3} q^{k^2+k} \equiv 0 \pmod{[n]\Phi_n(q)^3}.$$

The similar argument leads to the $n \equiv 3 \pmod 4$ case of Theorem 2. □

4 Two open problems

Numerical calculations indicate the following two open problems related to Corollaries 1-6.

Conjecture 1. *Let $n > 1$, $d > 1$, $r > 0$, $m \geq 0$ be integers with $\gcd(r, d) = 1$ and $n \equiv r \pmod d$ such that $n \geq r$. Then*

$$\sum_{k=0}^{(n-r)/d} (-1)^k [2dk+r]\frac{(q^r; q^d)_k^{2m+1}}{(q^d; q^d)_k^{2m+1}}\, q^{d\binom{k}{2}+m(d-r)k} \sum_{j=1}^{k}\left(\frac{q^{dj-d+r}}{[dj-d+r]^2} - \frac{q^{dj}}{[dj]^2}\right)$$
$$\equiv 0 \pmod{\Phi_n(q)}.$$

Conjecture 2. *Let $n > 1$, $d > 1$, $r > 0$, $m \geq 0$ be integers with $\gcd(r, d) = 1$ and $n \equiv r \pmod d$ such that $n \geq r$. Then*

$$\sum_{k=0}^{(n-r)/d} [2dk+r]\frac{(q^r; q^d)_k^{2m+2}}{(q^d; q^d)_k^{2m+2}}\, q^{m(d-r)k-rk} \sum_{j=1}^{k}\left(\frac{q^{dj-d+r}}{[dj-d+r]^2} - \frac{q^{dj}}{[dj]^2}\right)$$
$$\equiv 0 \pmod{\Phi_n(q)}.$$

Letting $n = p$ be an odd prime with $p \equiv r \pmod d$ and $\gcd(r, d) = 1$ such that $p \geq r$ and then letting $q \to 1$ in Conjectures 1 and 2, we are led to the following congruences:

$$\sum_{k=0}^{(p-r)/d} (-1)^k (2dk+r)\frac{(\frac{r}{d})_k^{2m+1}}{(1)_k^{2m+1}} \sum_{j=1}^{k}\left\{\frac{1}{(dj-d+r)^2} - \frac{1}{(dj)^2}\right\} \equiv 0 \pmod p,$$
$$\sum_{k=0}^{(p-r)/d} (2dk+r)\frac{(\frac{r}{d})_k^{2m+2}}{(1)_k^{2m+2}} \sum_{j=1}^{k}\left\{\frac{1}{(dj-d+r)^2} - \frac{1}{(dj)^2}\right\} \equiv 0 \pmod p.$$

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