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The Lehmer problem and Beatty sequences
by

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Abstract

Let a and q be positive integers. The D. H. Lehmer problem introduces the distribution of the set

$$\{a : a \leqslant q, (a, q) = 1, ab \equiv 1 \bmod q, 2 \nmid a + b\}.$$

Zhang gave the initial approach. Lu and Yi considered a generalization of the Lehmer problem, which restricts the integers in short intervals. In this paper, we study a more general problem. Let

$$\mathcal{B}_{\alpha,\beta} := (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty}$$

be the Beatty sequence. Let c be a positive integer with $(n, q) = (c, q) = 1$, $0 < \delta_1, \delta_2 \leqslant 1$. We investigate the distribution of the set

$$\{a : a \leqslant \delta_1 q, b \leqslant \delta_2 q, ab \equiv c \bmod q, n \nmid a + b, a \in \mathcal{B}_{\alpha,\beta}\}.$$

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1 Introduction

Let q be a positive integer. For each integer a with $1 \leqslant a < q, (a, q) = 1$, there is a unique integer b with $1 \leqslant b < q$ such that $ab \equiv 1 \pmod q$. We denote b by $\overline{a}$. Let

$$r(q) := \#\{a : 1 \leqslant a \leqslant q, (a, q) = 1, 2 \nmid a + \overline{a}\}.$$

The original problem is suggested by D. H. Lehmer (see [1, P. 251, F12]) to investigate a nontrivial estimation for $r(q)$ when q is an odd prime.

Zhang [12, 11, 10] gave the initial approach and obtained asymptotic formulas for $r(q)$, one of which reads as following:

$$r(q) = \frac{1}{2}\varphi(q) + O(q^{\frac{1}{2}} d^2(q) \log^2 q).$$

Liu and Zhang [4] considered two cases of the generalized Lehmer problems in special sets on r-th residues and primitive roots respectively, obtained two interesting hybrid mean value formulas of the error terms.

The Lehmer problem was generalized by Lu and Yi [5] in the sense of short intervals. Let $n \geqslant 2$ be a fixed positive integer, $q \geqslant 3$ and c be two integers with $(n,q) = (c,q) = 1$. Let

$$r_n(\delta_1, \delta_2, c, ; q) := \sideset{}{'}\sum_{\substack{a \leqslant \delta_1 q \\ ab \equiv c \bmod q \\ n \nmid a+b}} \sideset{}{'}\sum_{b \leqslant \delta_2 q} 1 \qquad (0 < \delta_1, \delta_2 \leq 1),$$

by $\sum'$ we indicate that the variable summed over takes values coprime to the number q. By several methods of character sums, Gauss sums and Kloosterman sums, they proved

$$r_n(\delta_1, \delta_2, c; q) = \left(1 - n^{-1}\right) \delta_1 \delta_2 \varphi(q) + O(q^{\frac{1}{2}} d^6(q) \log^2 q).$$

Based on the results obtained, we find that the Lehmer problem also has good distribution properties on some special sequences. It is interesting to generalize the Lehmer problem in short intervals like Liu and Zhang's paper [4] related to r-th residues and primitive roots. In this paper, we study the mean value distribution of the generalized Lehmer problem related to a generalized arithmetic progression.

For fixed real numbers α and β, the associated non-homogeneous Beatty sequence is the sequence of integers defined by

$$\mathcal{B}_{\alpha,\beta} := (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty},$$

where $\lfloor t \rfloor$ denotes the integer part of any $t \in \mathbb{R}$. Such sequences are also called generalized arithmetic progressions. If α is irrational, it follows from a classical exponential sum estimate of Vinogradov [9] that $\mathcal{B}_{\alpha,\beta}$ contains infinitely many prime numbers; in fact, one has the asymptotic estimate

$$\#\big\{\text{prime } p \leqslant x : p \in \mathcal{B}_{\alpha,\beta}\big\} \sim \alpha^{-1} \pi(x) \qquad \text{as} \quad x \to \infty,$$

where $\pi(x)$ is the prime counting function.

For any irrational number α, we define its type $\tau = \tau(\alpha)$ by the following definition

$$\tau := \sup \Big\{ t \in \mathbb{R} : \liminf_{n \to \infty} n^t \|\alpha n\| = 0 \Big\}.$$

Using Dirichlet's approximation theorem, one can see that $\tau \geqslant 1$ for every irrational number α. Thanks to the work of Khintchine [2] and Roth [6, 7], it is known that $\tau = 1$ for almost all real numbers, in the sense of the Lebesgue measure, and for all irrational algebraic numbers, respectively. Moreover, if α is an irrational number of type $\tau < \infty$, then so are $\alpha + \theta$ with θ a rational number, α^{-1} and $n\alpha^{-1}$ for all integer $n \geqslant 1$.

We denote

$$r_n(\delta_1, \delta_2, c, \alpha, \beta; q) := \sideset{}{'}\sum_{\substack{a \leqslant \delta_1 q \\ ab \equiv c \bmod q \\ n \nmid a+b \\ a \in \mathcal{B}_{\alpha,\beta}}} \sideset{}{'}\sum_{b \leqslant \delta_2 q} 1 \qquad (0 < \delta_1, \delta_2 \leq 1)$$

and obtain the following result.

Theorem 1.1. *Let $n \geq 2$ be a fixed positive integer, $q \geq 3$ and c be two integers with $(n,q) = (c,q) = 1$, δ_1, δ_2 be real numbers satisfying $0 < \delta_1, \delta_2 \leq 1$. Let $\alpha > 1$ be an irrational number of finite type τ. Then we have the following asymptotic formula*

$$r_n(\delta_1, \delta_2, c, \alpha, \beta; q) = \left(1 - n^{-1}\right) \alpha^{-1} \delta_1 \delta_2 \varphi(q) + O\left((\varphi(q))^{\frac{\tau}{\tau+1}+\varepsilon}\right),$$

where $\varphi(\cdot)$ is the Euler function, ε is a sufficiently small positive number and the implied constant only depends on n.

Since $\tau = 1$ for almost all real numbers, Theorem 1.1 gives an "almost all" result, which gives an error term corresponding to the error term in classical Lehmer problems.

Corollary 1.2. *Let $n \geq 2$ be a fixed positive integer, $q \geq 3$ and c be two integers with $(n,q) = (c,q) = 1$, δ_1, δ_2 be real numbers satisfying $0 < \delta_1, \delta_2 \leq 1$. For almost all irrational numbers $\alpha > 1$, we have that*

$$r_n(\delta_1, \delta_2, c, \alpha, \beta; q) = \left(1 - n^{-1}\right) \alpha^{-1} \delta_1 \delta_2 \varphi(q) + O\left(q^{\frac{1}{2}+\varepsilon}\right),$$

where $\varphi(\cdot)$ is the Euler function, ε is a sufficiently small positive number and the implied constant only depends on n.

2 Preliminaries

2.1 Notation

We denote by $\lfloor t \rfloor$ and $\{t\}$ the integer part and the fractional part of t, respectively. As is customary, we put

$$\mathbf{e}(t) := e^{2\pi i t} \qquad \text{and} \qquad \psi(t) := t - \lfloor t \rfloor - \frac{1}{2}.$$

The notation $\|t\|$ is used to denote the distance from the real number t to the nearest integer; that is,

$$\|t\| := \min_{n \in \mathbb{Z}} |t - n|.$$

Let $\mathbb{P}$ denote the set of primes in $\mathbb{N}$. The letter p always denotes a prime. For a Beatty sequence $(\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty}$, we denote $\omega := \alpha^{-1}$ and $\upsilon := \alpha^{-1}(1 - \beta)$. We use notation of the form $m \sim M$ as an abbreviation for $M < m \leqslant 2M$. Let χ^0 be the principal character modulo q.

For an arbitrary set $\mathcal{S}$, we use $\mathbf{1}_{\mathcal{S}}$ to denote its indicator function:

$$\mathbf{1}_{\mathcal{S}}(n) := \begin{cases} 1 & \text{if } n \in \mathcal{S}, \\ 0 & \text{if } n \notin \mathcal{S}. \end{cases}$$

We use $\mathbf{1}_{\alpha,\beta}$ to denote the characteristic function of numbers in a Beatty sequence:

$$\mathbf{1}_{\alpha,\beta}(n) := \begin{cases} 1 & \text{if } n \in \mathcal{B}_{\alpha,\beta}, \\ 0 & \text{if } n \notin \mathcal{B}_{\alpha,\beta}. \end{cases}$$

Throughout the paper, ε always denotes an arbitrarily small positive constant, which may not be the same at different occurrences; the implied constants in symbols O, $\ll$ and $\gg$ may depend (where obvious) on the parameters α, n, ε but are absolute otherwise. For given functions F and G, the notations $F \ll G$, $G \gg F$ and $F = O(G)$ are all equivalent to the statement that the inequality $|F| \leqslant \mathcal{C}|G|$ holds with some constant $\mathcal{C} > 0$.

2.2 Technical lemmas

We need the following well–known approximation of Vaaler [8].

Lemma 2.1. *For any $H \geqslant 1$, there exist numbers a_h, b_h such that*

$$\left|\psi(t) - \sum_{0<|h|\leqslant H} a_h\,\mathbf{e}(th)\right| \leqslant \sum_{|h|\leqslant H} b_h\,\mathbf{e}(th), \qquad a_h \ll \frac{1}{|h|}, \qquad b_h \ll \frac{1}{H}.$$

The following lemma provides a convenient characterization of the numbers that occur in the Beatty sequence $\mathcal{B}_{\alpha,\beta}$.

Lemma 2.2. *A natural number m has the form $\lfloor \alpha n + \beta \rfloor$ if and only if $\mathbf{1}_{\alpha,\beta}(m) = 1$, where $\mathbf{1}_{\alpha,\beta}(m) := \lfloor -\alpha^{-1}(m-\beta)\rfloor - \lfloor -\alpha^{-1}(m+1-\beta)\rfloor$.*

Proof. Note that an integer m has the form $m = \lfloor \alpha n + \beta \rfloor$ for some integer n if and only if

$$\frac{m-\beta}{\alpha} \leqslant n < \frac{m-\beta+1}{\alpha}.$$

□

Lemma 2.3. *Let $\alpha \in \mathbb{R}$, Q be an integer and P a positive integer. Then*

$$\left|\sum_{x=Q+1}^{Q+P} \mathbf{e}(\alpha x)\right| \leqslant \min\left(P, \frac{1}{2\|\alpha\|}\right).$$

Proof. See [3, Lemma 1]. □

2.3 Integers in Beatty sequences

Lemma 2.4. *Let a, q be positive integers, $\delta \in (0,1)$ be a real number, θ be a rational number. Let α be an irrational number of finite type τ and $H > 0$. We have*

$$\sum_{\substack{a\leqslant \delta q\\ a\in\mathcal{B}_{\alpha,\beta}}} \mathbf{e}(\theta a) = \alpha^{-1}\sum_{a\leqslant \delta_1 q} \mathbf{e}(\theta a) + O\left(\|\theta\|^{-1}q^{-\varepsilon} + q^{\varepsilon}\right).$$

Proof. We start by Lemma 2.2, then

$$\sum_{\substack{a\leqslant \delta q\\ a\in\mathcal{B}_{\alpha,\beta}}} \mathbf{e}(\theta a) = \sum_{a\leqslant \delta q} \mathbf{1}_{\alpha,\beta}(a)\mathbf{e}(\theta a),$$

where

$$\begin{aligned}\mathbf{1}_{\alpha,\beta}(m) &:= \lfloor -\alpha^{-1}(a-\beta)\rfloor - \lfloor -\alpha^{-1}(a+1-\beta)\rfloor \\ &= \alpha^{-1} + \psi(-(\omega(a+1-\beta))) - \psi(-\omega(a-\beta)).\end{aligned}$$

We deduce that

$$\sum_{\substack{a\leqslant \delta q\\ a\in\mathcal{B}_{\alpha,\beta}}} \mathbf{e}(\theta a) = \alpha^{-1}\sum_{a\leqslant\delta q}\mathbf{e}(\theta a) + S_1 + O(S_2),$$

where

$$S_1 = \sum_{a\leqslant\delta q}\mathbf{e}(\theta a)\sum_{0<|h|\leqslant H} a_h(\mathbf{e}(\omega h(a+1-\beta)) - \mathbf{e}(\omega h(a-\beta)))$$

and

$$S_2 = \sum_{a\leqslant\delta q}\mathbf{e}(\theta a)\sum_{|h|\leqslant H} b_h(\mathbf{e}(\omega h(a+1-\beta)) + \mathbf{e}(\omega h(a-\beta)))$$

by Lemma 2.1 and $H := q^{\varepsilon}$. Let

$$\upsilon_h := \mathbf{e}(-\omega h\beta)(\mathbf{e}(\omega h) - 1) \ll 1.$$

For S_1, we have that

$$S_1 = \sum_{0<|h|\leqslant H} a_h\upsilon_h \sum_{a\leqslant\delta q}\mathbf{e}((\theta+\omega h)a). \tag{1}$$

By Lemma 2.3, we have

$$\sum_{a\leqslant\delta q}\mathbf{e}((\theta+\omega h)a) \leqslant \min\left(\lfloor\delta q\rfloor, \frac{1}{2\|\theta+\omega h\|}\right). \tag{2}$$

For any sufficiently small $\varepsilon_0 > 0$, since $\theta/h+\omega$ is of type τ, there exists some constant $\mathfrak{c} > 0$ such that

$$\left\|\left(\frac{\theta}{h}+\omega\right)h\right\| > \mathfrak{c}h^{-\tau-\varepsilon_0}, \qquad h\geqslant 1. \tag{3}$$

Insert (2) and (3) to (1), it follows that

$$S_1 \ll \sum_{0<h<H} h^{-1}h^{\tau+\varepsilon_0} \ll H^{\tau+\varepsilon_0} \ll q^{\varepsilon}.$$

The contribution from $h = 0$ of S_2 is

$$\sum_{a\leqslant\delta q}\frac{1}{H}\mathbf{e}(\theta a) \ll H^{-1}\|\theta\|^{-1} \leqslant \|\theta\|^{-1}q^{-\varepsilon}.$$

The contribution from $h \neq 0$ of S_2 is similar to S_1, which is

$$\ll \sum_{0<h<H} H^{-1}h^{\tau+\varepsilon_0} \ll H^{\tau+\varepsilon} \ll q^{\varepsilon}.$$

$\square$

We remark that by taking
$$H = \|\theta\|^{-\frac{1}{\tau+1}+\varepsilon},$$
we have the optimal error term in Lemma 2.4, which gives that
$$\sum_{\substack{a\leqslant \delta q\\ a\in\mathcal{B}_{\alpha,\beta}}} \mathbf{e}(\theta a) = \alpha^{-1}\sum_{a\leqslant \delta_1 q}\mathbf{e}(\theta a) + O\left(\|\theta\|^{-\left(\frac{\tau}{\tau+1}+\varepsilon\right)}\right).$$
However, this optimization gives no better bound to our theorem. That is the reason we keep the easy estimation of Lemma 2.4.

Lemma 2.5. *Let a, q be positive integers, $\delta \in (0,1)$ be a real number, θ be a rational number. Let α be an irrational number of finite type τ. We have*
$$\sum_{\substack{a\leqslant \delta q\\ a\in\mathcal{B}_{\alpha,\beta}}}{}' 1 = \alpha^{-1}\delta\varphi(q) + O\left((\varphi(q))^{\frac{\tau}{\tau+1}+\varepsilon}\right).$$

Proof. The method is similar to the proof of Lemma 2.4. By Lemma 2.1 and Lemma 2.2
$$\sum_{\substack{a\leqslant \delta q\\ a\in\mathcal{B}_{\alpha,\beta}}}{}' 1 = \sum_{a\leqslant \delta q}{}' \mathbf{1}_{\alpha,\beta}(a) = T_1 + T_2 + T_3,$$
where
$$\begin{aligned} T_1 &:= \sum_{a\leqslant \delta q}{}' \alpha^{-1};\\ T_2 &:= \sum_{a\leqslant \delta q}{}' \sum_{0<|h|\leqslant H} a_h(\mathbf{e}(\omega h(a+1-\beta)) - \mathbf{e}(\omega h(a-\beta)));\\ T_3 &:= \sum_{a\leqslant \delta q}{}' \sum_{|h|\leqslant H} b_h(\mathbf{e}(\omega h(a+1-\beta)) + \mathbf{e}(\omega h(a-\beta))),\end{aligned}$$
with
$$H := (\varphi(q))^{\frac{1}{\tau+1}-\varepsilon}, \quad a_h \ll |h|^{-1}, \quad b_h \ll |H|^{-1}.$$
By a well-known estimation, it follows that
$$T_1 = \alpha^{-1}\delta\varphi(q) + O(1).$$
To be short, let
$$g(a) := \sum_{d|(a,q)} \mu(d),$$
and
$$\upsilon_h := \mathbf{e}(-\omega h\beta)(\mathbf{e}(\omega h) - 1) \ll 1.$$
We conclude that
$$T_2 = \sum_{a\leqslant \delta q}\sum_{0<|h|<H} g(a) a_h \upsilon_h \mathbf{e}(\omega h a)$$

$$\begin{aligned}
&= \sum_{0<|h|<H} a_h \upsilon_h \sum_{a\leqslant \delta q} g(a)\mathbf{e}(\omega ha) \\
&= \sum_{0<|h|<H} a_h \upsilon_h \sum_{d|q} \mu(d) \sum_{b\leqslant \delta q/d} \mathbf{e}(\omega hbd) \\
&= \sum_{d|q} \mu(d) \sum_{0<|h|<H} a_h \upsilon_h \sum_{b\leqslant \delta q/d} \mathbf{e}(\omega hbd).
\end{aligned}$$

By Lemma 2.3, we have

$$\sum_{b\leqslant \delta q/d} \mathbf{e}(\omega hbd) \leqslant \min\left(\left\lfloor \frac{\delta q}{d} \right\rfloor, \frac{1}{2\|\omega hd\|}\right). \tag{4}$$

For any sufficiently small $\varepsilon_0 > 0$, since ωd is of type τ, there exists some constant $\mathfrak{c} > 0$ such that

$$\|\omega hd\| > \mathfrak{c} h^{-\tau-\varepsilon_0}, \qquad h \geqslant 1. \tag{5}$$

Insert (5) to (4), we derive that

$$T_2 \ll \sum_{d|q} \sum_{0<h<H} h^{-1} h^{\tau+\varepsilon_0} \ll H^{\tau+\varepsilon_0} \sum_{d|q} 1 \ll H^{\tau+\varepsilon} \ll (\varphi(q))^{\frac{\tau}{\tau+1}+\varepsilon}.$$

The contribution from $h = 0$ of T_3 is

$$\ll H^{-1} {\sum_{a\leqslant \delta q}}' 1 \ll H^{-1}\varphi(q) \ll (\varphi(q))^{\frac{\tau}{\tau+1}+\varepsilon}.$$

The contribtuon from $h \neq 0$ of T_3 is similar to T_2, which is

$$\ll \sum_{d|q} \left| \mu(d) \sum_{0<h<H} H^{-1} h^{\tau+\varepsilon_0} \right| \ll H^{\tau+\varepsilon_0} \sum_{d|q} 1 \ll (\varphi(q))^{\frac{\tau}{\tau+1}+\varepsilon},$$

which is the same as T_2.

□

3 Proof of Theorem 1.1

We begin by the definition

$$r_n(\delta_1, \delta_2, c, \alpha, \beta; q) = \mathcal{S}_1 - \mathcal{S}_2,$$

where

$$\mathcal{S}_1 := {\sum_{\substack{a\leqslant \delta_1 q \\ ab\equiv c \bmod q \\ a\in\mathcal{B}_{\alpha,\beta}}}}' {\sum_{b\leqslant \delta_2 q}}' 1$$

and

$$\mathcal{S}_2 := \sideset{}{'}\sum_{\substack{a\leqslant\delta_1 q \\ ab\equiv c \bmod q \\ n|a+b \\ a\in\mathcal{B}_{\alpha,\beta}}} \sideset{}{'}\sum_{b\leqslant\delta_2 q} 1.$$

We work on $\mathcal{S}_1$, then

$$\begin{aligned}
\mathcal{S}_1 &= \sideset{}{'}\sum_{\substack{a\leqslant\delta_1 q \\ ab\equiv c \bmod q}} \sideset{}{'}\sum_{b\leqslant\delta_2 q} \mathbf{1}_{\alpha,\beta}(a) \\
&= \frac{1}{\varphi(q)} \sideset{}{'}\sum_{a\leqslant\delta_1 q} \sideset{}{'}\sum_{b\leqslant\delta_2 q} \sum_{\chi \bmod q} \chi(ab)\overline{\chi}(c)\mathbf{1}_{\alpha,\beta}(a) \\
&= \mathcal{S}_{11} + \mathcal{S}_{12},
\end{aligned}$$

where

$$\mathcal{S}_{11} := \frac{1}{\varphi(q)} \sideset{}{'}\sum_{a\leqslant\delta_1 q} \sideset{}{'}\sum_{b\leqslant\delta_2 q} \mathbf{1}_{\alpha,\beta}(a)$$

and

$$\mathcal{S}_{12} := \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi\neq\chi^0}} \overline{\chi}(c) \left(\sideset{}{'}\sum_{a\leqslant\delta_1 q} \chi(a)\mathbf{1}_{\alpha,\beta}(a) \right) \left(\sideset{}{'}\sum_{b\leqslant\delta_2 q} \chi(b) \right).$$

For $\mathcal{S}_2$, it follows that

$$\begin{aligned}
\mathcal{S}_2 &= \frac{1}{\varphi(q)} \sideset{}{'}\sum_{\substack{a\leqslant\delta_1 q \\ n|a+b}} \sideset{}{'}\sum_{b\leqslant\delta_2 q} \sum_{\chi \bmod q} \chi(ab)\overline{\chi}(c)\mathbf{1}_{\alpha,\beta}(a) \\
&= \mathcal{S}_{21} + \mathcal{S}_{22},
\end{aligned}$$

where

$$\mathcal{S}_{21} := \frac{1}{\varphi(q)} \sideset{}{'}\sum_{\substack{a\leqslant\delta_1 q \\ n|a+b}} \sideset{}{'}\sum_{b\leqslant\delta_2 q} \mathbf{1}_{\alpha,\beta}(a)$$

and

$$\mathcal{S}_{22} := \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi\neq\chi^0}} \overline{\chi}(c) \sideset{}{'}\sum_{\substack{a\leqslant\delta_1 q \\ n|a+b}} \sideset{}{'}\sum_{b\leqslant\delta_2 q} \chi(ab)\mathbf{1}_{\alpha,\beta}(a).$$

3.1 Estimation of $\mathcal{S}_{11}$

By the classical bound

$$\sideset{}{'}\sum_{a\leqslant\delta_1 q} 1 = \delta_1\varphi(q) + O(d(q)),$$

and Lemma 2.5, we have

$$\begin{aligned}\mathcal{S}_{11} &= \left(\delta_2 + O\left(\frac{d(q)}{\varphi(q)}\right)\right) \sideset{}{'}\sum_{a\leqslant \delta_1 q} \mathbf{1}_{\alpha,\beta}(a) \\ &= \left(\delta_2 + O\left(\frac{d(q)}{\varphi(q)}\right)\right) \left(\alpha^{-1}\delta_1\varphi(q) + O\left((\varphi(q))^{\frac{\tau}{\tau+1}+\varepsilon}\right)\right) \\ &= \alpha^{-1}\delta_1\delta_2\varphi(q) + O\left((\varphi(q))^{\frac{\tau}{\tau+1}+\varepsilon}\right). \end{aligned} \tag{6}$$

3.2 Estimation of $\mathcal{S}_{21}$

Our estimation follows from the argument of [5, Equation (9)], which is

$$\begin{aligned}\mathcal{S}_{21} &= \frac{1}{\varphi(q)} \sideset{}{'}\sum_{a\leqslant \delta_1 q} \mathbf{1}_{\alpha,\beta}(a) \sum_{\substack{b\leqslant \delta_2 q \\ b\equiv -a \bmod n}} \sum_{d|(b,q)} \mu(d) \\ &= \frac{1}{\varphi(q)} \sideset{}{'}\sum_{a\leqslant \delta_1 q} \mathbf{1}_{\alpha,\beta}(a) \sum_{d|q} \mu(d) \sum_{\substack{b\leqslant \delta_2 q \\ d|b \\ b\equiv -a \bmod n}} 1 \\ &= \frac{1}{\varphi(q)} \sideset{}{'}\sum_{a\leqslant \delta_1 q} \mathbf{1}_{\alpha,\beta}(a) \sum_{d|q} \mu(d) \left(\frac{\delta_2 q}{nd} + O(1)\right) \\ &= \frac{1}{\varphi(q)} \sideset{}{'}\sum_{a\leqslant \delta_1 q} \mathbf{1}_{\alpha,\beta}(a) \left(\frac{\delta_2\varphi(q)}{n} + O(d(q))\right) \\ &= \frac{1}{\varphi(q)} \left(\alpha^{-1}\delta_1\varphi(q) + O\left((\varphi(q))^{\frac{\tau}{\tau+1}+\varepsilon}\right)\right) \left(\frac{\delta_2\varphi(q)}{n} + O(d(q))\right) \\ &= \alpha^{-1}\delta_1\delta_2 n^{-1}\varphi(q) + O\left((\varphi(q))^{\frac{\tau}{\tau+1}+\varepsilon}\right). \end{aligned} \tag{7}$$

Combining (6) and (7), we have

$$\begin{aligned} r_n(\delta_1,\delta_2,c,\alpha,\beta;q) &= (1-n^{-1})\alpha^{-1}\delta_1\delta_2\varphi(q) \\ &\quad + S_{12} - S_{22} + O\left((\varphi(q))^{\frac{\tau}{\tau+1}+\varepsilon}\right). \end{aligned} \tag{8}$$

3.3 Estimation of $\mathcal{S}_{22}$ and $\mathcal{S}_{12}$

We begin with

$$\begin{aligned}\mathcal{S}_{22} &= \frac{1}{n\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi\neq\chi^0}} \overline{\chi}(c) \sideset{}{'}\sum_{a\leqslant \delta_1 q} \mathbf{1}_{\alpha,\beta}(a) \sideset{}{'}\sum_{b\leqslant \delta_2 q} \chi(ab) \sum_{l=1}^{n} \mathbf{e}\left(\frac{a+b}{n} l\right) \\ &= \frac{1}{n\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi\neq\chi^0}} \overline{\chi}(c) \sum_{l=1}^{n} \left(\sum_{a\leqslant \delta_1 q} \mathbf{1}_{\alpha,\beta}(a)\chi(a)\mathbf{e}\left(\frac{a}{n} l\right)\right)\end{aligned}$$

$$\cdot \left(\sum_{b \leqslant \delta_2 q} \chi(b) \mathbf{e}\left(\frac{b}{n} l\right) \right) \tag{9}$$

Let

$$G(r, \chi) := \sum_{h=1}^{q} \chi(h) \mathbf{e}\left(\frac{rh}{q}\right)$$

be the Gauss sum. For any nonprincipal character $\chi \bmod q$,

$$\chi(a) = \frac{1}{q} \sum_{r=1}^{q} G(r, \chi) \mathbf{e}\left(-\frac{ar}{q}\right) = \frac{1}{q} \sum_{r=1}^{q-1} G(r, \chi) \mathbf{e}\left(-\frac{ar}{q}\right)$$

and

$$\frac{l}{n} - \frac{r}{q} \neq 0$$

for $1 \leqslant l \leqslant n, 1 \leqslant r \leqslant q-1$ and $(n, q) = 1$. By the same argument of [5, Equation (13)], we have

$$\sum_{b \leqslant \delta_2 q} \chi(b) \mathbf{e}\left(\frac{b}{n} l\right) = \frac{1}{q} \sum_{r_2=1}^{q-1} G(r_2, \chi) \frac{f(\delta_2, l, r_2; n, q)}{\mathbf{e}\left(\frac{r_2}{q} - \frac{l}{n}\right) - 1}, \tag{10}$$

where

$$f(\delta, l, r; n, q) := 1 - \mathbf{e}\left(\left(\frac{l}{n} - \frac{r}{q}\right) \lfloor \delta q \rfloor\right)$$

and

$$|f(\delta_2, l, r; n, q)| \leqslant 2.$$

For a, by Lemma 2.4 we have

$$\begin{aligned}
&\sum_{a \leqslant \delta_1 q} \mathbf{1}_{\alpha,\beta}(a) \chi(a) \mathbf{e}\left(\frac{a}{n} l\right) \\
&= \frac{1}{q} \sum_{a \leqslant \delta_1 q} \mathbf{1}_{\alpha,\beta}(a) \sum_{r_1=1}^{q-1} G(r_1, \chi) \mathbf{e}\left(\left(\frac{l}{n} - \frac{r_1}{q}\right) a\right) \\
&= \frac{1}{q} \sum_{r_1=1}^{q-1} G(r_1, \chi) \sum_{a \leqslant \delta_1 q} \mathbf{1}_{\alpha,\beta}(a) \mathbf{e}\left(\left(\frac{l}{n} - \frac{r_1}{q}\right) a\right) \\
&= \frac{1}{\alpha q} \sum_{r_1=1}^{q-1} G(r_1, \chi) \left(\sum_{a \leqslant \delta_1 q} \mathbf{e}\left(\left(\frac{l}{n} - \frac{r_1}{q}\right) a\right) + O\left(\frac{q^{-\varepsilon}}{\|\frac{l}{n} - \frac{r_1}{q}\|} + q^{\varepsilon}\right) \right) \\
&= \frac{1}{\alpha q} \sum_{r_1=1}^{q-1} G(r_1, \chi) \left(\frac{f(\delta_1, l, r_1; n, q)}{\mathbf{e}\left(\frac{r_1}{q} - \frac{l}{n}\right) - 1} + O\left(\frac{q^{-\varepsilon}}{\|\frac{l}{n} - \frac{r_1}{q}\|} + q^{\varepsilon}\right) \right) \qquad (11)
\end{aligned}$$

Combining (9), (10) and (11), we bound $\mathcal{S}_{22}$ by bounding

$$\mathcal{S}_{23} := \frac{1}{\alpha n \varphi(q) q^2} \sum_{l=1}^{n} \sum_{\substack{\chi \bmod q \\ \chi \neq \chi^0}} \overline{\chi}(c) \sum_{r_1=1}^{q-1} G(r_1, \chi) \frac{f(\delta_1, l, r_1; n, q)}{\mathbf{e}\left(\frac{r_1}{q} - \frac{l}{n}\right) - 1}$$

$$\cdot \sum_{r_2=1}^{q-1} G(r_2,\chi) \frac{f(\delta_2,l,r_2;n,q)}{\mathbf{e}\left(\frac{r_2}{q}-\frac{l}{n}\right)-1},$$

$$\mathcal{S}_{24} := \frac{1}{\alpha n\varphi(q)q^{2+\varepsilon}} \sum_{l=1}^{n} \sum_{\substack{\chi \bmod q \\ \chi\neq\chi^0}} \overline{\chi}(c) \left(\sum_{r_1=1}^{q-1} G(r_1,\chi) \frac{1}{\|\frac{l}{n}-\frac{r_1}{q}\|}\right)$$

$$\cdot \left(\sum_{r_2=1}^{q-1} G(r_2,\chi) \frac{f(\delta_2,l,r_2;n,q)}{\mathbf{e}\left(\frac{r_2}{q}-\frac{l}{n}\right)-1}\right),$$

$$\mathcal{S}_{25} := \frac{1}{\alpha n\varphi(q)q^{2-\varepsilon}} \sum_{l=1}^{n} \sum_{\substack{\chi \bmod q \\ \chi\neq\chi^0}} \overline{\chi}(c) \sum_{r_1=1}^{q-1} G(r_1,\chi) \sum_{r_2=1}^{q-1} G(r_2,\chi) \frac{f(\delta_2,l,r_2;n,q)}{\mathbf{e}\left(\frac{r_2}{q}-\frac{l}{n}\right)-1}.$$

By the same argument of [5, Page 1273], it follows that

$$\mathcal{S}_{23} \ll \frac{d^2(q)}{q^{3/2}} \sum_{l=1}^{n} \left(\sum_{r=1}^{q-1} \frac{(r,q)}{\left|\mathbf{e}\left(\frac{r}{q}-\frac{l}{n}\right)-1\right|}\right)^2 \ll q^{1/2} d^6(q) \log^2 q,$$

$$\mathcal{S}_{24} \ll \frac{d^2(q)}{q^{3/2+\varepsilon}} \sum_{l=1}^{n} \left(\sum_{r=1}^{q-1} \frac{(r,q)}{\left|\mathbf{e}\left(\frac{r}{q}-\frac{l}{n}\right)-1\right|}\right)^2 \ll q^{1/2} d^6(q) \log^2 q,$$

$$\mathcal{S}_{25} \ll \frac{d^2(q)}{q^{3/2-\varepsilon}} \sum_{l=1}^{n} \sum_{r_1=1}^{q-1} (r_1,q) \sum_{r_2=1}^{q-1} \frac{(r_2,q)}{\left|\mathbf{e}\left(\frac{r_2}{q}-\frac{l}{n}\right)-1\right|} \ll q^{1/2+\varepsilon} d^4(q) \log q.$$

Hence eventually we derive

$$\mathcal{S}_{22} \ll q^{1/2+\varepsilon}.$$

Taking that $n=1$, we know that

$$\mathcal{S}_{12} \ll q^{1/2+\varepsilon}.$$

With (8), the proof is complete.

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