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Small values of the Ramanujan τ-function

by
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Abstract

Here, we show that if $\tau(n)$ is the Ramanujan τ-function, then there exists a function $f(n)$ tending to infinity such that $|\tau(n)|/n^{11/2} < (\log n)^{-f(n)}$ holds for an infinite sequence of positive integers n.

Key Words: The Ramanujan τ-function.
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1 Introduction

The Ramanujan τ-function are the coefficients of the following expansion

$$q\prod_{n\geq 1}(1-q^n)^{24}=\sum_{n\geq 1}\tau(n)q^n \qquad |q|<1.$$

Let $a(n)=\tau(n)/n^{11/2}$. A celebrated result of Deligne says that $|a_n|\leq d(n)$, where $d(n)$ is the number of divisors function of n. This was improved for most n to

$$|a_n|\leq(\log n)^{-1/2+\varepsilon}$$

for every $\varepsilon>0$ in [3]. All the above results hold in larger generality, like for the Fourier coefficients of normalized Hecke eigenforms of weight k for the full modular group (see also [2] for a more general set-up). Here, we address the issue of how small we can make $|a(n)|$, whether we can make it smaller than any negative power of $\log n$ for infinitely many n. Here is our theorem.

Theorem 1. *The inequality*

$$|a_n|<\exp\left(-(1/51+o(1))\log n\frac{\log\log\log n}{\log\log n}\right)$$

holds on an infinite set of positive integers n tending to infinity.

Note that $\exp\left((1/51+o(1))\log n(\log\log\log n/\log\log n)\right)$ tends to infinity faster than any power of the logarithm of n. The constant $1/51$ can certainly be improved but we did not make any effort to do so.

2 The proof

To start with, we write $n := \prod_{i=1}^{k} p_i^{m_i}$, where p_i are suitable primes and m_i are suitable exponents. We have

$$|a_{p_i^{m_i}}| = \left| \frac{\sin((m_i+1)\theta_{p_i})}{\sin \theta_{p_i}} \right|.$$

As most authors proceed, we write

$$|\sin((m_i+1)\theta_i)| = |\sin((m_i+1)\theta_i - \pi n_i)| = \sin |(m_i+1)\theta_{p_i} - \pi n_i|,$$

where

$$n_i = \left\lfloor \frac{(m_i+1)\theta_{p_i}}{\pi} \right\rceil$$

is the closest integer to $(m_i+1)\theta_{p_i}/\pi$. The number $x := (m_i+1)\theta_{p_i} - \pi n_i$ belongs to the interval $[-\pi/2, \pi/2]$, so we have $\sin |x| \le |x|$. Hence,

$$|a_{p_i^{m_i}}| \le \frac{\pi}{|\sin \theta_{p_i}|} \cdot (m_i+1) \left| \frac{\theta_{p_i}}{\pi} - \frac{n_i}{m_i+1} \right|.$$

To explain how to choose p_i, m_i, we appeal to an unconditional bound on the error term in the Sato–Tate law from [1]. There it is shown that for an interval I of $(0, \pi)$, writing $\pi_{f,I}(x) = \{p \le x : \theta_p \in I\}$, we have

$$\left| \frac{\pi_{f,I}(x)}{\pi(x)} - \mu_{ST}(I) \right| < 58.1 \frac{\log(11 \log x)}{(\log x)^{1/2}} \qquad \text{for} \qquad x \ge 3, \tag{2.1}$$

where

$$\mu_{ST}(I) = \frac{2}{\pi} \int_I \sin^2(\theta) d\theta.$$

We take $I = [9\pi/20 - 1/(\log x)^{1/3}, 9\pi/20 + 1/(\log x)^{1/3}]$. Then

$$\mu_{ST}(I) \ge \frac{2 \cdot 0.98^2}{\pi} \left(\frac{2}{(\log x)^{1/3}} \right) > \frac{1.2}{(\log x)^{1/3}}$$

for $x > x_0$, where from now on x_0 is a large number (not necessarily the same at every occurrence). Hence,

$$\pi_{f,I}(x) > \pi(x) \left(\frac{1.2}{(\log x)^{1/3}} - \frac{58.1 \log(11 \log x)}{(\log x)^{1/2}} \right) > \frac{1.1x}{(\log x)^{4/3}}$$

provided $x > x_0$. So, there are $> 1.1x/(\log x)^{4/3}$ primes $p \le x$ such that $\theta_p \in I$. Of these, the number of them which are smaller than $x/(\log x)$ is $\pi(x/(\log x)) < 2x/(\log x)^2 < 0.1x/(\log x)^{4/3}$ for $x > x_0$. Hence, there are $k = \lfloor x/(\log x)^{4/3} \rfloor$ such primes say $p_1, \ldots, p_k$ which all exceed $x/(\log x)$. Since $\theta_i \in I$ for $i = 1, \ldots, k$, we get that

$$\left| \frac{\theta_i}{\pi} - \frac{9}{20} \right| < \frac{1}{\pi (\log x)^{1/3}}.$$

Choosing $m_i = 19$, we can see that

$$\left\lfloor \frac{(m_i+1)\theta_{p_i}}{\pi} \right\rceil = 9.$$

We thus have that for $p_i^{m_i} = p_i^{19}$,

$$|a_{p_i^{19}}| < \frac{\pi}{|\sin \theta_{p_i}|} \cdot 20 \left| \frac{\theta_{p_i}}{\pi} - \frac{9}{20} \right| \le \frac{1}{(\log x)^{1/3+o(1)}}. \tag{2.2}$$

We take

$$N = \prod_{i=1}^{k} p_i^{19}.$$

Clearly, since $x/\log x < p_i \le x$, we have that $\log p_i = (1+o(1))\log x$ for $i = 1, \ldots, k$. Hence,

$$\log N = (1+o(1))19k \log x = (1+o(1))\frac{19x}{(\log x)^{1/3}}.$$

This shows that $\log x = (1+o(1))\log\log N$. Finally, by (2.2),

$$\begin{aligned} |a_N| &= \prod_{i=1}^{k} |a_{p_i^{19}}| \le \exp\left(-(1/3+o(1))k \log\log x\right) \\ &= \exp\left(-(1/51+o(1))\left(\frac{19x}{(\log x)^{1/3}}\right)\left(\frac{\log\log x}{\log x}\right)\right) \\ &= \exp\left(-(1/51+o(1))\frac{\log N \log\log\log N}{\log\log N}\right), \end{aligned}$$

which is what we wanted.

3 Comments

One of the referees noted that by following the main argument of [3] and working out the error terms using [1] one can get distributional results about positive integers n with small Ramanujan τ-function. We did not make an attempt to investigate this problem, we merely pointed out that recent results concerning the error term in the Sato-Tate law from [1] can be used to construct infinitely many integers with small values of $|\tau(n)|$. We thank the referee for this observation and leave the problem suggested by the referee as a future project. Furthermore, our argument applies to more general Fourier coefficients, namely Fourier coefficients of holomorphic cuspidal newforms $f(z)$ with even integral weight $k \ge 2$, level N, trivial nebentypus and no complex multiplication (CM), when either $f(z)$ corresponds to an elliptic curve defined over $\mathbb{Q}$ of arbitrary conductor or when f has square-free level. This case is the case of the Ramanujan function for which $k = 12$ and $N = 1$. Indeed, the main ingredient of our argument is inequality (2.1), which by the main result in [1] holds in this set-up with the coefficient 11 inside the logarithm in the right–hand side replaced by $(k-1)N$.

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