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On perturbation theory of semi Fredholm operators involving the measure of non strict singularity

by

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Abstract

This paper is concerned with stability analysis of left and right Atkinson types of unbounded linear operators on Banach spaces by using the measure of non strict singularity. Under quite general assumptions, we first present a new characterization on the left (right) Weyl spectra of unbounded linear operators which will be used to prove certain stability properties of left (right) Weyl operators with perturbation. Therefore, a new description of left (right) Weyl spectra of unbounded operator matrices with non maximal domain is presented to reach the validity of the theoretical results which led to significant advances in the spectral theory of operators matrices.

Key Words: Operator matrices, measure of non strict singularity, left (right) Weyl spectra.
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1 Introduction

The theory of the essential spectra of linear operators in Banach space is a modern section of the special analysis widely used in the mathematical and physical sense when resolving a number of application that can be formulated in terms of linear operators. More precisely, the perturbation theory of semi-Fredholm and Fredholm of linear operators in Banach spaces was done by different authors, see [3, 5, 11, 12, 14, 20]. More general than these classes of Fredholm operators are the classes of semi Fredholm operators which have complemented range and null space, which are namely known as the classes of the left and right Fredholm operators or left and right Atkinson operators. In the literature, the investigation of this kind of classes attracted the attention of many mathematicians and proved a vast body of important results in the theory of Fredholm operators. An interested reader can find some basic information on this kind of Fredholm theory in some works funded by G. Gonzalez in [10], R. Harte in [11], V. Müller in [20] or S. Č. Živković-Žlatanović in [28, 29].

On the other side, the notion of measure of non strict singularity, $\Delta(.)$, has witnessed an explosive development in spectral theory. Many years later, this kind of measure has been enriched in the literature by presenting a real development of some powerful methods for the study of some spectral problems such that the problem of the characterization and the stability of Schechter essential spectrum. This concept of study was performed via many subject of important works and was familiar to many mathematicians, we can quote especially the papers of [1, 2, 18, 19]. More precisely, in [18], N. Moalla used this class of perturbations in order to extend some earlier results of the characterization of the Weyl spectrum, $\sigma_w(.)$. In fact, the author proves, for $T \in \mathcal{C}(X)$, that:

$$\sigma_w(T) := \bigcap_{K \in \mathcal{S}_T(X)} \sigma(T+K) := \bigcap_{K \in \mathcal{D}_T(X)} \sigma(T+K),$$

where

$$\mathcal{S}_T(X) := \left\{K \in \mathcal{L}(X) : \Delta([(\lambda - T - K)^{-1}K]^n) < 1, \text{ for some } n \in \mathbb{N}^* \text{ and } \forall \lambda \in \rho(T+K)\right\}$$

and

$$\mathcal{D}_T(X) := \left\{K \in \mathcal{L}(X) : \Delta([K(\lambda - T - K)^{-1}]^n) < 1, \text{ for some } n \in \mathbb{N}^* \text{ and } \forall \lambda \in \rho(T+K)\right\}.$$

Motivated by the works mentioned above and keep the interesting, in the first part of this paper, of the investigation of left and right Weyl spectra in the theory of linear operators. Indeed, the use of this notion of measure remains a powerful tool in spectral theory since this class is not a two-sided closed ideal of the set of bounded operators. So, we aim in this paper to use this kind of the measure in order to enlarge some known results that are widely studied in [18]. In particular, we are interested to characterize the left and right Weyl spectra of closed, densely defined linear operator in Banach space as well as to develop some spectral properties of left and right Weyl operators via this kind of measure and the definition of the left and right Atkinson linear operators and their properties. In fact, we prove for $n \in \mathbb{N}^*$ and $T \in \mathcal{C}(X)$, that:

$$\sigma_w^l(T) := \bigcap_{K \in S_n^l(X)} \sigma^l(T+K) \quad \text{and} \quad \sigma_w^r(T) := \bigcap_{K \in S_n^r(X)} \sigma^r(T+K)$$

where:

$$S_n^l(X) := \{K \in \mathcal{L}(X) \ : \ \Delta([(\mu I - T - K)^l K]^n) < 1, \ \forall \, \mu \in \rho^l(T+K)\}$$

and

$$S_n^r(X) := \{K \in \mathcal{L}(X) \ : \ \Delta([K(\mu I - T - K)^r]^n) < 1, \ \forall \, \mu \in \rho^r(T+K)\}.$$

In the last section, we will apply the theoretical results obtained in Section 3 to provide a new description of some essential spectra in the theory of unbounded block 2×2 operator matrix with non maximal domain on the product of Banach spaces $X \times Y$ considered as the form:

$$\mathcal{A}_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for given unbounded operators entries and defined with non maximal domain that is with domain containing one supplemented condition relating their components entries with linear operators Ψ_X and Ψ_Y as:

$$\mathcal{D}(\mathcal{A}_0) := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in (\mathcal{D}(A) \cap \mathcal{D}(C)) \times (\mathcal{D}(B) \cap \mathcal{D}(D)) : \ \Psi_X f = \Psi_Y g \right\},$$

where the linear operators Ψ_X and Ψ_Y are defined as:

$$\Psi_X : X \longrightarrow Z \ \text{ and } \ \Psi_Y : Y \longrightarrow Z, \quad \text{for Banach space } Z.$$

Our central interest in Section 4 is to address a fine characterization on the left-right Weyl spectra of an unbounded operator matrix with non maximal domain and we gather some conditions that we must impose on their component entries, in order to improve and ameliorate under weaker assumptions some results recently obtained by the authors in [6, 19]. Specifically, we prove via this class of measure that:

$$\sigma_w^*(\mathcal{A}) = \sigma_w^*(A_1) \cup \sigma_w^*(D),$$

where A_1 denotes the restriction of A to $\mathcal{D}(A) \cap N(\Psi_X)$ and $\sigma_w^*(.) = \{\sigma_w^l(.), \sigma_w^r(.)\}$.

To make the paper easily accessible, we proceed as follows:
In Section 2, we shall some classical definitions and properties of linear operators are needed throughout the paper. Section 3 is dedicated to provide the perturbation problem of left and right Weyl of linear operators based on the theory of the measure of non strict singularity. The main results of this section are Theorems 3.2, 3.6 and 3.7. We apply the results obtained in the previous section, to study in Section 4, the stability and to give a new description of left and right Weyl spectra of unbounded operator matrix with non maximal domain involving the class of measure of non strictly singular perturbation.

2 Notations and basic definitions

To outline the main topics of this paper, we need firstly to introduce some standard notations and definitions.

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X, Y)$ (resp. $\mathcal{C}(X, Y)$) the set of all bounded (resp. closed, densely defined) linear operators from X into Y. The subspace of all compact operators of $\mathcal{L}(X, Y)$ is designed by $\mathcal{K}(X, Y)$. For $T \in \mathcal{C}(X, Y)$, we write $N(T) \subset X$ and $R(T)$ for the null-space and $R(T) \subset Y$ for the range of T. The nullity, $\alpha(T)$, of T is defined as the dimension of $N(T)$ and the deficiency $\beta(T)$ of T is defined as the codimension of $R(T)$.

Let $\mathcal{G}^l(X, Y)$ (resp. $\mathcal{G}^r(X, Y)$) denote the set of all left (resp. right) invertible operators from X into Y. It is well-known that $T \in \mathcal{G}^l(X, Y)$ (resp. $T \in \mathcal{G}^r(X, Y)$) if and only if T is injective and $R(T)$ is a closed and complemented subspace of Y (resp. T is onto and $N(T)$ is a complemented subspace of X).

The set of upper (resp. lower) semi-Fredholm operators from X into Y is defined as:

$$\Phi_+(X, Y) := \{T \in \mathcal{C}(X, Y) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } Y\}$$

(resp.

$$\Phi_-(X, Y) := \{T \in \mathcal{C}(X, Y) : \beta(T) < \infty\}).$$

Moreover, set of Fredholm (resp. semi-Fredholm) operators is defined by:

$$\Phi(X, Y) := \Phi_-(X, Y) \cap \Phi_+(X, Y), \quad (\text{resp. } \Phi_\pm(X, Y) := \Phi_-(X, Y) \cup \Phi_+(X, Y)).$$

For $T \in \Phi_\pm(X, Y)$, its index is defined by the quantity: $i(T) = \alpha(T) - \beta(T)$.

The set of left Fredholm (resp. right Fredholm) operators is defined as:

$$\Phi^l(X, Y) := \{T \in \Phi_+(X, Y) : \ R(T) \ \text{ is a complemented subset of } Y\}$$

(resp.

$$\Phi^r(X,Y) := \{T \in \Phi_-(X,Y): \ N(T) \ \text{ is a complemented subset of } X\}).$$

Clearly, the open sets $\Phi^l(X,Y)$ and $\Phi^r(X,Y)$ satisfying the following inclusions:

$$\Phi(X,Y) \subseteq \Phi^l(X,Y) \subseteq \Phi_+(X,Y) \quad \text{ and } \quad \Phi(X,Y) \subseteq \Phi^r(X,Y) \subseteq \Phi_-(X,Y).$$

The sets of left and right Weyl operators are defined by:

$$\mathcal{W}^l(X,Y) := \{T \in \mathcal{C}(X,Y) : T \in \Phi^l(X,Y) \text{ and } i(T) \leq 0\}$$

and

$$\mathcal{W}^r(X,Y) := \{T \in \mathcal{C}(X,Y) : T \in \Phi^r(X,Y) \text{ and } i(T) \geq 0\}.$$

Hence, the set of Weyl operators $\mathcal{W}(X,Y)$ is defined by:

$$\mathcal{W}(X,Y) := \mathcal{W}^l(X,Y) \cap \mathcal{W}^r(X,Y) := \{T \in \Phi(X,Y): \ i(T) = 0\}.$$

Remark 2.1. If $X = Y$, then the sets $\mathcal{L}(X,X)$, $\mathcal{C}(X,X)$, $\mathcal{K}(X,X)$, $\mathcal{G}^l(X,X)$, $\mathcal{G}^r(X,X)$, $\mathcal{G}(X,X)$, $\Phi_+(X,X)$, $\Phi_-(X,X)$, $\Phi^l(X,X)$, $\Phi^r(X,X)$, $\Phi(X,X)$, $\mathcal{W}^l(X,X)$, $\mathcal{W}^r(X,X)$ and $\mathcal{W}(X,X)$ are replaced, respectively, by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\mathcal{G}^l(X)$, $\mathcal{G}^r(X)$, $\mathcal{G}(X)$, $\Phi_+(X)$, $\Phi_-(X)$, $\Phi^l(X)$, $\Phi^r(X)$, $\Phi(X)$, $\mathcal{W}^l(X)$, $\mathcal{W}^r(X)$ and $\mathcal{W}(X)$.

Let $T \in \mathcal{C}(X)$. It follows from the closedness of T that $\mathcal{D}(T)$ endowed with the graph norm $\|.\|_T$ ($\|x\|_T = \|x\| + \|Tx\|$) is a Banach space denoted by X_T. Clearly, for $x \in \mathcal{D}(T)$ we have $\|Tx\| \leq \|x\|_T$. Therefore, $T \in \mathcal{L}(X_T, X)$. Let K be a linear operator on X. If $\mathcal{D}(T) \subset \mathcal{D}(K)$, then K will be called T-defined, its restriction to $\mathcal{D}(T)$ will be denoted by $\widehat{K}$. Moreover, if $\widehat{K} \in \mathcal{L}(X_T, X)$, we say that K is T-bounded.

Let K be an arbitrary T-bounded operator, hence we can consider T and K as operators from X_T into X, they are denoted by $\widehat{T}$ and $\widehat{K}$ respectively, they belong to $\mathcal{L}(X_T, X)$. Furthermore, we have the obvious relations:

$$\begin{cases} \alpha(\widehat{T}) = \alpha(T), \ \beta(\widehat{T}) = \beta(T), \ R(\widehat{T}) = R(T), \\ \alpha(\widehat{T} + \widehat{K}) = \alpha(T+K), \\ \beta(\widehat{T} + \widehat{K}) = \beta(T+K) \text{ and } R(\widehat{T} + \widehat{K}) = R(T+K). \end{cases} \tag{1}$$

Let $T \in \mathcal{C}(X)$, we define the resolvent set (resp. the spectrum) of T by:

$$\rho(T) := \{\lambda \in \mathbb{C} \ : \ \lambda I - T \text{ has a bounded inverse}\} \ \ (\text{resp. } \sigma(T) := \mathbb{C}\backslash\rho(T))$$

and the left (resp. right) spectrum of T by:

$$\sigma^l(T) := \{\lambda \in \mathbb{C} : \ \lambda I - T \notin \mathcal{G}^l(X)\} := \mathbb{C}\backslash\rho^l(T)$$

(resp.

$$\sigma^r(T) := \{\lambda \in \mathbb{C} : \ \lambda I - T \notin \mathcal{G}^r(X)\} := \mathbb{C}\backslash\rho^r(T)).$$

In this research work, we are basically concerned with the following essential spectra introduced in [7]:

Definition 2.2. Let $T \in \mathcal{C}(X)$. We define:
(i) the left Weyl spectrum by:

$$\sigma_w^l(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma^l(T+K).$$

(ii) the right Weyl spectrum by:

$$\sigma_w^r(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma^r(T+K).$$

Following Definition 2.2, we deduce the Weyl spectrum as:

$$\sigma_w(T) := \sigma_w^l(T) \cup \sigma_w^r(T).$$

The following result gives a characterization of left and right Weyl spectra by means of left and right Fredholm operators.

Proposition 2.3. [7] Let $T \in \mathcal{C}(X)$, then

$$\lambda \notin \sigma_w^l(T) \iff \lambda I - T \in \mathcal{W}^l(X) \iff \lambda I - T \in \Phi^l(X) \text{ and } i(\lambda I - T) \leq 0.$$

$$\lambda \notin \sigma_w^r(T) \iff \lambda I - T \in \mathcal{W}^r(X) \iff \lambda I - T \in \Phi^r(X) \text{ and } i(\lambda I - T) \geq 0.$$

The obvious Weyl spectra satisfies the following inclusions:

$$\sigma_{ef}^l(T) \subseteq \sigma_w^l(T) \subseteq \sigma_w(T) \quad \text{and} \quad \sigma_{ef}^r(T) \subseteq \sigma_w^r(T) \subseteq \sigma_w(T),$$

where the left (resp. right)-Fredholm spectrum of T, denoted by $\sigma_{ef}^l(T)$ (resp. $\sigma_{ef}^r(T)$), is defined as $\sigma_{ef}^l(T) := \mathbb{C}\backslash\Phi_T^l(X)$ (resp. $\sigma_{ef}^r(T) := \mathbb{C}\backslash\Phi_T^r(X)$), which $\Phi_T^*(X) := \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi^*(X)\}$, for $\Phi^*(X) = \{\Phi^l(X), \Phi^r(X)\}$.

At this level of analysis, we shall recall some basic definitions for bounded operators on Banach spaces that are useful in the remainder of this paper.

Definition 2.4. An operator $T \in \mathcal{L}(X,Y)$ is said to be strictly singular from X into Y if the restriction of T to any infinite-dimensional subspace of X is not an homeomorphism.

Let $\mathcal{SS}(X,Y)$ denote the set of strictly singular operators from X to Y. The concept of strictly singular operators was introduced in the pioneering paper by T. Kato [15]. In general, strictly singular operators are not compact (see [9, 15]). Note that, $\mathcal{SS}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$. If $X = Y$, $\mathcal{SS}(X) := \mathcal{SS}(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If X is a separable Hilbert space, then $\mathcal{SS}(X) = \mathcal{K}(X)$. For basic properties of strictly singular operators, we refer readers to [9, 17, 26, 27].

Definition 2.5. [16] Let $T : \mathcal{D}(T) \subset X \longrightarrow Y$ be a linear operator.
If T is closed and S is a closable linear operator with domain $\mathcal{D}(S)$ such that $\overline{S} = T$, then $\mathcal{D}(S)$ is called core of T.

3 A characterization of the left-right Weyl spectra involving measure of non strict singularity

The notion of measure of non strict singularity dating backs to 1972, was introduced in the pioneering paper of M. Schechter in [24] as generalization of the class of strictly singular operators as follows:

Definition 3.1. For $T \in \mathcal{L}(X)$, we define the measure of non strict singularity of T, denoted by $\Delta(T)$ as:

$$\Delta_M(T) := \inf_{N \subset M} \gamma(T|_N) \text{ and } \Delta(T) := \sup_{M \subset X} \Delta_M(T),$$

where M, N represent infinite dimensional subspace of X and $T|_N$ denotes the restriction of T with the subspace N and $\gamma(.)$ express the Hausdorff measure of non compactness of linear bounded operators.

In the following, we give a refinement characterization of left and right Weyl spectra of closed densely defined linear operators involving this concept of measure of non strict singularity.

Theorem 3.2. Let $T \in \mathcal{C}(X)$. Then, the following assertions hold, for some $n \in \mathbb{N}^*$:
(i)

$$\sigma_w^l(T) := \bigcap_{K \in S_n^l(X)} \sigma^l(T+K),$$

where:

$$S_n^l(X) := \{K \in \mathcal{L}(X) \ : \ \Delta([(\mu I - T - K)^l K]^n) < 1, \ \forall\, \mu \in \rho^l(T+K)\}.$$

(ii)

$$\sigma_w^r(T) := \bigcap_{K \in S_n^r(X)} \sigma^r(T+K),$$

where:

$$S_n^r(X) := \{K \in \mathcal{L}(X) \ : \ \Delta([K(\mu I - T - K)^r]^n) < 1, \ \forall\, \mu \in \rho^r(T+K)\}.$$

Proof.
(i) Based on the fact that $\mathcal{K}(X) \subset S_n^l(X)$, we infer that

$$\bigcap_{K \in S_n^l(X)} \sigma^l(T+K) \subset \bigcap_{K \in \mathcal{K}(X)} \sigma^l(T+K) := \sigma_w^l(T).$$

To prove the reverse inclusion, let us consider $\lambda \notin \bigcap_{K \in S_n^l(X)} \sigma^l(T+K)$. Thus, there exists $K \in S_n^l(X)$ such that $\lambda \in \rho^l(T+K)$. Then, there exist $(\lambda I - T - K)^l \in \mathcal{L}(X, \mathcal{D}(T))$ such that

$$(\lambda I - T - K)^l (\lambda I - T - K) = I_{\mathcal{D}(T)}.$$

This shows that

$$(\lambda I - T - K)^l (\lambda I - T) = I_{\mathcal{D}(T)} + (\lambda I - T - K)^l K \subset I_X + (\lambda I - T - K)^l K.$$

Since $K \in S_n^l(X)$, we infer that $\Delta([(\lambda I - T - K)^l K]^n) < 1$. According to Proposition 2.3 in [18], we obtain $I_X + (\lambda I - T - K)^l K \in \Phi(X)$ with $i(I + (\lambda I - T - K)^l K) = 0$. So, we deduce that $(\lambda I - T - K)^l (\lambda I - T) \in \Phi(X) \subset \Phi^l(X)$. Consequently, the use of Theorem 2.3 in [10], allows us to conclude that

$$\lambda I - T \in \Phi^l(X).$$

Since $T + K \in \mathcal{C}(X)$, we can make $\mathcal{D}(T + K) = \mathcal{D}(T)$ into a Banach space by equipping it with the graph norm $\|.\|_T$. Let $X_T = (\mathcal{D}(T), \|.\|_T)$ be the Banach space for the graph norm $\|.\|_T$. Hence, we can regard T as an operator from X_T into X. This will be denoted by $\widehat{T}$. Clearly, $\widehat{T}+\widehat{K}$ and $\widehat{K}$ are bounded operators from X_T into X and $(\lambda I-\widehat{T}-\widehat{K})^l \in \mathcal{L}(X, X_T)$. Clearly,

$$(\lambda I - \widehat{T} - \widehat{K})^l (\lambda I - \widehat{T} - \widehat{K}) = I_{X_T}$$

is a Fredholm operator satisfying

$$i((\lambda I - \widehat{T} - \widehat{K})^l (\lambda I - \widehat{T} - \widehat{K})) = 0.$$

So, one has Eq.(1), we have:

$$i((\lambda I - \widehat{T} - \widehat{K})^l (\lambda I - \widehat{T})) = i((\lambda I - T - K)^l (\lambda I - T)).$$

It is clear that $(\lambda I - \widehat{T} - \widehat{K})^l$ is a Fredholm operator, if and only if, $(\lambda I - \widehat{T} - \widehat{K})$ is a Fredholm operator, if and only if, $(\lambda I - \widehat{T})$ is too. Thus, to reach this Fredholmness condition, we discuss two cases:

Case I: $\lambda I - \widehat{T} \in \Phi^l(X_T, X) \setminus \Phi(X_T, X)$. In this case, we derive from Eq. (1), that $\alpha(\lambda I - T) = 0$ and $\beta(\lambda I - T) = +\infty$. So,

$$i(\lambda I - T) = -\infty \leq 0.$$

Case II: $\lambda I - \widehat{T} \in \Phi(X_T, X)$. Combining Theorem 5.13 in [23] with the fact that $(\lambda I - \widehat{T} - \widehat{K})^l (\lambda I - \widehat{T})$ is a Fredholm operator on X_T, we deduce that $(\lambda I - \widehat{T} - \widehat{K})^l \in \Phi(X, X_T)$. According to Eq. (1), we get

$$i(\lambda I - T) = i(\lambda I - \widehat{T}) = i(\lambda I - \widehat{T} - \widehat{K}) = i(\lambda I - T - K) \leq 0.$$

So, Proposition 2.3 reveals that $\lambda \notin \sigma_w^l(T)$.

(ii) Since $\mathcal{K}(X) \subset S_n^r(X)$, we infer that

$$\bigcap_{K \in S_n^r(X)} \sigma^r(T + K) \subset \bigcap_{K \in \mathcal{K}(X)} \sigma^r(T + K) := \sigma_w^r(T).$$

Next, we may prove the reverse inclusion:
Indeed, let $\lambda \notin \bigcap_{K \in S_n^r(X)} \sigma^r(T + K)$, then there exists $K \in S_n^r(X)$ such that $\lambda \notin \sigma^r(T + K)$. That is, there exists $K \in S_n^r(X)$ such that $\lambda I - T - K \in \mathcal{G}^r(X)$. According to Eq. (1), we conclude that there exists $(\lambda I - \widehat{T} - \widehat{K})^r \in \mathcal{L}(X, X_T)$ such that

$$(\lambda I - \widehat{T})(\lambda I - \widehat{T} - \widehat{K})^r = I_X + \widehat{K}(\lambda I - \widehat{T} - \widehat{K})^r.$$

As $\widehat{K} \in S_n^r(X_T, X)$, Proposition 2.3 in [18] gives that $(\lambda I - \widehat{T})(\lambda I - \widehat{T} - \widehat{K})^r \in \Phi(X)$. Using Theorem 2.3 [10] together with the fact that $R((\lambda I - \widehat{T} - \widehat{K})^r) \subset \mathcal{D}(T)$, $(\lambda I - \widehat{T} - \widehat{K})^r(\lambda I - \widehat{T})$ and $\widehat{K}(\lambda I - \widehat{T} - \widehat{K})^r(\lambda I - \widehat{T})$ are continuous, we infer that

$$\lambda I - \widehat{T} \in \Phi^r(X_T, X).$$

In the following, we will discuss two cases to regard the index of the operator $\lambda I - \widehat{T}$:

Case I: $\lambda I - \widehat{T} \in \Phi^r(X_T, X) \setminus \Phi(X_T, X)$. Then, by Eq. (1) we deduce in the same ways as the previous item that:

$$i(\lambda I - T) = i(\lambda I - \widehat{T}) = +\infty \geq 0.$$

Case II: $\lambda I - \widehat{T} \in \Phi(X_T, X)$. In such case, the use of Theorem 5.13 in [23] revels that $(\lambda I - \widehat{T} - \widehat{K})^r \in \Phi(X, X_T)$. Hence, by Eq. (1), we infer that

$$i(\lambda I - T) = i(\lambda I - \widehat{T}) = i(\lambda I - \widehat{T} - \widehat{K}) = i(\lambda I - T - K) \geq 0.$$

Which completes the proof of theorem. □

Remark 3.3. **(i)** $SS(X) \subset S_n^l(X) \cap S_n^r(X)$.

(ii) The study of the stability and the invariance problem of left (resp. right) Weyl spectrum of linear operators under the set of $S_n^l(X)$ (resp. $S_n^r(X)$) seems not to be fulfilled, since these sets are not a two-sided closed ideal of the set of bounded operators. This bring us to introduce the following subset of $S_n^l(X)$ (resp. $S_n^r(X)$) containing $\mathcal{SS}(X)$ to reach the validity of this problem as:

$$\Omega_n^l(X, Y) := \{T \in \mathcal{L}(X, Y) \ : \ \Delta((KT)^n) < 1, \ \forall K \in \mathcal{L}(Y, X)\}$$

(resp.

$$\Omega_n^r(X, Y) := \{T \in \mathcal{L}(X, Y) \ : \ \Delta((TK)^n) < 1, \ \forall K \in \mathcal{L}(Y, X)\}),$$

for $n \in \mathbb{N}^*$.

(iii) If $X = Y$, we denote the sets $\Omega_n^l(X, X) := \Omega_n^l(X)$ and $\Omega_n^r(X, X) := \Omega_n^r(X)$.

In order to give a precise description of left and right Fredholm operators, we need first to establish an useful lemma.

Lemma 3.4. Let X and Y be two Banach spaces. Then, for $n \in \mathbb{N}^*$, we have:

(i) $T \in \Omega_n^l(X, Y)$ and $K \in \mathcal{L}(Y, X)$ imply $KT \in \Omega_n^l(X)$.
$T \in \Omega_n^l(X, Y)$ and $K \in \mathcal{SS}(X, Y)$ imply $T + S \in \Omega_n^l(X, Y)$.

(ii) $T \in \Omega_n^r(X, Y)$ and $K \in \mathcal{L}(Y, X)$ imply $KT \in \Omega_n^r(Y)$.
$T \in \Omega_n^r(X, Y)$ and $K \in \mathcal{SS}(X, Y)$ imply $T + S \in \Omega_n^r(X, Y)$. □

Proof.

(i) The first implication can be easily obtained from the definition of the set $\Omega_n^r(X)$.
Now, we will prove: $\Omega_n^l(X, Y) + \mathcal{SS}(X, Y) \subset \Omega_n^l(X, Y)$.

Indeed, let $T \in \Omega_n^l(X,Y)$ and $S \in \mathcal{SS}(X,Y)$.
A short computation reveals, for $n \in \mathbb{N}^*$ and $K \in \mathcal{L}(Y,X)$, that:

$$\begin{array}{rcl}\Delta((K(T+S))^n) & = & \Delta((KT+KS)^n) \\ & = & \Delta((KT)^n + n(KT)^{n-1}KS + ... + nKT(KS)^{n-1} + (KS)^n) \\ & = & \Delta((KT)^n + M(S)).\end{array}$$

While $M(S)$ is strictly singular operator. Thus, we conclude by the property of the stability the strict singularity, that:

$$\Delta((K(T+S))^n) = \Delta((KT)^n) < 1.$$

Which asserts the desired result.
(ii) The results of this assertion follows in the same way as item **(i)**. □

The second main result of this subsection shows that the classes $\mathcal{W}^l(X)$ and $\mathcal{W}^r(X)$ are stable under small perturbations.

Proposition 3.5. Let $T \in \mathcal{C}(X)$. Hence, we have the following assertions for some $n \in \mathbb{N}^*$:
(i) Assume that $K \in \Omega_n^l(X)$, then:

$$T \in \mathcal{W}^l(X) \quad \text{if and only if} \quad T+K \in \mathcal{W}^l(X).$$

(ii) Assume that $K \in \Omega_n^r(X)$, then:

$$T \in \mathcal{W}^r(X) \quad \text{if and only if} \quad T+K \in \mathcal{W}^r(X).$$

Proof.
(i) Let $T \in \mathcal{W}^l(X)$. Since $T \in \Phi^l(X)$, then we infer from Corollary 1.52 in [3] that there exist $T^l \in \mathcal{L}(X, \mathcal{D}(T))$ and $F \in \mathcal{K}(X)$ such that

$$T^lT = I - F \text{ on } \mathcal{D}(T) \quad \text{with} \quad i(T) \leq 0.$$

Thus, we write for $K \in \Omega_n^l(X)$,

$$T^l(T+K) = I + T^lK - F.$$

One has $K \in \Omega_n^l(X)$ and $T^l \in \mathcal{L}(X, \mathcal{D}(T))$, we get from Lemma 3.4, $\Delta((T^lK)^n) < 1$, for $n \in \mathbb{N}^*$. Therefore, according to Proposition 2.3 in [18], we conclude that $I + T^lK \in \Phi(X)$ with $i(I+T^lK) = 0$. As $F \in \mathcal{K}(X)$, then we can deduce that $T^l(T+K) \in \Phi(X)$. Following Theorem 2.3 in [10], it is proven that:

$$T + K \in \Phi^l(X).$$

To reach the desired result, we need to provide that $i(T+K) \leq 0$. For this purpose, let $\widehat{T}$, $\widehat{T} + \widehat{K}$ and $\widehat{K}$ the bounded operators from X_T into X, $\widehat{F} \in \mathcal{K}(X_T)$ and $\widehat{T^l} \in \mathcal{L}(X, X_T)$. According to Eq. (1), we infer that $\widehat{T^l}\widehat{T} = I_{X_T} - \widehat{F}$ is a Fredholm operator satisfying $i(\widehat{T^l}\widehat{T}) = 0$ and $i(\widehat{T^l}(\widehat{T} + \widehat{K})) = i(T^l(T+K))$.

It is clear that $\widehat{T^l}$ is a Fredholm operator if and only if $\widehat{T}$ is a Fredholm operator if and only if $\widehat{T}+\widehat{K}$ is too. So, we will discuss two cases to reach the desired property of one of this operator.

Case I: We suppose that $\widehat{T}+\widehat{K} \in \Phi^l(X_T, X)\backslash\Phi(X_T, X)$. That is $\alpha(\widehat{T}+\widehat{K}) = 0$ and $\beta(\widehat{T}+\widehat{K}) = +\infty$, then by Eq.(1), we get

$$i(T+K) = i(\widehat{T}+\widehat{K}) = i(\widehat{T}) = i(T) \leq 0.$$

Case II: Assume that $\widehat{T}+\widehat{K} \in \Phi(X_T, X)$. So, in this case, the result may be checked from Theorem 5.13 in [23]. That is $\widehat{T}^l$ is a Fredholm operator and therefore by Eq. (1),

$$i(T+K) \leq 0.$$

Conversely, let $T+K \in \mathcal{W}^l(X)$. It is trivial to see that $T+K \in \mathcal{C}(X)$ and $-K \in \Omega_n^l(X)$. Hence, applying the above reasoning by replacing T by $T+K$ and K by $-K$, we get $T+K-K = T \in \mathcal{W}^l(X)$.

(ii) The result flows from the same reasoning of the first item and the proof of Theorem 3.2-**(ii)**. $\square$

As a straightforward consequence of the previous Proposition and Proposition 2.3, we explore some stability results of the left and right Weyl spectrum of perturbed unbounded operator involving the measuring concept of non strict singularity:

Theorem 3.6. Let $T \in \mathcal{C}(X)$. Then, we get:
(i) $\sigma_w^l(T+K) = \sigma_w^l(T), \quad$ for all $K \in \Omega_n^l(X)$.
(ii) $\sigma_w^r(T+K) = \sigma_w^r(T), \quad$ for all $K \in \Omega_n^r(X)$.

The following result provides a practical criterion for the stability of left and right Weyl essential spectra by the concept of the measure of non strict singularity perturbation.

Theorem 3.7. Let $(T_1, T_2) \in \mathcal{C}^2(X)$ such that $\rho(T_1) \cap \rho(T_2) \neq \emptyset$.
Thus, we have:
(i) If the operator $(\lambda I - T_1)^{-1} - (\lambda I - T_2)^{-1} \in \Omega_n^l(X)$, for some $\lambda \in \rho(T_1) \cap \rho(T_2)$, then

$$\sigma_w^l(T_1) = \sigma_w^l(T_2).$$

(ii) If the operator $(\lambda I - T_1)^{-1} - (\lambda I - T_2)^{-1} \in \Omega_n^r(X)$, for some $\lambda \in \rho(T_1) \cap \rho(T_2)$, then

$$\sigma_w^r(T_1) = \sigma_w^r(T_2).$$

Proof. Without loss of generality, we may assume that $\lambda = 0$. Hence, $0 \in \rho(T_1) \cap \rho(T_2)$. Therefore, we can write for $\mu \neq 0$

$$\mu I - T_1 = -\mu(\mu^{-1}I - T_1^{-1})T_1.$$

Since T_1 is one to one and onto, then

$$\alpha(\mu I - T_1) = \alpha(\mu^{-1}I - T_1^{-1})$$

$$R(\mu I - T_1) = R(\mu^{-1} I - T_1^{-1})$$

and

$$\beta(\mu I - T_1) = \beta(\mu^{-1} I - T_1^{-1}).$$

(i) This shows that $\mu I - T_1 \in \Phi^l(X)$ if and only if $\mu^{-1} I - T_1^{-1} \in \Phi^l(X)$ and $i(\mu I - T_1) = i(\mu^{-1} I - T_1^{-1})$. Therefore,

$$\begin{array}{lll} \mu \notin \sigma_w^l(T_1) & \text{if and only if} & \mu I - T_1 \in \Phi^l(X) \text{ and } i(\mu I - T_1) \leq 0 \\ & \text{if and only if} & \mu^{-1} I - T_1^{-1} \in \Phi^l(X) \text{ and } i(\mu^{-1} I - T_1^{-1}) \leq 0 \\ & \text{if and only if} & \mu^{-1} I - T_1^{-1} \in \mathcal{W}^l(X). \end{array}$$

Combining Lemma 3.5-**(i)** and the fact that $T_1^{-1} - T_2^{-1} \in \Omega_n^l(X)$, we conclude that $\mu \notin \sigma_w^l(T_1)$ if and only if $\mu^{-1} I - T_2^{-1} \in \mathcal{W}^l(X)$. Consequently, we deduce that:

$$\mu \notin \sigma_w^l(T_1) \text{ if and only if } \mu \notin \sigma_w^l(T_2).$$

(ii) The proof of this assertion may be conducted in a similar way to the one in last assertion **(i)**. □

4 Left and right Weyl spectra of unbounded block 2×2 operator matrix

To reach the validity and the applicability of the theoretical results, we will derived it to present an amelioration of the spectral analysis of unbounded block 2×2 operator matrix with non maximal domain. Precisely, our aim consists to investigate a new description of the left and right Weyl spectra of this kind of operator matrices via the concept of measure of non strict singularity perturbations, which has led to significant advances in the theory of operators matrices.

4.1 Description of the unbounded operator matrix with non maximal domain

Let X, Y and Z be three Banach spaces. We consider linear operators:

$$\Psi_X : X \longrightarrow Z \quad \text{and} \quad \Psi_Y : Y \longrightarrow Z.$$

We define the linear unbounded operator matrix $\mathcal{A}_0$, in the space $X \times Y$ as follows:

$$\mathcal{A}_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

defined on its non maximal domain, $\mathcal{D}(\mathcal{A}_0)$ as:

$$\mathcal{D}(\mathcal{A}_0) \quad := \quad \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in (\mathcal{D}(A) \cap \mathcal{D}(C)) \times (\mathcal{D}(B) \cap \mathcal{D}(D)) : \ \Psi_X f = \Psi_Y g \right\},$$

where the operator entries:
* A (resp. D) acts on the Banach space X (resp. Y) and has domain $\mathcal{D}(A)$ (resp. $\mathcal{D}(D)$).

* B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts from Y into X (resp. from X into Y).

Next, we start by enacting some essential hypotheses on the components entries of this kind of operator matrix $\mathcal{A}_0$ introduced by S. Charfi et al. in [8]:
$(\mathcal{H}_1)$ The operator A is densely defined and closable.
It follows from this hypothesis that, $\mathcal{D}(A)$, equipped with the graph norm

$$\|x\|_A := \|x\| + \|Ax\|, \quad x \in \mathcal{D}(A),$$

can be completed to a Banach space X_A, which coincides with $\mathcal{D}(\overline{A})$, the domain of the closure of A and which is contained in X.
$(\mathcal{H}_2)$ $\mathcal{D}(A) \subset \mathcal{D}(\Psi_X) \subset X_A$ and Ψ_X is bounded as a mapping from X_A into Z.
$(\mathcal{H}_3)$ The set $\mathcal{D}(A) \cap N(\Psi_X)$ is dense in X with $\rho(A_1) \neq \emptyset$, for $A_1 := A|_{\mathcal{D}(A)\cap N(\Psi_X)}$.

We recall the following results needed to formulate our theoretical results.

Lemma 4.1. [4, Lemma 2.1-2.2] Under the hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_3)$, for any $\lambda \in \rho(A_1)$, we have the following assertions:
(i) $\mathcal{D}(A) := \mathcal{D}(A_1) \oplus N(\lambda I - A)$.
(ii) The restriction of the subset $\Psi_X|_{N(\lambda I - A)}$ is injective and denoted by Ψ_λ .
(iii) $\mathcal{R}(\Psi_\lambda) = \Psi_X(N(\lambda I - A)) = \Psi_X(\mathcal{D}(A))$ does not depend on λ.

As a consequence, we have Ψ_λ is invertible for $\lambda \in \rho(A_1)$ with inverse denoted by K_λ:

$$K_\lambda := (\Psi_\lambda)^{-1} := (\Psi_X|_{N(\lambda I - A)})^{-1} : \Psi_X(\mathcal{D}(A)) \longrightarrow N(\lambda I - A).$$

Concerning the operator C, we will suppose the following assumption:
$(\mathcal{H}_4)$ $\mathcal{D}(A) \subset \mathcal{D}(C) \subset X_A$ and $C(\lambda I - A_1)^{-1}$ is a bounded operator from X_A into Y.

Remark 4.2. **(i)** Combining the closed graph theorem with above assumption, we infer for $\lambda \in \rho(A_1)$, that the operator $C_\lambda := C(\lambda I - A_1)^{-1}$ is bounded from X into Y.
(ii) If the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ are satisfied, then for $\lambda \in \rho(A_1)$ and $x \in \mathcal{D}(A)$, we have:

$$(\lambda I - A)x = (\lambda I - A_1)(I - K_\lambda \Psi_X)x.$$

When dealing with the case of operator matrix with non maximal domain, we need to assume further assumptions as well:
$(\mathcal{H}_5)$ For some (hence for all) $\lambda \in \rho(A_1)$, K_λ is a bounded operator from $\Psi_X(\mathcal{D}(A))$ into X. Its extension by continuity to $\overline{\Psi_X(\mathcal{D}(A))}$ is denoted by $\overline{K}_\lambda$.
$(\mathcal{H}_6)$ $\mathcal{D}(B) \cap \mathcal{D}(D) \subset \mathcal{D}(\Psi_Y)$. The sets

$$Y_1 := \{y \in \mathcal{D}(B) \cap \mathcal{D}(D) : \Psi_Y y \in \Psi_X(\mathcal{D}(A))\}$$

and

$$Y_2 := \{y \in \mathcal{D}(B) \cap \mathcal{D}(\Psi_Y) : \Psi_Y y \in \Psi_X(\mathcal{D}(A)) = Z_1\}$$

are dense in Y.
* The restriction of Ψ_Y to the set Y_2, $\Psi_Y|_{Y_2}$, is a bounded operator from Y_2 into Z.

* The continuous extension of $\Psi_Y|_{Y_1}$ on the all space Y will be denoted by $\overline{\Psi}^{\circ}_Y$.

$(\mathcal{H}_7)$ The set Y_1 is a core of the densely defined linear operator D, with $\rho(D) \neq \emptyset$.

$(\mathcal{H}_8)$ The operator B is densely defined on Y satisfying that for some (hence for all) $\lambda \in \rho(A_1)$, the operators:
* $(\lambda I - A_1)^{-1}B$ is bounded on its domain,
* $C[-K_\lambda \Psi_Y + (\lambda I - A_1)^{-1}B]$ is bounded on Y_2.

For $\lambda \in \rho(A_1)$, the Schur-complement associated to this kind of operator matrix is defined on the set Y_1 as:

$$M_\lambda := D - C[-K_\lambda \Psi_Y + (\lambda I - A_1)^{-1}B].$$

Remark 4.3. **(i)** As a justifications of the systematic analysis of all assumptions on the entries of this kind of operator matrix $\mathcal{A}_0$ of the above form, we refer the readers to some physical examples like: Delay equation or integro differential equation that are studied in details in the works [6, 8, 13, 25].
(ii) The assumptions $(\mathcal{H}_6)$ and $(\mathcal{H}_8)$ assert that:

$$\overline{R}_\lambda := -\overline{C[-K_\lambda \Psi_Y + (\lambda I - A_1)^{-1}B]} \in \mathcal{L}(Y).$$

(iii) Under the assumptions $\mathcal{H}_i, i = \{3, ., 8\}$, the operator M_λ is closable for some (hence for all) $\lambda \in \rho(A_1)$, with closure $\overline{M}_\lambda$ satisfying:

$$\overline{M}_\lambda := D + \overline{R}_\lambda.$$

Moreover, assume further that $\rho(\overline{M}_\lambda) \neq \emptyset$, for $\lambda \in \rho(A_1)$. Thus, the resolvent expression of the operator $\overline{M}_\lambda$ may be expressed by:

$$\begin{aligned}(\lambda I - \overline{M}_\lambda)^{-1} &= (\lambda I - D)^{-1} + (\lambda I - \overline{M}_\lambda)^{-1}[\overline{M}_\lambda - D](\lambda I - D)^{-1} \\ &= (\lambda I - D)^{-1} + (\lambda I - \overline{M}_\lambda)^{-1}\overline{R}_\lambda(\lambda I - D)^{-1}, \end{aligned} \tag{2}$$

for all $\lambda \in \rho(A_1) \cap \rho(D) \cap \rho(\overline{M}_\lambda)$.

The above assumptions are essential and fruitful to describe a fine closure of such kind of operator matrix $\mathcal{A}_0$.

Theorem 4.4. [8] Assume that the hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_8)$ are fulfilled.
Then, the operator $\mathcal{A}_0$ is closable for some (hence for all) $\lambda \in \rho(A_1)$, with closure $\mathcal{A}$ may be described as:

$$\begin{aligned}\lambda I - \mathcal{A} &= \begin{pmatrix} I & 0 \\ C_\lambda & I \end{pmatrix}\begin{pmatrix} \lambda I - A_1 & 0 \\ 0 & \lambda I - \overline{M}_\lambda \end{pmatrix}\begin{pmatrix} I & G_\lambda \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ C_\lambda & I \end{pmatrix}\begin{pmatrix} \lambda I - A_1 & 0 \\ 0 & \lambda I - D \end{pmatrix}\begin{pmatrix} I & G_\lambda \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \overline{R}_\lambda \end{pmatrix}. \end{aligned} \tag{3}$$

Our interest in this last part of this subsection consists at showing what are the conditions that we will be required on the components entries of $\lambda I - \mathcal{A}$ which make it invertible.

Proposition 4.5. Assume that the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_8)$ are fulfilled such that $\rho(\overline{M}_\lambda) \neq \emptyset$, for $\lambda \in \rho(A_1)$. Then, the operator $\lambda I - \mathcal{A}$ is invertible in $X \times Y$ and its inverse may be expressed by:

$$
\begin{aligned}
(\lambda I-\mathcal{A})^{-1} &= \begin{pmatrix} (\lambda I-A_1)^{-1}+G_\lambda(\lambda I-\overline{M}_\lambda)^{-1}C_\lambda & -G_\lambda(\lambda I-\overline{M}_\lambda)^{-1} \\ -(\lambda I-\overline{M}_\lambda)^{-1}C_\lambda & (\lambda I-\overline{M}_\lambda)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} (\lambda I-A_1)^{-1}+T_1C_\lambda & -T_1 \\ -T_2 & (\lambda I-D)^{-1} \end{pmatrix} \\
&+ \begin{pmatrix} G_\lambda(\lambda I-\overline{M}_\lambda)^{-1}\overline{R}_\lambda T_2 & -G_\lambda(\lambda I-\overline{M}_\lambda)^{-1}\overline{R}_\lambda(\lambda I-D)^{-1} \\ -(\lambda I-\overline{M}_\lambda)^{-1}\overline{R}_\lambda T_2 & (\lambda I-\overline{M}_\lambda)^{-1}\overline{R}_\lambda(\lambda I-D)^{-1} \end{pmatrix}, \quad (4)
\end{aligned}
$$

where $T_1 := G_\lambda(\lambda I-D)^{-1}$ and $T_2 := (\lambda I-D)^{-1}C_\lambda$, for $\lambda \in \rho(A_1)\cap\rho(D)\cap\rho(\overline{M}_\lambda)$.

Proof. Let $\lambda \in \rho(A_1)$. The identity of Eq. (3) and the invertibility of the outer factors:

$$
\begin{pmatrix} I & 0 \\ C_\lambda & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -C_\lambda & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & G_\lambda \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -G_\lambda \\ 0 & I \end{pmatrix},
$$

show that, in view of the fact that $\lambda I - A_1$ and $\lambda I-\overline{M}_\lambda$ are invertible in X and Y, respectively, we infer that the operator $\lambda I-\mathcal{A}$ is too in $X\times Y$. Thus, for $\lambda\in\rho(A_1)\cap\rho(\overline{M}_\lambda)$, we have $\lambda\in\rho(\mathcal{A})$.
Therefore, the resolvent of the operator $\mathcal{A}$ may be expressed as well:

$$
\begin{aligned}
(\lambda I-\mathcal{A})^{-1} &= \begin{pmatrix} I & G_\lambda \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} \lambda I-A_1 & 0 \\ 0 & \lambda I-\overline{M}_\lambda \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ C_\lambda & I \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \begin{array}{c}(\lambda I-A_1)^{-1} \\ +G_\lambda(\lambda I-\overline{M}_\lambda)^{-1}C_\lambda\end{array} & -G_\lambda(\lambda I-\overline{M}_\lambda)^{-1} \\ -(\lambda I-\overline{M}_\lambda)^{-1}C_\lambda & (\lambda I-\overline{M}_\lambda)^{-1} \end{pmatrix}.
\end{aligned}
$$

Following with the Remark 4.3-**(iii)**, we infer that for $\lambda \in \rho(A_1)\cap\rho(D)\cap\rho(\overline{M}_\lambda)$, the resolvent of the operator matrix $\mathcal{A}$ obey to the following form:

$$
(\lambda I-\mathcal{A})^{-1} = \begin{pmatrix} (\lambda I-A_1)^{-1}+G_\lambda(\lambda I-D)^{-1}C_\lambda & -G_\lambda(\lambda I-D)^{-1} \\ -(\lambda I-D)^{-1}C_\lambda & (\lambda I-D)^{-1} \end{pmatrix}
$$

$$
+\begin{pmatrix} G_\lambda(\lambda I-\overline{M}_\lambda)^{-1}\overline{R}_\lambda(\lambda I-D)^{-1}C_\lambda & -G_\lambda(\lambda I-\overline{M}_\lambda)^{-1}\overline{R}_\lambda(\lambda I-D)^{-1} \\ -(\lambda I-\overline{M}_\lambda)^{-1}\overline{R}_\lambda(\lambda I-D)^{-1}C_\lambda & (\lambda I-\overline{M}_\lambda)^{-1}\overline{R}_\lambda(\lambda I-D)^{-1} \end{pmatrix}
$$

□

4.2 Left and right Weyl spectra of operator matrix via the measure of non strict singularity

The purpose of this subsection is to describe the left and right Weyl spectra of unbounded block 2×2 operator matrix with non maximal domain regardless of these diagonal operators entries by means of measure of non strict singularity. To do this, we will see that the following result on the notion of a measure of non strict singularity of 2×2 block operator matrix established by N. Moalla in [18], which turns out to be necessary for proving our desired result.

Lemma 4.6. [18, Lemma 4.1] For all bounded operator matrix

$$T := \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

on $X \times Y$, we consider

$$\Lambda(T) = \max\{\Delta(T_1) + \Delta(T_2), \Delta(T_3) + \Delta(T_4)\}.$$

Then, Λ defines a measure of non-strict-singularity of the operator matrix T on the space $X \times Y$.

Theorem 4.7. Let the matrix operator $\mathcal{A}_0$ satisfies the assumptions $(\mathcal{H}_i)$, $1 \leq i \leq 8$, for $\lambda \in \rho(A_1)$, the operator $\overline{R}_\lambda$ is supposed strictly singular in Y and $\rho(\overline{M}_\lambda) \neq \emptyset$.
Assume that for some $\lambda \in \rho(A_1) \cap \rho(D) \cap \rho(\overline{M}_\lambda)$ and for all bounded operators H and K satisfying:

(i) If $\Delta(HT_1KT_1) < \frac{1}{9}$, $\Delta(HT_1KT_2) < \frac{1}{9}$, $\Delta(HT_2KT_2) < \frac{1}{9}$ and $\Delta(HT_2KT_1) < \frac{1}{9}$, then

$$\sigma_w^l(\mathcal{A}) \subset \sigma_w^l(A_1) \cup \sigma_w^l(D).$$

Moreover, if ${}^C\sigma_{ef}^l(A_1)$ and ${}^C\sigma_{ef}^l(D)$ are connected (where ${}^C\Omega$ will denote the complement of a subset $\Omega \subset \mathbb{C}$), then

$$\sigma_w^l(\mathcal{A}) = \sigma_w^l(A_1) \cup \sigma_w^l(D).$$

(ii) If $\Delta(T_1KT_1H) < \frac{1}{12}$, $\Delta(T_1KT_2H) < \frac{1}{12}$, $\Delta(T_2KT_2H) < \frac{1}{6}$ and $\Delta(T_2KT_1H) < \frac{1}{6}$, then

$$\sigma_w^r(\mathcal{A}) \subset \sigma_w^r(A_1) \cup \sigma_w^r(D).$$

Assume further that ${}^C\sigma_{ef}^r(A_1)$ and ${}^C\sigma_{ef}^r(D)$ are connected, then

$$\sigma_w^r(\mathcal{A}) = \sigma_w^r(A_1) \cup \sigma_w^r(D).$$

(iii) If $\Delta(HT_1KT_1) < \frac{1}{9}$, $\Delta(HT_1KT_2) < \frac{1}{9}$, $\Delta(HT_2KT_2) < \frac{1}{9}$, $\Delta(HT_2KT_1) < \frac{1}{9}$,
$\Delta(T_1KT_1H) < \frac{1}{12}$, $\Delta(T_1KT_2H) < \frac{1}{12}$, $\Delta(T_2KT_2H) < \frac{1}{6}$ and $\Delta(T_2KT_1H) < \frac{1}{6}$, then

$$\sigma_w(\mathcal{A}) \subset \sigma_w(A_1) \cup \sigma_w(D).$$

Moreover, since ${}^C\sigma_{ef}(A_1)$ is connected, then

$$\sigma_w(\mathcal{A}) = \sigma_w(A_1) \cup \sigma_w(D).$$

Proof. Let consider the diagonal operator matrix

$$\mathcal{Q} := \begin{pmatrix} A_1 & 0 \\ 0 & D \end{pmatrix}.$$

For $\lambda \in \rho(A_1) \cap \rho(D) \cap \rho(\overline{M}_\lambda)$, we have $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{Q})$. So, the representation of $(\lambda I - \mathcal{A})^{-1} - (\lambda I - \mathcal{Q})^{-1}$ may be written from Eq. (4) as well:

$$(\lambda I - \mathcal{A})^{-1} - (\lambda I - \mathcal{Q})^{-1} := \mathcal{J}_\lambda + \mathcal{O}_\lambda,$$

where the blocks operators matrices $\mathcal{J}_\lambda$ and $\mathcal{O}_\lambda$ are formally given by:

$$\mathcal{J}_\lambda := \begin{pmatrix} T_1 C_\lambda & -T_1 \\ & \\ -T_2 & 0 \end{pmatrix} = \begin{pmatrix} T_3 & -T_1 \\ & \\ -T_2 & 0 \end{pmatrix}$$

and

$$\mathcal{O}_\lambda := \begin{pmatrix} -G_\lambda(\lambda I - \overline{M}_\lambda)^{-1}\overline{R}_\lambda(\lambda I - D)^{-1}C_\lambda & G_\lambda(\lambda I - \overline{M}_\lambda)^{-1}\overline{R}_\lambda(\lambda I - D)^{-1} \\ & \\ (\lambda I - \overline{M}_\lambda)^{-1}\overline{R}_\lambda(\lambda I - D)^{-1}C_\lambda & -(\lambda I - \overline{M}_\lambda)^{-1}\overline{R}_\lambda(\lambda I - D)^{-1} \end{pmatrix}.$$

(i) Since $\overline{R}_\lambda \in \mathcal{SS}(Y)$ and the subset $\mathcal{SS}(Y)$ is a closed two sided-ideal of $\mathcal{L}(Y)$, we infer that $\mathcal{O}_\lambda \in \mathcal{SS}(X \times Y)$. Keeping the stability under strict singular operator of the subset $\Omega^l_n(X \times Y)$ (see Remark 3.2 in [18] for more details), to reach the result for $(\lambda I - \mathcal{A})^{-1} - (\lambda I - \mathcal{Q})^{-1}$ it remains to provide it only for the operator $\mathcal{J}_\lambda$.

To do this, let us consider the bounded block operator matrix $K := \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$.

A short computation, reveals that:

$$[K\mathcal{J}_\lambda]^2 = \begin{pmatrix} \begin{matrix} (K_1T_3)^2 + (K_2T_2)^2 + 2K_1T_3K_2T_2 \\ +K_1T_1K_3T_3 + K_1T_1K_4T_2 \end{matrix} & \begin{matrix} K_1T_3K_1T_1 + K_2T_2K_1T_1 \\ +K_1T_1K_3T_3 \end{matrix} \\ & \\ \begin{matrix} K_3T_3K_1T_3 + K_3T_3K_2T_2 + K_4T_2K_1T_3 \\ +K_4T_2K_2T_2 + K_3T_1K_3T_3 + K_3T_1K_4T_2 \end{matrix} & \begin{matrix} K_4T_3K_1T_1 + K_4T_2K_1T_1 \\ +K_4T_1K_3T_1 \end{matrix} \end{pmatrix}.$$

Referring to Lemma 4.6 under the fact that $\Lambda\left([K\mathcal{J}_\lambda]^2\right) < 1$, we deduce that $\mathcal{J}_\lambda \in \Omega^l_2(X \times Y)$. Hence, Remark 3.2 in [18] asserts that $\mathcal{J}_\lambda + \mathcal{O}_\lambda \in \Omega^l_2(X \times Y)$.
Now, applying the result of Theorem 3.7-**(i)**, we infer that

$$\sigma^l_w(\mathcal{A}) = \sigma^l_w(\mathcal{Q}) \quad \text{with} \quad i(\mathcal{A}) = i(\mathcal{Q}).$$

As $\mathcal{Q}$ is a diagonal operator matrix, it is trivial to see that:

$$\sigma^l_w(\mathcal{Q}) \subset \sigma^l_w(A_1) \cup \sigma^l_w(D) \quad \text{with} \quad i(\mathcal{Q}) = i(A_1) + i(D).$$

Consequently, we get:

$$\sigma_w^l(\mathcal{A}) \subset \sigma_w^l(A_1) \cup \sigma_w^l(D).$$

To accomplish the result of the reverse inclusion, assume that $\lambda \notin \sigma_w^l(\mathcal{A})$. That is $\lambda I - \mathcal{A} \in \Phi^l(X \times Y)$ with $i(\lambda I - \mathcal{A}) \leq 0$. Hence, $\lambda I - A_1 \in \Phi^l(X)$ and $\lambda I - D \in \Phi^l(Y)$ such that $i(\lambda I - A_1) + i(\lambda I - D) \leq 0$.
To provide that $i(\lambda I - A_1) \leq 0$ and $i(\lambda I - D) \leq 0$, we will discuss two cases:

Case I: For $\lambda I - A_1 \in \Phi(X)$ and $\lambda I - D \in \Phi(Y)$. Using the fact that $\rho(A_1) \neq \emptyset$ and $\rho(D) \neq \emptyset$, we conclude that there exists $\mu_0 \in \rho(A_1)$ and $\mu_1 \in \rho(D)$. Hence, $\mu_0 I - A_1 \in \Phi(X)$ with $i(\mu_0 - A_1) = 0$ and $\mu_1 I - D \in \Phi(Y)$ with $i(\mu_1 I - D) = 0$. Moreover, while ${}^C\sigma_{ef}^l(A_1)$ and ${}^C\sigma_{ef}^l(D)$ are connected, we deduce from the property of the index that:

$$i(\lambda I - A_1) = i(\mu_0 I - A_1) = 0 \quad \text{ and } \quad i(\lambda I - D) = i(\mu_1 I - D) = 0,$$

for all $\lambda I - A_1 \in \Phi(X)$ and $\lambda I - D \in \Phi(Y)$.

Case II: Let $\lambda I - A_1 \in \Phi^l(X) \setminus \Phi(X)$ and $\lambda I - D \in \Phi^l(Y) \backslash \Phi(Y)$, then, we get $\alpha(\lambda I - A_1) < +\infty$, $\beta(\lambda I - A_1) = +\infty$, $\alpha(\lambda I - D) < +\infty$ and $\beta(\lambda I - D) = +\infty$. That is,

$$i(\lambda I - A_1) < 0 \quad \text{ and } \quad i(\lambda I - D) < 0.$$

Consequently, $i(\lambda I - A_1) = -\infty \leq 0$ and $i(\lambda I - D) = -\infty \leq 0$. This proved that $\lambda \notin \sigma_w^l(A_1) \cup \sigma_w^l(D)$.

(ii) To accomplish the result for the right Weyl spectrum of the operator matrix $\mathcal{A}$, we will proceed by steps.

Firstly, we will provide that $\mathcal{J}_\lambda \in \Omega_2^r(X \times Y)$.
Indeed, we compute $\Lambda((\mathcal{J}_\lambda K)^2)$, for the bounded operator $K := \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$. That is,

$$\Lambda\left([\mathcal{J}_\lambda K]^2\right) = \Lambda\left(\begin{pmatrix} \begin{array}{c} (T_3K_1)^2 + (T_1K_3)^2 + 2T_3K_1T_1K_3 \\ +T_3K_2T_2K_1 + T_1K_4T_2K_1 \end{array} & \begin{array}{c} T_3K_1T_3K_2 + T_3K_1T_1K_4 \\ +T_1K_3T_3K_2 + T_1K_3T_1K_4 \\ +T_3K_2T_2K_2 + T_1K_4T_2K_2 \end{array} \\ \begin{array}{c} T_2K_1T_3K_1 + T_2K_2T_1K_3 \\ +T_2K_2T_2K_1 \end{array} & \begin{array}{c} T_2K_1T_3K_2 + T_2K_1T_1K_4 \\ +(T_2K_2)^2 \end{array} \end{pmatrix}\right)$$

$$= \Lambda\left(\begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}\right) = \max\left\{\Delta(Q_1) + \Delta(Q_2), \Delta(Q_3) + \Delta(Q_4)\right\},$$

where

$$\begin{array}{rcl} Q_1 & := & (T_3K_1)^2 + (T_1K_3)^2 + 2T_3K_1T_1K_3 + T_3K_2T_2K_1 + T_1K_4T_2K_1 \\ Q_2 & := & T_3K_1T_3K_2 + T_3K_1T_1K_4 + T_1K_3T_3K_2 + T_1K_3T_1K_4 + T_3K_2T_2K_2 + T_1K_4T_2K_2 \\ Q_3 & := & T_2K_1T_3K_1 + T_2K_2T_1K_3 + T_2K_2T_2K_1 \\ Q_4 & := & T_2K_1T_3K_2 + T_2K_1T_1K_4 + (T_2K_2)^2. \end{array}$$

Secondly, we will compute $\Delta(Q_1) + \Delta(Q_2)$ and $\Delta(Q_3) + \Delta(Q_4)$.

In fact, a short computation reveals that:

$$\begin{aligned}
\Delta(Q_1)+\Delta(Q_2) &= \Delta\Big((T_3K_1)^2+(T_1K_3)^2+2T_3K_1T_1K_3+T_3K_2T_2K_1+T_1K_4T_2K_1\Big)\\
&+ \Delta\Big(T_3K_1T_3K_2+T_3K_1T_1K_4+T_1K_3T_3K_2+T_1K_3T_1K_4\\
&\quad +T_3K_2T_2K_2+T_1K_4T_2K_2\Big)\\
&\leq \Delta(T_1\widetilde{K}_1T_1\widetilde{K}_1)+\Delta(T_1K_3T_1K_3)+2\Delta(T_1\widetilde{K}_1T_1K_3)+\Delta(T_1\widetilde{K}_2T_2K_1)\\
&+ \Delta(T_1K_4T_2K_1)+\Delta(T_1\widetilde{K}_1T_1\widetilde{K}_2)+\Delta(T_1\widetilde{K}_1T_1K_4)+\Delta(T_1K_3T_1\widetilde{K}_2)\\
&+ \Delta(T_1K_3T_1K_4)+\Delta(T_1\widetilde{K}_2T_2K_2)+\Delta(T_1K_4T_2K_2).
\end{aligned}$$

Similarly, we get:

$$\begin{aligned}
\Delta(Q_3)+\Delta(Q_4) &= \Delta\Big(T_2K_1T_3K_1+T_2K_2T_1K_3+T_2K_2T_2K_1\Big)\\
&+ \Delta\Big(T_2K_1T_3K_2+T_2K_1T_1K_4+(T_2K_2)^2\Big)\\
&\leq \Delta(T_2K_1T_1\widetilde{K}_1)+\Delta(T_2K_2T_1K_3)+\Delta(T_2K_2T_2K_1)\\
&+ \Delta(T_2K_1T_1\widetilde{K}_2)+\Delta(T_2K_1T_1K_4)+\Delta(T_2K_2T_2K_2),
\end{aligned}$$

where $\widetilde{K}_* := C_\lambda K_*$, for $* = \{1,2\}$.
Based on the assumptions $\Delta(T_1KT_1H) < \frac{1}{12}$ and $\Delta(T_1KT_2H) < \frac{1}{12}$, for all bounded operators H and K, we infer that $\Delta(Q_1)+\Delta(Q_2) < 1$. On the other side, the assumptions $\Delta(T_2KT_2H) < \frac{1}{6}$ and $\Delta(T_2KT_1H) < \frac{1}{6}$, assert that $\Delta(Q_3)+\Delta(Q_4) < 1$. Consequently, we deduce that

$$\max\left(\Delta(Q_1)+\Delta(Q_2), \Delta(Q_3)+\Delta(Q_4)\right) < 1.$$

So, one checks $\mathcal{J}_\lambda \in \Omega_2^r(X\times Y)$ and $\mathcal{O}_\lambda \in \mathcal{SS}(X\times Y)$. Hence, following Lemma 2.3-**(ii)**, we infer that $\mathcal{J}_\lambda+\mathcal{O}_\lambda \in \Omega_2^r(X\times Y)$. That is, $(\lambda I-\mathcal{A})^{-1}-(\lambda I-\mathcal{Q})^{-1} \in \Omega_2^r(X\times Y)$. Finally, the result follows immediately from Theorem 3.7-**(ii)**, that is:

$$\sigma_w^r(\mathcal{A}) = \sigma_w^r(\mathcal{Q}) \subset \sigma_w^r(A_1)\cup\sigma_w^r(D).$$

The rest of the proof may be checked in the similar ways as the item **(i)**.

(iii) The first inclusion of this item is an immediate consequence of the items **(i)** and **(ii)**. That is

$$\begin{aligned}
\sigma_w(\mathcal{A}) &= \sigma_w^l(\mathcal{A})\cup\sigma_w^r(\mathcal{A})\\
&\subset \sigma_w^l(A_1)\cup\sigma_w^l(D)\cup\sigma_w^r(A_1)\cup\sigma_w^r(D)\\
&\subset \sigma_w(A_1)\cup\sigma_w(D).
\end{aligned}$$

For the opposite inclusion, let consider $\lambda\notin\sigma_w(\mathcal{A})$. In other terms, $\lambda I-\mathcal{A}$ is a Fredholm operator in $X\times Y$ with $i(\lambda I-\mathcal{A}) = 0$. Which asserts that $\lambda I-A_1$ and $\lambda I-D$ are two Fredholm operators in X respectively in Y such that

$$i(\lambda I-A_1)+i(\lambda I-D) = 0.$$

Furthermore, since $\rho(A_1) \neq \emptyset$, then there exist $\mu \in \rho(A_1)$ such that $\mu I - A_1 \in \Phi(X)$ with $i(\mu I - A_1) = 0$. Accordingly with the component connexe of ${}^C\sigma_{ef}(A_1)$ and the property of the index, we deduce that

$$i(\lambda I - A_1) = i(\mu I - A_1) = 0, \ \ \forall \lambda \in {}^C\sigma_{ef}(A_1).$$

Consequently, we conclude that $i(\lambda I - D) = 0$. This in turn yields that $\lambda \notin \sigma_w(A_1) \cup \sigma_w(D)$. $\square$

Remark 4.8. **(i)** The results of Theorem 4.7 remain true if we replace the assumptions for

$$\begin{cases} \Delta(HT_1KT_1) < \frac{1}{9}, & \Delta(HT_1KT_2) < \frac{1}{9}, \\ \Delta(HT_2KT_2) < \frac{1}{9}, & \Delta(HT_2KT_1) < \frac{1}{9} \end{cases} \text{ and } \begin{cases} \Delta(T_1KT_1H) < \frac{1}{12}, & \Delta(T_1KT_2H) < \frac{1}{12}, \\ \Delta(T_2KT_2H) < \frac{1}{6}, & \Delta(T_2KT_1H) < \frac{1}{6} \end{cases}$$

by:

$$\begin{cases} \gamma(HT_1KT_1) < \frac{1}{9}, \ \gamma(HT_1KT_2) < \frac{1}{9}, \\ \gamma(HT_2KT_2) < \frac{1}{9}, \ \gamma(HT_2KT_1) < \frac{1}{9}, \end{cases} \quad \text{and} \quad \begin{cases} \gamma(T_1KT_1H) < \frac{1}{12}, \ \gamma(T_1KT_2H) < \frac{1}{12}, \\ \gamma(T_2KT_2H) < \frac{1}{6}, \ \gamma(T_2KT_1H) < \frac{1}{6} \end{cases}$$

respectively by:

$$\begin{cases} \|HT_1KT_1\| < \frac{1}{9}, \ \|HT_1KT_2\| < \frac{1}{9}, \\ \|HT_2KT_2\| < \frac{1}{9}, \ \|HT_2KT_1\| < \frac{1}{9}, \end{cases} \quad \text{and} \quad \begin{cases} \|T_1KT_1H\| < \frac{1}{12}, \ \|T_1KT_2H\| < \frac{1}{12}, \\ \|T_2KT_2H\| < \frac{1}{6}, \ \|T_2KT_1H\| < \frac{1}{6}, \end{cases}$$

while the interaction between the notion of measures of non compactness and non strict singularity initiated in the paper [18] for $T \in \mathcal{L}(X, Y)$, as:

$$\Delta(T) \leq \gamma(T).$$

(ii) If T_1 and T_2 are strictly singular operators (resp. compacts operators), then all the assumptions of Theorem 4.7 are still verified.
(iii) The results of Theorem 4.7 are still valid for unbounded operator matrix with maximal domain case. That is, if we take $\Psi_X = \Psi_Y \equiv 0$, then we obtain that

$$\sigma_w^*(\mathcal{A}) \subset \sigma_w^*(A) \cup \sigma_w^*(D), \ \ \text{for } \sigma_w^*(.) = \{\sigma_w^l(.), \sigma_w^r(.)\}.$$

So, in this case a generalization of the results given in [18, 19] can be obtained in the case of left and right Weyl spectra.

Conclusion: During the last years, the measuring concept of non strict singularity has proved to be useful in different areas, in particular in the study of spectral problems. This concept of perturbation has attracted the attention of various researchers which can be regarded as a fine measure of the classical Hausdorff measure of non compactness [22], we refer the readers to see some references on this concept [1, 2, 18, 19, 21]. On the other hand, during the past few years, the notion of left and right Atkinson in the context of linear operators was widely studied. Various techniques and new tools have been introduced for the study of some spectral properties of this kind of operators (see, for example, the references [3, 10, 11, 28, 29]). Besides the specificity of this classes in the theory of linear operators, the interaction between this notion and the measure of non strict singularity concept demonstrates those efficiency to formulate our goal in this paper. More precisely, we will use them to provide some advances on the characterization of the left and right Weyl spectra of closed densely defined linear operator and to develop some new in the study of spectral analysis of unbounded operators matrices with non maximal domain problems in order to enlarge some known results that are widely studied in [18, 19].

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